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Prescribing scalar curvatures: non compactness versus critical points at infinity

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Abstract: We illustrate an example of a generic, positive function K on a Riemannian manifold to be conformally prescribed as the scalar curvature, for which the corresponding Yamabe type L^2 -gradient flow exhibits non compact flow lines, while a slight modification of it is compact.

Keywords: Conformal geometry, scalar curvature, critical points at infinity, geometric flows

MSC: 35B33, 35R01, 53A30, 53C44

1 Introduction

Within the setting of conformally prescribing the scalar curvature on a Riemannian manifold and in the context of the calculus of variations, i.e. by considering an associated energy functional, we shall illustrate in a very particular case the difference of non compact flow lines of a given gradient flow to critical points at infinity, as we have discussed in [16], namely showing, that the volume preserving L^2 -gradient flow (1.1), which is a natural analogon to the Yamabe flow and was studied in [15], exhibits one specific, single bubbling non compactness for exactly one energetic value of the variationally associated prescribed scalar curvature functional, while a suitable modification of this flow eliminates any non compactness. And, as we shall see, the same holds true for the strong gradient type flow (1.3) modified to preserve the conformal volume just like (1.1). Hence as a take away those non compact flow lines do not induce critical points at infinity, cf. [16], i.e. these flows lead to variationally unmotivated singularities and are hence as geometric flows evidently not the best choice in the context of the calculus of variations, i.e. for energetic deformations.

However such gradient type flows, whether weak or strong, i.e. with respect to a L^2 - or $W^{1,2}$ -gradient, are of interest in their own right apart from their usefulness in proving mere existence results to the underlying elliptic problem of prescribing the scalar curvature on a Riemannian manifold conformally, in particular due to the naturality of L^2 -gradient flows for a geometric problem.

We wish to mention some works relevant to the flow analysis.

- (i) The most simple case evidently is, when the function K to be prescribed is constant, e.g. $K = 1$, and the underlying manifold is the standard sphere $\mathbb{S}^n$, in which case flow convergence is known, cf. [2], [19], with exponential speed, cf. [7].
- (ii) Later on and based on the positive mass theorem also on non spherical manifolds flow convergence in the Yamabe case $K = 1$ was established, cf. [19], [18], [8], with a subsequent analysis on upper and lower bounds of the speed of convergence, cf. [9].
- (iii) Returning to the spherical case $M = \mathbb{S}^n$, but considering a non constant function K to be conformally prescribed as the scalar curvature, flows and their lack of compactness were first analysed and charac-

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terised in [2], [3] and [5] in case $n = 3$. For higher dimensional cases we refer to [6] for $n = 4$ and to [16] for $n \geq 5$, see also [12], [13] and [14].

- (iv) Finally the case of a general Riemannian manifold M with non constant K to be prescribed, to which the present work belongs, has been less studied with respect to an analysis of gradient flows. We point in case of a positive Yamabe invariant of M to [16] for a classification of non compactness in dimensions $n \geq 5$ and to [15] for some compactness results in dimensions $n = 3, 4, 5$. In case of a negative Yamabe invariant flow convergence was proven in [1] recently.

In order to introduce the relevant notions, consider a smooth, closed Riemannian manifold

$$M = (M^n, g_0), \quad n = 3, 4, 5$$

with volume measure μ_{g_0} and scalar curvature R_{g_0} . The Yamabe invariant

$$Y(M) = \inf_{\mathcal{A}} \frac{\int c_n |\nabla u|_{g_0}^2 + R_{g_0} u^2 d\mu_{g_0}}{(\int u^{\frac{2n}{n-2}} d\mu_{g_0})^{\frac{n-2}{n}}} \quad \text{with } c_n = 4 \frac{n-1}{n-2},$$

where

$$\mathcal{A} = \{u \in W_{g_0}^{1,2}(M) : u \geq 0, u \not\equiv 0\},$$

is assumed to be positive. Then the conformal Laplacian

$$L_{g_0} = -c_n \Delta_{g_0} + R_{g_0}$$

is a positive, selfadjoint operator with Green's function G_{g_0} . We may assume

$$R_{g_0} > 0 \quad \text{and} \quad \int K d\mu_{g_0} = 1$$

for the background metric g_0 . For a conformal metric

$$g = g_u = u^{\frac{4}{n-2}} g_0$$

there holds $d\mu = d\mu_{g_u} = u^{\frac{2n}{n-2}} d\mu_{g_0}$ for the volume element and

$$R = R_{g_u} = u^{-\frac{n+2}{n-2}} (-c_n \Delta_{g_0} u + R_{g_0} u) = u^{-\frac{n+2}{n-2}} L_{g_0} u$$

for the scalar curvature. We may define

$$\|u\|^2 = \int L_{g_0} u u d\mu_{g_0}$$

and use $\|\cdot\|$ as an equivalent norm on $W^{1,2}(M)$. Let $0 < K \in C^\infty(M)$ and

$$r = r_u = \int R d\mu, \quad k = k_u = \int K d\mu, \quad \bar{K} = \bar{K}_u = \frac{K}{k}.$$

In [15] we have studied the L^2 -pseudo gradient flow

$$\partial_t u = -\left(\frac{R}{\bar{K}} - \frac{r}{k}\right)u \quad \text{on } X = \{u \in C^\infty(M, \mathbb{R}_+) : k = 1\}, \quad (1.1)$$

which evidently coincides with the Yamabe flow in case $K = 1$. Obviously

$$\partial_t k = 0,$$

i.e. the unit volume $k \equiv 1$ is preserved. Let us consider the scaling invariant energy

$$J(u) = \frac{\int c_n |\nabla u|_{g_0}^2 + R_{g_0} u^2 d\mu_{g_0}}{(\int K u^{\frac{2n}{n-2}} d\mu_{g_0})^{\frac{n-2}{n}}} = \frac{\int L_{g_0} u u d\mu_{g_0}}{(\int K u^{\frac{2n}{n-2}} d\mu_{g_0})^{\frac{n-2}{n}}} \quad \text{for } u \in \mathcal{A}, \quad (1.2)$$

omitting from now on $d\mu_{g_0}$, when integrating with respect to it.

Proposition 1.1. We have $J(u) = \frac{r}{k^{\frac{n-2}{n}}}$ and

$$\begin{aligned} (i) \quad & \frac{1}{2} \partial J(u) v = \frac{1}{k^{\frac{n-2}{n}}} \left[\int L_{g_0} u v - \frac{r}{k} \int K u^{\frac{n+2}{n-2}} v \right] = \frac{1}{k^{\frac{n-2}{n}}} \int \left(R - \frac{r}{k} K \right) u^{\frac{n+2}{n-2}} v \\ (ii) \quad & \frac{1}{2} \partial^2 J(u) v w = \frac{1}{k^{\frac{n-2}{n}}} \left[\int L_{g_0} v w - \frac{n+2}{n-2} \frac{r}{k} \int K u^{\frac{4}{n-2}} v w \right] \\ & - \frac{2}{k^{\frac{n-2}{n}+1}} \left[\int L_{g_0} u v \int K u^{\frac{n+2}{n-2}} w + \int L_{g_0} u w \int K u^{\frac{n+2}{n-2}} v \right] \\ & + 4 \frac{n-1}{n-2} \frac{r}{k^{\frac{n-2}{n}+2}} \int K u^{\frac{n+2}{n-2}} v \int K u^{\frac{n+2}{n-2}} w. \end{aligned}$$

Moreover J is $C_{loc}^{2,\alpha}$ and uniformly Hölder continuous on each

$$U_\varepsilon = \{u \in \mathcal{A} : \varepsilon < \|u\|, J(u) \leq \varepsilon^{-1}\} \subset \mathcal{A}.$$

In particular the problem of conformally prescribing the scalar curvature is variational and

$$\frac{1}{2} |\partial J(u)| \leq \frac{1}{k^{\frac{n-2}{n}}} \|R - r\bar{K}\|_{L_\mu^{\frac{2n}{n+2}}} \leq \frac{1}{k^{\frac{n-2}{n}}} \|R - r\bar{K}\|_{L_\mu^2},$$

where $|\partial J(u)| = |\partial J(u)|_{W_{g_0}^{-1,2}(M)}$. Then by a slight abuse of notation we define

$$|\delta J|(u) = 2k^{\frac{2-n}{n}} \|R - r\bar{K}\|_{L_\mu^2}$$

as a natural majorant of $|\partial J(u)|$ and along a flow line we have

$$\partial_t J(u) \lesssim -|\delta J(u)|^2.$$

From Theorem 1 in [15] we know at least in cases $n = 3, 4, 5$, that every flow line for (1.1) exists positively for all times. Consequently we have a priori

$$\int_0^\infty |\delta J(u)|^2 dt < \infty,$$

as J by positivity of the Yamabe invariant is lower bounded. Similarly we may consider the gradient flow

$$\partial_t u = -\nabla J(u), \quad \nabla = \nabla^{L_{g_0}},$$

for which $\partial_t \|u\| = 0$ instead of $\partial_t k = 0$. This describes a strong gradient flow, since by definition

$$\forall w \in W^{1,2}(M) : \langle \nabla J(u), w \rangle_{L_{g_0}} = \partial J(u) w \text{ and } \|\cdot\|_{L_{g_0}} \simeq \|\cdot\|_{W^{1,2}},$$

and we write $\nabla J(u) = L_{g_0}^- \partial J(u)$. For the sake of easy comparability to (1.1) consider

$$\partial_t u = -\frac{r}{2k} \left(\nabla J(u) - \frac{\int K u^{\frac{n+2}{n-2}} \nabla J(u)}{k} u \right) \quad (1.3)$$

as a strong pseudo gradient flow. Then $\partial_t k = 0$ and, since by scaling invariance we have

$$\partial J(u) u = 0,$$

there holds under (1.3) on X

$$\partial_t J(u) = -\frac{r}{2k} \|\nabla J(u)\|^2 = -\frac{r}{2} |\partial J(u)|^2.$$

In particular and by positivity of the Yamabe invariant we have along each flow line

$$c(K) \leq J(u) = r_u = r = \int L_{g_0} u u = \|u\|^2 \leq J(u_0). \quad (1.4)$$

Then, since

$$\nabla J(u) = L_{g_0}^- \partial J(u) = k^{\frac{2-n}{n}} L_{g_0}^- (L_{g_0} u - r \bar{K} u^{\frac{n+2}{n-2}}) \leq \frac{u}{k^{\frac{n-2}{n}}} \quad (1.5)$$

by positivity of $L_{g_0}^- = G_{g_0}$, we find under (1.3)

$$\partial_t u \geq -C(1 + |\partial J(u)|)u,$$

so $u > 0$ is preserved. Indeed due to $k = 1$ and (1.4) we find from Proposition 1.1, that $|\partial J(u)|$ is a priori bounded along flow lines. Therefore each flow line exists positively for all times and

$$\partial_t J(u) \simeq -|\partial J(u)|^2,$$

whence

$$\int_0^\infty \|\nabla J(u)\|^2 = \int_0^\infty |\partial J(u)|^2 < \infty.$$

We thus see, that (1.3) defines a pseudo gradient flow on X as well. Note, that (1.3) falls into the class of ordinary differential equations, hence long time existence is a non issue in contrast to the L^2 -type flow (1.1). The difference, when considering (1.1) in contrast to (1.3) apart from the distinguishing quadratic a priori integrability of $|\delta J|$ versus $|\partial J|$ lies in the ease of adaptability. In fact considering a bounded and for instance smooth vectorfield W on X satisfying $\langle \nabla J, W \rangle \geq 0$ we may modify (1.3) to

$$\partial_t u = -\frac{r}{2k} (\nabla J(u) + W - \frac{\int K u^{\frac{n+2}{n-2}} (\nabla J(u) + W)}{k} u), \quad (1.6)$$

as we shall do in Section 3.3. We then still decrease energy, find quadratic a priori integrability of $|\partial J|$, preserve $\partial_t k = 0$ and $u > 0$ and finally also (1.6) falls into the class of ordinary differential equations, hence also (1.6) defines a flow on X . In contrast the long time existence of (1.1) relies on higher order integrability properties of $R - r\bar{K}$, cf. [8],[15], which may be destroyed by even slight adaptations.

In any case, i.e. (1.1),(1.3) or (1.6), the volume $k = 1$ is preserved and the lower bounded energy J decreased, whence along a flow line u

$$\int L_{g_0} u u = r = k^{\frac{n-2}{n}} J(u) = J(u) < J(u)|_{t=0} < \infty,$$

i.e. we have norm control along each flow line. Moreover under (1.1) there holds

$$|\partial J(u)| \lesssim |\delta J(u)| \rightarrow 0 \text{ as } t \rightarrow \infty, \quad (1.7)$$

cf. Proposition 2.11 in [15]. Likewise there holds under (1.6)

$$|\partial J(u)| = \|\nabla J(u)\| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Indeed $\int_0^\infty |\partial J(u)|^2 < \infty$ necessitates

$$|\partial J(u_{t_k})| \rightarrow 0 \text{ and } \int_{t_k}^\infty |\partial J(u)|^2 \rightarrow 0 \text{ as } k \rightarrow \infty$$

for a least a sequence $t_k \rightarrow \infty$ as $k \rightarrow \infty$ in time and thus for any $t > t_k$

$$|\partial J(u_t)|^4 \leq |\partial J(u_{t_k})|^4 + C \int_{t_k}^\infty |\partial J(u)|^2 \rightarrow 0 \text{ as } k \rightarrow \infty \quad (1.8)$$

using a priori uniform boundedness of $|\partial J(u)|$ and $|\partial^2 J(u)|$, cf. Proposition 1.1, along flow lines.

Based on a fine description of a possible lack of compactness of flow lines, we had extracted suitable assumptions to guarantee compactness of the flow on X induced by (1.1), cf. Theorem 2 from [15]. For instance for $n = 5$ under

Cond₅: M is not conformally equivalent to the standard sphere $\mathbb{S}^5$ and

$$\langle \nabla \Delta K, \nabla K \rangle > \frac{1}{3} |\Delta K|^2$$

on $\{\Delta K < 0\} \cap U$ for an open neighbourhood U of $\{\nabla K = 0\}$

every flow line for (1.1) is compact and hence converges to a solution of $\partial J = 0$ in X . We will restrict our attention to the very simple scenario

Condition 1.2. *Let $n = 5$ and*

- (i) $M \not\cong \mathbb{S}^5$ conformally
- (ii) $\exists x_0 \in M : \{x_0\} = \{K = \max_M K\}$
- (iii) $\Delta K > 0$ on $\{x_1, \dots, x_q\} = \{\nabla K = 0\} \setminus \{x_0\}$
- (iv) *in a conformal normal coordinate system around $x_0 \simeq 0$ we have*

$$K(x) = 1 - |x|^4, \text{ where } |x| = \left(\sum_i x_i^2\right)^{\frac{1}{2}}.$$

We refer to [11] and [10] for the notion of conformal normal coordinates. Also note, that we only slightly violate *Cond₅*, since indeed close to x_0 we have

$$\langle \nabla \Delta K, \nabla K \rangle = \frac{2}{n+2} |\Delta K|^2 < \frac{1}{3} |\Delta K|^2,$$

in particular *Cond₅* from [15] guaranteeing flow convergence is pretty sharp. As a consequence the only possible non compactness, i.e. non compact flow lines for (1.1) or (1.3), correspond to a bubbling close to x_0 with critical energy

$$J_\infty = J(\varphi_{x_0, \infty}) = \frac{c_0}{K^{\frac{n-2}{n}}(x_0)}.$$

This unique bubbling then occurs both for (1.1) and (1.3) and we will compare these flows in detail. However by a slight modification of the latter flow in the spirit of (1.6) this non compactness will be completely removed.

Theorem 1.3. *Let $M = (M^n, g_0)$ be a Riemannian manifold of dimension $n = 5$ and positive Yamabe invariant. Then under Condition 1.2 the flows generated by*

- (i) *the Yamabe type, L^2 -gradient flow (1.1) and*
- (ii) *its normalised, strong gradient type analogon (1.3)*

for the prescribed scalar curvature functional (1.2) exhibit exclusively non compact flow lines of single bubble type at the unique maximum of K , while there exists a compact pseudo gradient for the latter functional, i.e. a pseudo gradient, all of whose flow lines are compact and hence converging.

Proof. We have seen above, that (1.1) and (1.3) induce a flow Φ on X , whose flow lines

$$u = u_t = \Phi(t, u_0)$$

up to a time sequence are Palais-Smale. Then up to a subsequence in time

$$\forall \varepsilon > 0 \exists N = N(\varepsilon) \in \mathbb{N} \forall n \geq N : u_{t_n} \in V(\omega, p, \varepsilon)$$

for

- (i) either $\omega = 0$ and $p \in \mathbb{N}_{\geq 1}$
- (ii) or a solution $\omega > 0$ to $\partial J(\omega) = 0$ and $p \in \mathbb{N}_{\geq 0}$,

cf. Definition 2.5 and Proposition 2.6. In fact $\omega = 0$ and $p = 0$ would imply

$$u_{t_n} \xrightarrow{n \rightarrow \infty} 0 \text{ strongly}$$

contradicting the normalisation $k = k_u = 1$. The latter statement is sharpened via Proposition 2.17 to

$$\forall \varepsilon > 0 \exists T = T(\varepsilon) > 0 \forall t \geq T : u = u_t \in V(\omega, p, \varepsilon).$$

Hence convergence in case $p = 0$. By Section 3.1 only $p = 1$ is possible in case $p > 0$ and then

$$a \xrightarrow{t \rightarrow \infty} x_0 = \{K = \max K\}$$

for the single blow-up point a of

$$u = \alpha \varphi_{a,\lambda} + v \in V(p, \varepsilon) = V(0, p, \varepsilon).$$

Lemma 3.4 then shows, that indeed $\lambda \rightarrow \infty$ for suitable initial data. Hence we have proven the exclusive existence of non compact flow lines as a single bubbling at x_0 .

Finally for the modified flow on X induced by (3.17), which is a pseudo gradient flow by virtue of Lemma 3.5, the only possibility for a non compact flow line is as before a single bubbling scenario, cf. (3.18), which is ruled out in Section 3.4. Hence (3.17) induces a compact flow. $\square$

The plan of this work is as follows. In Section 2 we recall some preliminary notions already introduced in [15] for the study of such flows. In particular in Section 2.1 we study the difference or rather the strict similarities of the shadow flow for (1.1) and (1.3), i.e. the dynamics of those variables relevant to the underlying finite dimensional reduction. Subsequently we recall in Section 2.2 some first and easy properties on flow lines based on this reduction. After this lengthy exposition of introduction and preliminary results in Sections 1 and 2 we study in Section 3 all possibilities of non compact flow lines for the flows induced by (1.1) and (1.3) and afterwards of a slight modification of the latter. Precisely we exclude in Section 3.1 all possibilities for non compact flow lines for (1.1) and (1.3), which are not of single bubble type and concentrating at the maximum point of K . Subsequently in Section 3.2 we show, that the latter remaining possibility is realised, i.e. that in fact such non compact flow lines exist for both flows. Finally we modify the latter flows in Section 3.3 and thus introduce a new pseudo gradient flow, which in Section 3.4 is shown to be compact. Last and for the sake of readability we collect in the Appendix 4 some statements from [15] and a proof from Section 2.

2 Preliminaries

As we had seen via (1.7) and (1.8), every flow line for (1.1) and (1.6) up to the choice of a time sequence constitutes a Palais-Smale sequence for J , whose possible lack of compactness we now describe.

Definition 2.1. For $a \in M$ let $u_a > 0$ via $g_a = u_a^{\frac{4}{n-2}} g_0$ introduce conformal normal coordinates and let G_{g_a} be the Green's function of the conformal Laplacian L_{g_a} . For $\lambda > 0$ let

$$\varphi_{a,\lambda} = u_a \left(\frac{\lambda}{1 + \lambda^2 \gamma_n G_a^{\frac{2}{2-n}}} \right)^{\frac{n-2}{2}}, \quad G_a = G_{g_a}(a, \cdot), \quad \gamma_n = (4n(n-1)\omega_n)^{\frac{2}{2-n}}.$$

One may expand $G_a = \frac{1}{4n(n-1)\omega_n} (r_a^{2-n} + H_a)$ with $r_a = d_{g_a}(a, \cdot)$ and decompose

$$H_a = H_{r,a} + H_{s,a}, \quad H_{r,a} \in C_{loc}^{2,\alpha}, \quad H_{s,a} = O \begin{pmatrix} 0 & \text{for } n = 3 \\ r_a^2 \ln r_a & \text{for } n = 4 \\ r_a & \text{for } n = 5 \end{pmatrix}.$$

In addition the positive mass theorem tells, that $H_a(a) \geq 0$ for all $a \in M$ and

$$H_a(a) = 0 \text{ for } M \simeq \mathbb{S}^n, \text{ while } H_a(a) > 0 \text{ for } M \not\simeq \mathbb{S}^n$$

in the sense of conformal equivalence.

We abbreviate some notation.

Definition 2.2. For $k, l = 1, 2, 3$ and $\lambda_i > 0$, $a_i \in M$, $i = 1, \dots, p$ define

- (i) $\varphi_i = \varphi_{a_i, \lambda_i}$ and $(d_{1,i}, d_{2,i}, d_{3,i}) = (1, -\lambda_i \partial_{\lambda_i}, \frac{1}{\lambda_i} \nabla_{a_i})$
- (ii) $\phi_{1,i} = \varphi_i$, $\phi_{2,i} = -\lambda_i \partial_{\lambda_i} \varphi_i$, $\phi_{3,i} = \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i$, so $\phi_{k,i} = d_{k,i} \varphi_i$
- (iii) $K_i = K(a_i)$, $\nabla K_i = \nabla K(a_i)$ and so on.

Let us collect some standard interaction estimates for these bubbles.

Lemma 2.3. Let $k, l = 1, 2, 3$ and $i, j = 1, \dots, p$. We have

- (i) $|\phi_{k,i}|, |\lambda_i \partial_{\lambda_i} \phi_{k,i}|, |\frac{1}{\lambda_i} \nabla_{a_i} \phi_{k,i}| \leq C \varphi_i$
- (ii) $\int \varphi_i^{\frac{4}{n-2}} \phi_{k,i} \phi_{k,i} = c_k \cdot id + O(\lambda_i^{2-n} + \lambda_i^{-2})$, $c_k > 0$
- (iii) $\int \varphi_i^{\frac{n+2}{n-2}} \phi_{k,j} = b_k d_{k,i} \varepsilon_{i,j} + o_\varepsilon(\varepsilon_{i,j}) = \frac{n+2}{n-2} \int \phi_{k,i} \varphi_i^{\frac{4}{n-2}} \varphi_j$, $b_k > 0$, $i \neq j$
- (iv) $\int \varphi_i^{\frac{4}{n-2}} \phi_{k,i} \phi_{l,i} = O(\lambda_i^{2-n} + \lambda_i^{-2})$ for $k \neq l$, $\int \varphi_i^{\frac{2n}{n-2}} = c_1 + O(\lambda_i^{2-n})$ and $\int \varphi_i^{\frac{n+2}{n-2}} \phi_{k,i} = O(\lambda_i^{2-n})$ for $k = 2, 3$
- (v) $\int \varphi_i^\alpha \varphi_j^\beta = O(\varepsilon_{i,j}^\beta)$ for $i \neq j$ and $\alpha + \beta = \frac{2n}{n-2}$, $\alpha > \frac{n}{n-2} > \beta \geq 1$
- (vi) $\int \varphi_i^{\frac{n}{n-2}} \varphi_j^{\frac{n}{n-2}} = O(\varepsilon_{i,j}^{\frac{n}{n-2}} \ln \varepsilon_{i,j})$, $i \neq j$
- (vii) $(1, \lambda_i \partial_{\lambda_i}, \frac{1}{\lambda_i} \nabla_{a_i}) \varepsilon_{i,j} = O(\varepsilon_{i,j})$, $i \neq j$,

where

- 1.) $\varepsilon = \min\{\frac{1}{\lambda_i}, \frac{1}{\lambda_j}, \varepsilon_{i,j}\}$, $\varepsilon_{i,j} = (\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i, a_j))^{\frac{2-n}{2}}$
- 2.) $c_1 = \int_{\mathbb{R}^n} \frac{1}{(1+r^2)^n}$, $c_2 = \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|r^2-1|^2}{(1+r^2)^{n+2}}$, $c_3 = \frac{(n-2)^2}{n} \int_{\mathbb{R}^n} \frac{r^2}{(1+r^2)^{n+2}}$.

Proof. Cf. 3.4 in [15]. □

For a better description of the gradient we decompose the second variation. To that end we recall from [15], cf. Lemma 3.6 and Proposition 3.7,

Lemma 2.4. For $\omega > 0$ solving

$$L_{g_0} \omega = K \omega^{\frac{n+2}{n-2}}$$

there exist $\varepsilon > 0$, an open neighbourhood U of ω and

$$h : B_\varepsilon^{\mathbb{R}^{m+1}}(0) \longrightarrow H_0(\omega)^{\perp_{L_{g_0}}}, \quad H_0(\omega) = \ker \partial^2 J(\omega)$$

smooth such, that

$$\{w \in U : \Pi_{H_0(\omega)^{\perp_{L_{g_0}}}} \nabla J(w) = 0\} = \{u_{\alpha, \beta} = (1 + \alpha)\omega + \beta^i e_i + h(\alpha, \beta) : (\alpha, \beta) \in B_\varepsilon^{m+1}(0)\},$$

where $\{\omega, e_i : i = 1, \dots, m\} \in ONB_{L_{g_0}}(\ker \partial^2 J(\omega))$ and

$$\|h(\alpha, \beta)\| = O(|\alpha|^2 + |\beta|^2).$$

We call $w \in U$ a pseudo critical point related to ω , if

$$\Pi_{H_0(\omega)^{\perp_{L_{g_0}}}} \nabla J(w) = 0.$$

Moreover there holds $|h(\alpha, \beta)|_{C^k} \longrightarrow 0$ as $|\alpha| + |\beta| \longrightarrow 0$ for any $k \in \mathbb{N}$.

Note, that due to scaling invariance

$$\Pi \nabla J(\omega) = 0 \Leftrightarrow \forall \alpha > 0 : \Pi \nabla J(\alpha \omega) = 0.$$

Thus we may reparametrise the pseudo critical points related to ω as

$$u_{\alpha,\beta} = \alpha(\omega + \beta^i e_i + h(\beta)), h(\beta) \perp_{L_{g_0}} H_0(\omega),$$

where $\|h(\beta)\| = O(\|\beta\|^2)$ and $\|h(\beta)\|_{C^k} \rightarrow 0$ as $\|\beta\| \rightarrow 0$.

We may thereby define a neighbourhood of, where a loss of compactness, if present, has to occur.

Definition 2.5. Let $\omega \geq 0$ solve $L_{g_0} \omega = K\omega^{\frac{n+2}{n-2}}$, $p \in \mathbb{N}$ and $\varepsilon > 0$. Let for $u \in X$

$$A_u(\omega, p, \varepsilon) = \{(\alpha, \beta_k, \alpha_i, \lambda_i, a_i) \in (\mathbb{R}_+, \mathbb{R}^m, \mathbb{R}_+^p, \mathbb{R}_+^p, M^p) :$$

$$\forall_{i \neq j} \lambda_i^{-1}, \lambda_j^{-1}, \varepsilon_{i,j}, |1 - \frac{r\alpha_i^{\frac{4}{n-2}} K(a_i)}{4n(n-1)k}|, |1 - \frac{r\alpha_j^{\frac{4}{n-2}}}{k}|, |\beta|, \|u - u_{\alpha,\beta} - \alpha^i \varphi_{a_i, \lambda_i}\| < \varepsilon\}.$$

We define

$$V(\omega, p, \varepsilon) = \{u \in X : A_u(\omega, p, \varepsilon) \neq \emptyset\}$$

and call $V(\omega, p, \varepsilon)$ in case $p > 0$ a neighbourhood of a potential critical point at infinity.

Note, that $u_{\alpha,\beta} = 0$, if $\omega = 0$, and the conditions on α and β_k become trivial. Moreover either $w \equiv 0$ or $w > 0$ due to the strong maximum principle.

Proposition 2.6. Every Palais-Smale sequence of J in X is precompact in some $V(\omega, p, \varepsilon)$, i.e.

$$\forall t_k \rightarrow \infty \exists (t_{k_l}) \subset (t_k) : u_{t_{k_l}} \in V(\omega, p, \varepsilon),$$

eventually for every $\varepsilon > 0$.

This characterisation of lack of compactness is classical like the subsequent reduction by minimisation and we refer to [4], [15] and [17].

Proposition 2.7. For every $\varepsilon_0 > 0$ there exists $\varepsilon_1 > 0$ such, that for

$$u \in V(\omega, p, \varepsilon) \text{ with } \varepsilon < \varepsilon_1$$

the minimisation problems

$$\begin{aligned} \text{(i)} \quad & \inf_{(\tilde{\alpha}, \tilde{\beta}_k, \tilde{\alpha}_i, \tilde{\lambda}_i) \in A_u(\omega, p, 2\varepsilon_0)} \int K u^{\frac{4}{n-2}} |u - u_{\tilde{\alpha}, \tilde{\beta}} - \tilde{\alpha}^i \varphi_{\tilde{a}_i, \tilde{\lambda}_i}|^2 \\ \text{(ii)} \quad & \inf_{(\tilde{\alpha}, \tilde{\beta}_k, \tilde{\alpha}_i, \tilde{\lambda}_i) \in A_u(\omega, p, 2\varepsilon_0)} \|u - u_{\tilde{\alpha}, \tilde{\beta}} - \tilde{\alpha}^i \varphi_{\tilde{a}_i, \tilde{\lambda}_i}\|^2 \end{aligned}$$

admit each a unique minimiser $(\alpha, \beta_k, \alpha_i, a_i, \lambda_i) \in A_u(\omega, p, \varepsilon_0)$ and we define

$$\varphi_i = \varphi_{a_i, \lambda_i}, v = u - u_{\alpha, \beta} - \alpha^i \varphi_i, \quad \varepsilon_{i,j} = \left(\frac{\lambda_j}{\lambda_i} + \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i, a_j) \right)^{\frac{2-n}{2}}$$

depending on the chosen minimisation. Moreover

$$(\alpha, \beta_k, \alpha_i, a_i, \lambda_i) \text{ and } v$$

depend smoothly on u .

The above minimisations evidently induce orthogonal properties for

$$v = u - u_{\alpha, \beta} - \alpha^i \varphi_i$$

with respect to the scalar products

$$\langle a, b \rangle_{Ku^{\frac{4}{n-2}}} = \int K u^{\frac{4}{n-2}} ab \text{ or } \langle a, b \rangle_{L_{g_0}} = \int L_{g_0} ab$$

respectively. This justifies to define the orthogonal spaces, on which v lives.

Definition 2.8. For $u \in V(\omega, p, \varepsilon)$ let

$$H_u(\omega, p, \varepsilon) = \langle u_{\alpha, \beta}, \partial_{\beta_i} u_{\alpha, \beta}, \varphi_i, -\lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i \rangle^{\perp_{Ku \frac{4}{n-2}}}$$

or respectively

$$H_u(\omega, p, \varepsilon) = \langle u_{\alpha, \beta}, \partial_{\beta_i} u_{\alpha, \beta}, \varphi_i, -\lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i \rangle^{\perp_{L_{g_0}}}$$

in case $\omega > 0$. In case $\omega = 0$ let $H_u(0, p, \varepsilon) = H_u(p, \varepsilon)$ and

$$H_u(p, \varepsilon) = \langle \varphi_i, -\lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i \rangle^{\perp_{Ku \frac{4}{n-2}}}$$

or respectively

$$H_u(p, \varepsilon) = \langle \varphi_i, -\lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i \rangle^{\perp_{L_{g_0}}}.$$

Recalling Definition 2.2 and $u_{\alpha, \beta} = 0$ in case $\omega = 0$ we may simply write

$$H_u(\omega, p, \varepsilon) = \langle u_{\alpha, \beta}, \partial_{\beta_i} u_{\alpha, \beta}, \phi_{k,i} \rangle^{\perp}, \text{ in particular } H_u(p, \varepsilon) = \langle \phi_{k,i} \rangle^{\perp}.$$

depending on the chosen minimisation. These orthogonalities differ only a little, as the next Lemma, whose proof we delay to Appendix 4, quantifies.

Lemma 2.9. Let $v_1 \in H_u(\omega, p, \varepsilon) = \langle u_{\alpha, \beta}, \partial_{\beta_i} u_{\alpha, \beta}, \phi_{k,i} \rangle^{\perp_{Ku \frac{4}{n-2}}}$. Then

- (i) $\Pi_{\langle \phi_{k,i} \rangle}^{\perp_{L_{g_0}}} v_1 = O((\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n-2}} + \sum_{j \neq i} \varepsilon_{i,j} + \|v\|) \|v_1\|)$ for $\omega = 0$
- (ii) $\Pi_{\langle u_{\alpha, \beta}, \partial_{\beta_i} u_{\alpha, \beta}, \phi_{k,i} \rangle}^{\perp_{L_{g_0}}} v_1 = O((\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{j \neq i} \varepsilon_{i,j} + \|v\|) \|v_1\|)$ for $\omega > 0$.

Conversely for $v_2 \in H_u(\omega, p, \varepsilon) = \langle u_{\alpha, \beta}, \partial_{\beta_i} u_{\alpha, \beta}, \phi_{k,i} \rangle^{\perp_{L_{g_0}}}$ there holds

- (i) $\Pi_{\langle \phi_{k,i} \rangle}^{\perp_{Ku \frac{4}{n-2}}} v_2 = O((\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n-2}} + \sum_{j \neq i} \varepsilon_{i,j} + \|v\|) \|v_2\|)$ for $\omega = 0$
- (ii) $\Pi_{\langle u_{\alpha, \beta}, \partial_{\beta_i} u_{\alpha, \beta}, \phi_{k,i} \rangle}^{\perp_{Ku \frac{4}{n-2}}} v_2 = O((\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{j \neq i} \varepsilon_{i,j} + \|v\|) \|v_2\|)$ for $\omega > 0$.

The foregoing Lemma will help us to carry over several estimates from [15], which were based on a representation $u = \alpha^i \varphi_i + v$ with orthogonalities

$$\langle \phi_{k,i}, v \rangle_{Ku \frac{4}{n-2}} = 0$$

from the first minimisation problem in Proposition 2.7.

Proposition 2.10. There exist $\gamma, \varepsilon_0 > 0$ such, that for any $0 < \varepsilon < \varepsilon_0$ and

$$u = \alpha^i \varphi_i + v \in V(p, \varepsilon)$$

there holds $\partial^2 J(\alpha^i \varphi_i)|_{H^+} > \gamma$ for $H = H_u(p, \varepsilon)$.

This positivity property is well known in either case

$$H_u(p, \varepsilon) = \langle \phi_{k,i} \rangle^{\perp_{Ku \frac{4}{n-2}}} \text{ or } H_u(p, \varepsilon) = \langle \phi_{k,i} \rangle^{\perp_{L_{g_0}}}$$

and evidently one case follows from the other by virtue of Lemma 2.9. Likewise in case $u \in V(\omega, p, \varepsilon)$, cf. Proposition 5.5 from [15].

Proposition 2.11. *There exist $\gamma, \varepsilon_0 > 0$ such, that for any*

$$u = u_{\alpha, \beta} + \alpha^i \varphi_i + v \in V(\omega, p, \varepsilon)$$

with $0 < \varepsilon < \varepsilon_0$ we may decompose

$$H_u(\omega, p, \varepsilon) = H = H_+ \oplus_{L_{g_0}} H_- \text{ with } \dim H_- < \infty$$

and for any $h_+ \in H_+, h_- \in H_-$ there holds

- (i) $\partial^2 J(u_{\alpha, \beta} + \alpha^i \varphi_i)|_{H_+} > \gamma$ and $\partial^2 J(u_{\alpha, \beta} + \alpha^i \varphi_i)|_{H_-} < -\gamma$
- (ii) $\partial^2 J(u_{\alpha, \beta} + \alpha^i \varphi_i)h_+ h_- = o_\varepsilon(\|h_+\| \|h_-\|)$.

The invertibility of the second variation on the orthogonal space, on which v lives, then provides a priori estimates.

Proposition 2.12. *For $\varepsilon > 0$ small we have*

- (i) $\|v\| = O(\sum_r \frac{|\nabla K_r|}{\lambda_r} + \frac{|\Delta K_r|}{\lambda_r^2} + \lambda_r^{2-n} + \sum_{r \neq s} \varepsilon_{r,s} + |\partial J(u)|)$ on $V(p, \varepsilon)$
- (ii) $\|v\| = O(\sum_r \frac{|\nabla K_r|}{\lambda_r} + \lambda_r^{\frac{2-n}{2}} + \sum_{r \neq s} \varepsilon_{r,s} + |\partial J(u)|)$ on $V(\omega, p, \varepsilon)$

Proof. The statement for $V(p, \varepsilon)$ follows by expanding

$$\partial J(u)v = \partial J(\alpha^i \varphi_i + v)v$$

in v and applying Propositions 2.10 and 4.2. Likewise the statement for $V(\omega, p, \varepsilon)$ follows by expanding

$$\partial J(u)v_\pm = \partial J(u_{\alpha, \beta} + \alpha^i \varphi_i + v)v_\pm$$

in v and applying Proposition 4.3 and 2.11, where we denote by

$$v_+ = \Pi_{H_+}^{\perp L_{g_0}} v \text{ and } v_- = \Pi_{H_-}^{\perp L_{g_0}} v$$

the corresponding projections onto H_+ and H_- in Proposition 2.11. □

These estimates on v are upon the appearance of $|\partial J(u)|$ instead of $|\delta J(u)|$ the same as in [15], cf. Corollaries 4.6 and 5.6 therein. In fact in the latter work we had too graciously estimated against $|\delta J(u)|$ in many cases. In what follows we will simply give the correct statements without repeating the various proofs from [15].

2.1 The shadow flows

We recall some standard testings of the first variation

$$\partial J(u) = \frac{2}{k^{\frac{n-2}{n}}} \left[\int L_{g_0} u v - \frac{r}{k} \int K u^{\frac{n+2}{n-2}} v \right],$$

cf. Proposition 1.1.

Proposition 2.13. *For $u \in V(\omega, p, \varepsilon)$ and $\varepsilon > 0$ sufficiently small let*

$$\sigma_{k,i} = - \int (L_{g_0} u - r \bar{K} u^{\frac{n+2}{n-2}}) \phi_{k,i}, \quad i = 1, \dots, p, \quad k = 1, 2, 3.$$

Then in case $\omega = 0$ we have with constants $b_2, \dots, e_4 > 0$

- (i) $\sigma_{2,i} = d_2 \alpha_i \frac{H_i}{\lambda_i^{\frac{n-2}{2}}} + e_2 \frac{r \alpha_i^{\frac{n+2}{n-2}}}{k} \frac{\Delta K_i}{\lambda_i^2} - b_2 \frac{r \alpha_i^{\frac{4}{n-2}} K_i}{k} \sum_{j=1}^p \alpha_j \lambda_i \partial_{\lambda_i} \varepsilon_{i,j}$

$$(ii) \quad \sigma_{3,i} = \frac{r\alpha_i^{\frac{n+2}{n-2}}}{k} [e_3 \frac{\nabla K_i}{\lambda_i} + e_4 \frac{\nabla \Delta K_i}{\lambda_i^3}] + b_3 \frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} \sum_{i \neq j=1}^p \frac{\alpha_j}{\lambda_i} \nabla a_i \varepsilon_{i,j}$$

up to some

$$o_\varepsilon(\lambda_i^{2-n} + \sum_{i \neq j=1}^p \varepsilon_{i,j}) + O(\sum_{r \neq s} \varepsilon_{r,s}^2 + \|v\|^2 + |\partial J(u)|^2),$$

whereas in case $\omega > 0$ with constants $d_2, \dots, b_3 > 0$

$$(i) \quad \sigma_{2,i} = d_2 \frac{r\alpha_i^{\frac{4}{n-2}}}{k} \frac{\alpha \omega_i}{\lambda_i^{\frac{n-2}{2}}} - b_2 \frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} \sum_{i \neq j=1}^p \alpha_j \lambda_i \partial_{\lambda_i} \varepsilon_{i,j}$$

$$(ii) \quad \sigma_{3,i} = d_3 \frac{r\alpha_i^{\frac{n+2}{n-2}}}{k} \frac{\nabla K_i}{\lambda_i} + b_3 \frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} \sum_{i \neq j=1}^p \alpha_j \frac{1}{\lambda_i} \nabla a_i \varepsilon_{i,j}$$

up to some

$$o_\varepsilon(\lambda_i^{\frac{2-n}{2}} + \sum_{i \neq j=1}^p \varepsilon_{i,j}) + O(\sum_{r \neq s} \varepsilon_{r,s}^2 + \|v\|^2 + |\partial J(u)|^2).$$

Proof. Cf. Corollaries 4.3 and 5.3 in [15]. □

So far and in contrast to [15] we have removed the appearance of $|\delta J|$. In fact only in the computation of the shadow flow, i.e. the description of the movements of α_i, λ_i and a_i this error term inevitably enters.

Proposition 2.14. For $u \in V(p, \varepsilon)$ with $\varepsilon > 0$ small we have

$$(i) \quad -\frac{\dot{\lambda}_i}{\lambda_i} = \frac{r}{k} [\frac{d_2}{c_2} \frac{H_i}{\lambda_i^{n-2}} + \frac{e_2}{c_2} \frac{\Delta K_i}{K_i \lambda_i^2} - \frac{b_2}{c_2} \sum_{i \neq j=1}^p \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j}] (1 + o_{\frac{1}{\lambda_i}}(1))$$

$$(ii) \quad \lambda_i \dot{a}_i = \frac{r}{k} [\frac{e_3}{c_3} \frac{\nabla K_i}{K_i \lambda_i} + \frac{e_4}{c_3} \frac{\nabla \Delta K_i}{K_i \lambda_i^3} + \frac{b_3}{c_3} \sum_{i \neq j=1}^p \frac{\alpha_j}{\alpha_i} \frac{1}{\lambda_i} \nabla a_i \varepsilon_{i,j}] (1 + o_{\frac{1}{\lambda_i}}(1))$$

up to some $o_\varepsilon(\lambda_i^{2-n} + \sum_{i \neq j=1}^p \varepsilon_{i,j})$ and

$$O(\sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{|\Delta K_r|^2}{\lambda_r^4} + \lambda_r^{-2(n-2)} + \varepsilon_{r,s}^2) + \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

For $u \in V(\omega, p, \varepsilon)$ with $\varepsilon > 0$ small we have

$$(i) \quad -\frac{\dot{\lambda}_i}{\lambda_i} = \frac{r}{k} [\frac{d_2}{c_2} \frac{\alpha \omega_i}{\alpha_i K_i \lambda_i^{\frac{n-2}{2}}} - \frac{b_2}{c_2} \sum_{i \neq j=1}^p \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j}] (1 + o_{\frac{1}{\lambda_i}}(1))$$

$$(ii) \quad \lambda_i \dot{a}_i = \frac{r}{k} [\frac{d_3}{c_3} \frac{\nabla K_i}{K_i \lambda_i} + \frac{b_3}{c_3} \sum_{i \neq j=1}^p \frac{\alpha_j}{\alpha_i} \frac{1}{\lambda_i} \nabla a_i \varepsilon_{i,j}] (1 + o_{\frac{1}{\lambda_i}}(1))$$

up to some $o_\varepsilon(\lambda_i^{\frac{2-n}{2}} + \sum_{i \neq j=1}^p \varepsilon_{i,j})$ and

$$O(\sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \lambda_r^{2-n} + \varepsilon_{r,s}^2) + \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

The statements concerning the Yamabe type flow (1.1) are exactly those of Corollaries 4.7, 5.7 in [15] and they are proven by testing the flow via $\langle \partial_t u, \phi_{l,j} \rangle$. In case of (1.1) the natural scalar product is

$$\langle a, b \rangle_{Ku^{\frac{4}{n-2}}} = \int Ku^{\frac{4}{n-2}} ab.$$

Hence letting $\check{\xi}_{k,i} = (\frac{\dot{\alpha}_i}{\alpha_i}, -\frac{\dot{\lambda}_i}{\lambda_i}, \lambda_i \dot{a}_i)$ we have to evaluate on $V(p, \varepsilon)$ under (1.1) for instance

$$\begin{aligned} I_1 + I_2 &= \alpha_i \langle \phi_{k,i}, \phi_{l,j} \rangle_{Ku^{\frac{4}{n-2}}} \check{\xi}_{k,i}^{k,i} + \langle \partial_t v, \phi_{l,j} \rangle_{Ku^{\frac{4}{n-2}}} = \langle \partial_t u, \phi_{l,j} \rangle_{Ku^{\frac{4}{n-2}}} \\ &= - \langle (\frac{R}{K} - \frac{r}{k})u, \phi_{l,j} \rangle_{Ku^{\frac{4}{n-2}}} = I_3, \end{aligned}$$

where

$$(i) \quad \int Ku^{\frac{4}{n-2}} \phi_{k,i} \phi_{l,j} = c_k \alpha_i^{\frac{4}{n-2}} K_i \delta_{k,l} \delta_{i,j} \text{ up to some}$$

$$O\left(\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n-2}}\right) \delta_{i,j} + O\left(\sum_{i \neq m=1}^p \varepsilon_{i,m} + \|v\|\right),$$

cf. the proof of Lemma 4.1 in [15].

$$(ii) \quad I_2 = \int Ku^{\frac{4}{n-2}} \partial_t v \phi_{l,j} = - \int K \partial_t u^{\frac{4}{n-2}} v \phi_{l,j} + O(\|v\|)_{k,l} \delta_{i,j} \dot{\xi}^{k,i} \text{ and}$$

$$\int K \partial_t u^{\frac{4}{n-2}} v \phi_{l,j} = \frac{4}{n-2} \int (R - r\bar{K}) u^{\frac{4}{n-2}} v \phi_{l,j}$$

$$(iii) \quad I_3 = - \int (R - r\bar{K}) u^{\frac{n+2}{n-2}} \phi_{l,j} = -\frac{1}{2} \partial J(u) \phi_{l,j}, \text{ cf. Proposition 1.1 and recalling } k = 1$$

In contrast under (1.3) the natural scalar product is

$$\langle a, b \rangle_{L_{g_0}} = \int L_{g_0} a b$$

and we have to evaluate

$$\begin{aligned} I_1 + I_2 &= \alpha_i \langle \phi_{k,i}, \phi_{l,j} \rangle_{L_{g_0}} \dot{\xi}^{k,i} + \langle \partial_t v, \phi_{l,j} \rangle_{L_{g_0}} = \langle \partial_t u, \phi_{l,j} \rangle_{L_{g_0}} \\ &= -\frac{r}{2k} \langle \nabla J(u) - \frac{\int Ku^{\frac{n+2}{n-2}} \nabla J(u)}{k} u, \phi_{l,j} \rangle_{L_{g_0}} = I_3, \end{aligned}$$

where

$$\begin{aligned} (i) \quad & \int L_{g_0} \phi_{k,i} \phi_{l,j} = 4n(n-1) c_k \delta_{k,l} \delta_{i,j} + O\left(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n-2}}\right) \delta_{i,j} + O(\varepsilon_{i,j}) \\ (ii) \quad & I_2 = \int L_{g_0} \partial_t v \phi_{l,j} = O(\|v\|)_{k,l} \delta_{i,j} \dot{\xi}^{k,i} \\ (iii) \quad & I_3 = -\frac{r}{2k} \partial J(u) \phi_{l,j} + O\left(\int Ku^{\frac{n+2}{n-2}} \nabla J(u)\right) \text{ and due to } \partial J(u)u = 0 \end{aligned}$$

$$\int Ku^{\frac{n+2}{n-2}} \nabla J(u) = O(|\partial J(u)|^2).$$

In order to compare (i)-(iii), note, that by virtue of Propositions 4.1 we have

$$K_i \alpha_i^{\frac{4}{n-2}} = 4n(n-1) \frac{k}{r} \quad (2.1)$$

up to some

$$O\left(\sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{|\Delta K_r|}{\lambda_r^2} + \frac{1}{\lambda_r^{n-2}} + \varepsilon_{r,s} + \|v\| + |\partial J(u)|\right),$$

since

$$\sigma_{1,i} = O(|\partial J(u)|),$$

cf. Proposition 4.1, also (5.13) in [15] for the analogon in case $\omega \neq 0$. Consequently

$$\Xi_{k,i,l,j} \dot{\xi}^{k,i} = \frac{r}{k} \sigma_{l,j} + \begin{cases} O\left(\int (R - r\bar{K}) u^{\frac{4}{n-2}} v \phi_{l,j}\right) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}$$

with invertible

$$\begin{aligned} \Xi_{k,i,l,j} &= 4n(n-1) \alpha_i c_k \delta_{k,l} \delta_{i,j} + O\left(\frac{1}{\lambda_i^2}\right) \delta_{i,j} \\ &\quad + O\left(\sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{|\Delta K_r|}{\lambda_r^2} + \frac{1}{\lambda_r^{n-2}} + \varepsilon_{r,s} + \|v\| + |\partial J(u)|\right) \end{aligned} \quad (2.2)$$

and hence, since $\sigma_{l,j} = O(|\partial J(u)|)$

$$\begin{aligned} \dot{\xi}_{k,i} &= \frac{r}{k} \frac{\sigma_{k,i}}{4n(n-1)\alpha_i c_k} (1 + O(\frac{1}{\lambda_i^2})) + \begin{cases} O(\sum_r \int |R - r\tilde{K}| u^{\frac{4}{n-2}} |v| \varphi_r) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases} \\ &\quad + O(\sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{|\Delta K_r|^2}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^2 + \|v\|^2 + |\partial J(u)|^2). \end{aligned}$$

Here enters the difference from (1.1) to (1.3). In fact we have to estimate

$$\begin{aligned} \int |R - r\tilde{K}| u^{\frac{4}{n-2}} |v| \varphi_r &\lesssim \int |R - r\tilde{K}| u^{\frac{4}{n-2}} |v| (u + |v|) \\ &\lesssim \|R - r\tilde{K}\|_{L_{su}^{\frac{2n}{n+2}}} \|v\| + \|R - r\tilde{K}\|_{L_{su}^{\frac{n}{2}}} \|v\|^2 \\ &\leq |\delta J(u)|^2 + (1 + \|R - r\tilde{K}\|_{L_{su}^{\frac{n}{2}}}^2) \|v\|^2, \end{aligned}$$

i.e. there appears $|\delta J(u)|$ instead of $|\partial J(u)|$. Also note, that we have

$$\|R - r\tilde{K}\|_{L_{su}^{\frac{n}{2}}} \longrightarrow 0 \text{ as } t \longrightarrow \infty$$

along each flow line by virtue of Proposition 2.11 from [15]. We thus obtain

$$\begin{aligned} \dot{\xi}_{k,i} &= \frac{r}{k} \frac{\sigma_{k,i}}{4n(n-1)\alpha_i c_k} (1 + o_{\frac{1}{\lambda_i}}(1)) + O\left(\sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^2 \right. \\ &\quad \left. + \|v\|^2 + \begin{cases} |\delta J(u)|^2 & \text{under (1.1)} \\ |\partial J(u)|^2 & \text{under (1.3)} \end{cases} \right). \end{aligned}$$

Hence Proposition 2.14 for $u \in V(p, \varepsilon)$ follows from Proposition 2.13 and (2.1) absorbing $\|v\|$ via Proposition 2.12. The case $u \in V(\omega, p, \varepsilon)$ is analogous.

2.2 Principal behaviour

Let us recall some generic notions and results in the statements below.

Definition 2.15. We call ∂J principally lower bounded, if for every $p \geq 1$ there exist $c, \varepsilon > 0$ such, that

- (i) $|\partial J(u)| \geq c(\sum_r \frac{|\nabla K_r|}{K_r \lambda_r} + \frac{|\Delta K_r|}{K_r \lambda_r^2} + \lambda_r^{2-n} + \sum_{r \neq s} \varepsilon_{r,s})$ for all $u \in V(p, \varepsilon)$
- (ii) $|\partial J(u)| \geq c(\sum_r \frac{|\nabla K_r|}{K_r \lambda_r} + \lambda_r^{\frac{2-n}{2}} + \sum_{r \neq s} \varepsilon_{r,s})$ for all $u \in V(\omega, p, \varepsilon)$.

Under this mild assumption we have uniformity in $V(\omega, p, \varepsilon)$ as follows.

Proposition 2.16. Assume ∂J to be principally lower bounded. For

$$u = u_{\alpha,\beta} + \alpha^i \varphi_i + v \in V(\omega, p, \varepsilon)$$

with $k_u \equiv 1$ we then have

$$\lambda_i^{-1}, \varepsilon_{i,j}, |1 - \frac{r_\infty \alpha_i^{\frac{4}{n-2}} K_i}{4n(n-1)}|, |(\frac{r}{k})_{u_{1,\beta}} - r_\infty \alpha^{\frac{4}{n-2}}|, |\partial J(u_{1,\beta})|, \|v\| \longrightarrow 0$$

uniformly as $|\partial J(u)| \longrightarrow 0$ and $J(u) = r \longrightarrow J_\infty = r_\infty$.

Proof. Cf. Proposition 6.2 in [15]. □

As a consequence we obtain limiting uniqueness of non compact flow lines in analogy to the unique limit of compact flow lines.

Proposition 2.17. Assume ∂J to be principally lower bounded.

If a sequence $u(t_k)$ along (1.1) or (1.3) diverges in the sense, that

$$\exists p > 1, \varepsilon_k \searrow 0 : u(t_k) \in V(\omega, p, \varepsilon_k),$$

then u diverges as well in the sense, that

$$\exists p > 1 \forall \varepsilon > 0 \exists T > 0 \forall t > T : u(t) \in V(\omega, p, \varepsilon).$$

Proof. Cf. Proposition 6.3 from [15] □

Remark 2.18. In the statement of Proposition 2.17 and in contrast to its corresponding counterpart Proposition 6.3 in [15] we have replaced

"...converging to a critical point at infinity in the sense, that ..."

by

"...diverges in the sense, that...".

In fact, as we have exposed in [16] and will see in the present paper, not every non compact or diverging flow line leads to a critical point at infinity.

Note, that Proposition 2.17 in combination with Proposition 2.6 tells us, that every non compact, i.e. diverging flow line has to remain in some $V(\omega, p, \varepsilon)$ eventually for every $\varepsilon > 0$.

Lemma 2.19. If ∂J is principally lower bounded, then under (1.1) or (1.3)

$$K(a_i) \longrightarrow K_{i_\infty} \text{ and } |\nabla K(a_i)| \longrightarrow 0 \text{ for all } i = 1, \dots, p$$

for every non compact, i.e. in some $V(\omega, p, \varepsilon)$ diverging flow line.

Proof. Cf. Proposition 6.3 in [15]. □

Finally we note, that

Proposition 2.20. ∂J is principally lower bounded under Condition 1.2.

Proof. We just have to adapt the corresponding proof of Proposition 6.5 in [15] to this situation. In case $\omega = 0$ Propositions 2.12, 2.13 and (2.1) show

$$\begin{aligned} \text{(i)} \quad \sigma_{2,i} &= \tilde{\gamma}_1 \alpha_i \frac{H_i}{\lambda_i^{n-2}} + \gamma_2 \alpha_i \frac{\Delta K_i}{K_i \lambda_i^2} - \tilde{\gamma}_5 \sum_{i \neq j=1}^p \alpha_j \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \\ \text{(ii)} \quad \sigma_{3,i} &= \tilde{\gamma}_3 \alpha_i \frac{\nabla K_i}{K_i \lambda_i} + \gamma_4 \alpha_i \frac{\nabla \Delta K_i}{K_i \lambda_i^3} + \gamma_6 \sum_{i \neq j=1}^p \frac{\alpha_j}{\lambda_i} \nabla a_i \varepsilon_{i,j} \end{aligned}$$

up to some $o_\varepsilon(\lambda_i^{2-n} + \sum_{i \neq j=1}^p \varepsilon_{i,j})$ and

$$O\left(\sum_r \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{|\Delta K_r|^2}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \sum_{r \neq s} \varepsilon_{r,s}^2 + |\partial J(u)|^2\right).$$

Letting $0 < \underline{\kappa} \leq \kappa_i \leq \bar{\kappa} < \infty$ for $|\nabla K_i| \neq 0$ and $\kappa_i = 0$ for $|\nabla K_i| = 0$ we get

$$\begin{aligned} & \sum_i C^i \left(\frac{\sigma_{2,i}}{\alpha_i} + \kappa_i \left\langle \frac{\sigma_{3,i}}{\alpha_i}, \frac{\nabla K_i}{|\nabla K_i|} \right\rangle \right) \\ & \geq \sum_i C^i \left[\gamma_1 \frac{H_i}{\lambda_i^{n-2}} + \gamma_2 \frac{\Delta K_i}{K_i \lambda_i^2} + \gamma_3 \kappa_i \frac{|\nabla K_i|}{K_i \lambda_i} + \gamma_4 \kappa_i \frac{\langle \nabla \Delta K_i, \nabla K_i \rangle}{K_i |\nabla K_i| \lambda_i^3} \right] \\ & \quad - \tilde{\gamma}_5 \sum_{i \neq j} C^i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + o_\varepsilon \left(\sum_{r \neq s} \varepsilon_{r,s} \right) + O \left(\sum_{i \neq j} \frac{C^i}{\lambda_i} |\nabla_{a_i} \varepsilon_{i,j}| \right) \\ & \quad + O \left(\sum_r \frac{|\Delta K_r|^2}{\lambda_r^4} + |\partial J(u)|^2 \right). \end{aligned} \quad (2.3)$$

Ordering $\frac{1}{\lambda_1} \geq \dots \geq \frac{1}{\lambda_p}$ we then have for $\varepsilon \ll 1$ and $C \gg 1$

$$- \sum_{i \neq j} C^i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \geq c \sum_{i > j} C^i \varepsilon_{i,j} \quad (2.4)$$

and

$$\sum_{i \neq j} \frac{C^i}{\lambda_i} |\nabla_{a_i} \varepsilon_{i,j}| = O \left(\sum_{i > j} C^j \varepsilon_{i,j} \right). \quad (2.5)$$

To prove (2.4) and (2.5) note, that

$$\sum_{i \neq j} C^i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} = \sum_{i > j} \left[C^i \frac{\alpha_j}{\alpha_i} - C^j \frac{\alpha_i}{\alpha_j} \right] \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + \sum_{i < j} C^i \frac{\alpha_j}{\alpha_i} [\lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + \lambda_j \partial_{\lambda_j} \varepsilon_{i,j}].$$

One has $-\lambda_i \partial_{\lambda_i} \varepsilon_{i,j} - \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} = (n-2) \varepsilon_{i,j}^{\frac{n}{2-n}} \lambda_i \lambda_j \gamma_n G^{\frac{2}{2-n}}(a_i, a_j) > 0$ and

$$-\lambda_i \partial_{\lambda_i} \varepsilon_{i,j} = \frac{n-2}{2} \varepsilon_{i,j}^{\frac{n}{2-n}} \left(\frac{\lambda_i}{\lambda_j} - \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \gamma_n G^{\frac{2}{2-n}}(a_i, a_j) \right) \geq \frac{n-2}{4} \varepsilon_{i,j} \quad (2.6)$$

for $i > j$. Thus (2.4) follows. Finally note, that

$$\sum_{i \neq j} \frac{C^i}{\lambda_i} |\nabla_{a_i} \varepsilon_{i,j}| = \frac{n-2}{2} \sum_{i < j} C^i \varepsilon_{i,j} \left| \frac{\left(\frac{\lambda_j}{\lambda_i} \right)^{\frac{1}{2}} (\lambda_i \lambda_j)^{\frac{1}{2}} \gamma_n \nabla_{a_i} G^{\frac{2}{2-n}}(a_i, a_j)}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \gamma_n G^{\frac{2}{2-n}}(a_i, a_j)} \right|$$

up to some $o(\sum_{i \neq j} \varepsilon_{i,j})$, whence we immediately obtain (2.5).

Plugging (2.4) and (2.5) into (2.3) we obtain for $C > 1$ sufficiently large

$$\begin{aligned} & \sum_i C^i \left(\frac{\sigma_{2,i}}{\alpha_i} + \kappa_i \left\langle \frac{\sigma_{3,i}}{\alpha_i}, \frac{\nabla K_i}{|\nabla K_i|} \right\rangle \right) \\ & \geq \sum_i C^i \left[\gamma_1 \frac{H_i}{\lambda_i^{n-2}} + \gamma_2 \frac{\Delta K_i}{K_i \lambda_i^2} + \gamma_3 \kappa_i \frac{|\nabla K_i|}{K_i \lambda_i} + \gamma_4 \kappa_i \frac{\langle \nabla \Delta K_i, \nabla K_i \rangle}{K_i |\nabla K_i| \lambda_i^3} \right] \\ & \quad + \gamma_5 \sum_{i > j} C^i \varepsilon_{i,j} + O \left(\frac{|\Delta K_r|^2}{\lambda_r^4} + |\partial J(u)|^2 \right). \end{aligned}$$

In case $\Delta K_i \geq 0$ or $|\nabla K_i| > \varepsilon$ for $\varepsilon > 0$ small we immediately obtain

$$\begin{aligned} & \gamma_i \frac{H_i}{\lambda_i^{n-2}} + \gamma_2 \frac{\Delta K_i}{K_i \lambda_i^2} + \gamma_3 \kappa_i \frac{|\nabla K_i|}{K_i \lambda_i} + \gamma_4 \kappa_i \frac{\langle \nabla \Delta K_i, \nabla K_i \rangle}{K_i |\nabla K_i| \lambda_i^3} \\ & \geq c \left[\frac{H_i}{\lambda_i^{n-2}} + \frac{|\Delta K_i|}{K_i \lambda_i^2} + \frac{|\nabla K_i|}{K_i \lambda_i} \right] \end{aligned} \quad (2.7)$$

for some $c > 0$ and all $\lambda_i > 0$ sufficiently large choosing κ_i such, that

$$\gamma_i \frac{H_i}{\lambda_i^{n-2}} + \gamma_4 \kappa_i \frac{\langle \nabla \Delta K_i, \nabla K_i \rangle}{K_i |\nabla K_i| \lambda_i^3} \geq c \frac{H_i}{\lambda_i^{n-2}}.$$

Also (2.7) follows in case $\Delta K_i < 0$ and $|\nabla K_i| < \varepsilon$, unless

$$d_{g_0}(a_i, x_0) \ll 1.$$

In particular (2.7) follows in case $\Delta K_i < 0$ and $|\nabla K_i| = 0$, since then by Condition 1.2

$$a_i = x_0 \text{ and } \nabla K_i = 0, \Delta K_i = 0, \nabla \Delta K_i = 0.$$

Finally in case $\Delta K_i < 0$ and $0 \neq |\nabla K_i| < \varepsilon$ we have

$$\langle \nabla \Delta K_i, \nabla K_i \rangle = 32(n+2)|a_i|^4 = \frac{2}{n+2}|\Delta K_i|^2$$

and thus by Cauchy-Schwarz inequality

$$\frac{\Delta K_i}{K_i \lambda_i^2} > -\frac{1}{2} \sqrt{\frac{n+2}{2}} \frac{|\nabla K_i|}{K_i \lambda_i} - \frac{1}{2} \sqrt{\frac{n+2}{2}} \frac{\langle \nabla \Delta K_i, \nabla K_i \rangle}{K_i |\nabla K_i| \lambda_i^3}.$$

Choosing therefore κ_i such, that

$$\frac{1}{2} \sqrt{\frac{n+2}{2}} \gamma_2 < \gamma_3 \kappa_i \text{ and } \frac{1}{2} \sqrt{\frac{n+2}{2}} \gamma_2 < \gamma_4 \kappa_i,$$

then (2.7) holds true as well and thus in any case. We conclude

$$\sum_i C^i \left(\frac{\sigma_{2,i}}{\alpha_i} + \kappa_i \left\langle \frac{\sigma_{3,i}}{\alpha_i}, \frac{\nabla K_i}{|\nabla K_i|} \right\rangle \right) \geq \sum_i \left[\frac{H_i}{\lambda_i^{n-2}} + \frac{|\Delta K_i|}{K_i \lambda_i^2} + \frac{|\nabla K_i|}{K_i \lambda_i} \right] + \sum_{i>j} \varepsilon_{i,j}$$

up to some $O(|\partial J(u)|^2)$. Since $\sigma_{k,i} = O(|\partial J(u)|)$ by definition, the claim follows noticing $H_i > c > 0$ due to $M \not\cong \mathbb{S}^n$ and by means of the positive mass theorem. The case $\omega > 0$ is proven analogously. $\square$

3 Divergence and Compactification

Throughout this section we assume Condition 1.2 to hold true and identify the lack of compactness of the flows on X generated by (1.1) and (1.3). Subsequently we will perform a slight variation of these flows and thereby restore compactness.

3.1 Compact regions

In order to describe how non compact flow lines under (1.1) or (1.3) look like, we first exclude most of the generic possibilities of diverging flow lines within $V(\omega, p, \varepsilon)$, since by virtue of Propositions 2.6 and 2.17 we know, that every non compact flow line has to remain in some $V(\omega, p, \varepsilon)$ eventually, provided ∂J is principally lower bounded, cf. Definition 2.15 and this we ensure by Condition 1.2 via Proposition 2.20. Moreover Lemma 2.19 then allows us to distinguish non compact flow lines with respect to their end configuration. In fact, since we assume $\{x_0, \dots, x_q\} = \{\nabla K = 0\}$ and there holds

$$|\nabla K_i| = |\nabla K(a_i)| \longrightarrow 0 \text{ as } t \longrightarrow \infty$$

by virtue of Lemma 2.19, we find $a_i \longrightarrow x_{j_i}$ as $t \longrightarrow \infty$.

Lemma 3.1. *Every non zero weak limit flow line, i.e. eventually*

$$u \notin V(p, \varepsilon),$$

is compact.

Proof. Since every flow line constitutes up to a subsequence in time a Palais-Smale sequence, cf. (1.7) and (1.8), Propositions 2.6 and 2.17 tell us, that we may assume $u \in V(\omega, p, \varepsilon)$ for all times to come for some $V(\omega, p, \varepsilon)$ and $u \rightarrow \omega$ strongly in case $\omega > 0$ and $p = 0$, in which latter case u as a flow line is compact. Hence we may assume, that eventually $u \in V(\omega, p, \varepsilon)$ for $\omega > 0$ and $p \geq 1$. Then Proposition 2.14 and the principal lower bound on ∂J , cf. Definition 2.15, give

$$-\frac{\dot{\lambda}_i}{\lambda_i} = \frac{r}{k} \left[\frac{d_2}{c_2} \frac{\alpha \omega_i}{\alpha_i K_i \lambda_i^{\frac{n-2}{2}}} - \frac{b_2}{c_2} \sum_{i \neq j=1}^p \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \right] (1 + o_{\frac{1}{\lambda_i}}(1))$$

up to some

$$o_\varepsilon(\lambda_i^{\frac{2-n}{2}} + \sum_{i \neq j=1}^p \varepsilon_{i,j}) + \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

Then ordering $\frac{1}{\lambda_1} \geq \dots \geq \frac{1}{\lambda_p}$ and recalling (2.4) and $\omega_i = \omega(a_i) > 0$ we find for $\psi = \sum_i C^i \ln \frac{1}{\lambda_i}$

$$\psi' \geq \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

Then the right hand side is integrable in time, while necessarily $\psi \rightarrow -\infty$ as some $\lambda_i \rightarrow \infty$. Hence all λ_i have to stay bounded, which due to the principal lower bound on ∂J prevents $|\partial J(u)| \rightarrow 0$, hence contradicting the time integrability of $|\partial J(u)|^2$. $\square$

Lemma 3.2. Every flow line away from x_0 , i.e. eventually

$$u \notin V(p, \varepsilon) \cap \{\forall 1 \leq i \leq p : a_i \xrightarrow{t \rightarrow \infty} x_0\},$$

is compact.

Proof. We may assume $u \in V(p, \varepsilon)$ eventually. Then Proposition 2.14 and the principal lower bound on ∂J , cf. Definition 2.15, give

$$-\frac{\dot{\lambda}_i}{\lambda_i} = \frac{r}{k} \left[\frac{d_2}{c_2} \frac{H_i}{\lambda_i^{n-2}} + \frac{e_2}{c_2} \frac{\Delta K_i}{K_i \lambda_i^2} - \frac{b_2}{c_2} \sum_{i \neq j=1}^p \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \right] (1 + o_{\frac{1}{\lambda_i}}(1))$$

up to some

$$o_\varepsilon(\lambda_i^{2-n} + \sum_{i \neq j=1}^p \varepsilon_{i,j}) + \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

Moreover by assumption

$$\{1, \dots, p\} = P \supseteq Q = \{1 \leq i \leq p : a_i \xrightarrow{t \rightarrow \infty} x_{j_i} \neq x_0\} \neq \emptyset.$$

Then ordering $\frac{1}{\lambda_{l_1}} \geq \dots \geq \frac{1}{\lambda_{l_q}}$ for $Q = \{l_1, \dots, l_q\}$ we consider

$$\psi = \sum_{i=1}^q C^i \ln \frac{1}{\lambda_{l_i}}.$$

Since $\Delta K_{l_i} > 0$ for $l_i \in Q$, as $\{\Delta K \leq 0\} \cap \{\nabla K = 0\} = \{x_0\}$, we have

$$\psi' \geq c \sum_{i=1}^q \frac{1}{\lambda_{l_i}^2} - \frac{r}{k} \frac{b_2}{c_2} \sum_{Q \ni l_i \neq j \in P} C^i \frac{\alpha_j}{\alpha_{l_i}} \lambda_{l_i} \partial_{\lambda_{l_i}} \varepsilon_{l_i,j} + \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

Recalling (2.4) we then find

$$- \sum_{Q \ni l_i \neq j \in Q} C^i \frac{\alpha_j}{\alpha_{l_i}} \lambda_{l_i} \partial_{\lambda_{l_i}} \varepsilon_{l_i,j} \geq c \sum_{Q \ni l_i \neq j \in Q} \varepsilon_{l_i,j}$$

and secondly

$$- \sum_{Q \ni l_i \neq j \in P \setminus Q} C^i \frac{\alpha_j}{\alpha_{l_i}} \lambda_{l_i} \partial_{\lambda_{l_i}} \varepsilon_{l_i, j} \geq c \sum_{Q \ni l_i \neq j \in P \setminus Q} \varepsilon_{l_i, j},$$

since for $l_i \in Q$ and $j \in P \setminus Q$ by definition

$$d(a_{l_i}, x_{j_i}) \ll 1 \text{ for some } x_{j_i} \neq x_0, \text{ while } d(a_j, x_0) \ll 1,$$

hence a_{l_i} and a_j are far from each other and therefore, cf. Lemma 2.3,

$$\lambda_{l_i} \partial_{\lambda_{l_i}} \varepsilon_{l_i, j} = \frac{2-n}{2} \varepsilon_{l_i, j}^{\frac{n-2}{2}} \left(\frac{\lambda_{l_i}}{\lambda_j} - \frac{\lambda_j}{\lambda_{l_i}} + \lambda_{l_i} \lambda_j \gamma_n G_{g_0}^{\frac{2}{2-n}}(a_{l_i}, a_j) \right) = \frac{2-n}{2} \varepsilon_{l_i, j} (1 + o(1)).$$

Hence, while $\psi \rightarrow -\infty$ as some $\lambda_{l_i} \rightarrow \infty$, we have

$$\psi' \geq \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}$$

in contradiction to, that necessarily $\lambda_i \xrightarrow{t \rightarrow \infty} \infty$. $\square$

Lemma 3.2 tells us, that every diverging flow line can only concentrate at $x_0 = \max_M K$. We now exclude tower bubbling at x_0 as well.

Lemma 3.3. *Every non single bubbling flow line at x_0 , i.e. eventually*

$$u \notin V(1, \varepsilon) \cap \{a \xrightarrow{t \rightarrow \infty} x_0\},$$

is compact.

Proof. We may assume $u \in V(p, \varepsilon)$ eventually and $\forall_i a_i \rightarrow x_0$. Then Proposition 2.14 and the principal lower bound on ∂J , cf. Definition 2.15, give

$$\begin{aligned} \text{(i)} \quad -\frac{\dot{\lambda}_i}{\lambda_i} &= \frac{r}{k} \left[\frac{d_2}{c_2} \frac{H_i}{\lambda_i^{n-2}} + \frac{e_2}{c_2} \frac{\Delta K_i}{K_i \lambda_i^2} - \frac{b_2}{c_2} \sum_{i \neq j=1}^p \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \right] (1 + o_{\frac{1}{\lambda_i}}(1)) \\ \text{(ii)} \quad \lambda_i \dot{a}_i &= \frac{r}{k} \left[\frac{e_3}{c_3} \frac{\nabla K_i}{K_i \lambda_i} + \frac{e_4}{c_3} \frac{\nabla \Delta K_i}{K_i \lambda_i^3} + \frac{b_3}{c_3} \sum_{i \neq j=1}^p \frac{\alpha_j}{\alpha_i} \frac{1}{\lambda_i} \nabla a_i \varepsilon_{i,j} \right] (1 + o_{\frac{1}{\lambda_i}}(1)) \end{aligned}$$

up to some

$$o_\varepsilon(\lambda_i^{2-n} + \sum_{i \neq j=1}^p \varepsilon_{i,j}) + \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

More precisely by Condition 1.2 and recalling $K_i = K(a_i)$ et cetera we have

$$\nabla K_i = -4|a_i|^2 a_i, \quad \Delta K_i = -4 \cdot 7|a_i|^2 \quad \text{and} \quad \nabla \Delta K_i = -8 \cdot 7 a_i.$$

Consequently putting $\frac{d_2}{c_2} = \gamma_1$, $\frac{e_2}{c_2} = \gamma_2$, $\gamma_3 = \frac{e_3}{c_3}$ and $b = \frac{b_2}{c_2}$ we find

$$\begin{aligned} \text{(i)} \quad -\frac{\dot{\lambda}_i}{\lambda_i} &= \frac{r}{k} \left[\gamma_1 \frac{H_i}{\lambda_i^{n-2}} - 4 \cdot 7 \gamma_2 \frac{|a_i|^2}{\lambda_i^2} - b \sum_{i \neq j=1}^p \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \right] (1 + o_{\frac{1}{\lambda_i} + |a_i|}(1)) \\ \text{(ii)} \quad \lambda_i \dot{a}_i &= \frac{r}{k} \left[-4 \gamma_3 \frac{|a_i|^2 a_i}{\lambda_i} + O\left(\sum_{i \neq j=1}^p \left| \frac{\nabla a_i}{\lambda_i} \varepsilon_{i,j} \right| \right) \right] (1 + o_{\frac{1}{\lambda_i} + |a_i|}(1)) \end{aligned}$$

up to some

$$o_\varepsilon(\lambda_i^{2-n} + \sum_{i \neq j=1}^p \varepsilon_{i,j}) + \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

We first order $\lambda_1 |a_1|^5 \leq \dots \leq \lambda_p |a_p|^5$ and study for $C \gg 1 \gg \varepsilon > 0$

$$\Theta = \sum_i C^i \eta \left(\frac{\lambda_i |a_i|^5}{\varepsilon} \right) \ln \frac{\lambda_i |a_i|^5}{\varepsilon} \quad (3.1)$$

with a cut-off function $\eta \in C^\infty(\mathbb{R}, [0, 1])$ satisfying

$$\eta|_{(0,1)} = 0, \quad \eta|_{(2,\infty)} = 1 \quad \text{and} \quad \eta'|_{(1,2)} > 0.$$

Then clearly $\theta \geq 0$ and there holds

$$\theta' = \sum_i C^i \vartheta_i \partial_t \ln(\lambda_i |a_i|^5),$$

where

$$\vartheta_i = \eta\left(\frac{\lambda_i |a_i|^5}{\epsilon}\right) + \eta'\left(\frac{\lambda_i |a_i|^5}{\epsilon}\right) \frac{\lambda_i |a_i|^5}{\epsilon} \ln \frac{\lambda_i |a_i|^5}{\epsilon} \quad (3.2)$$

and hence

$$\vartheta_i \text{ is } \begin{cases} = 0 & \text{on } \lambda_i |a_i|^5 \leq \epsilon \\ > 0 & \text{on } \epsilon < \lambda_i |a_i|^5 \leq 2\epsilon \\ = 1 & \text{on } \lambda_i |a_i|^5 \geq 2\epsilon \end{cases}.$$

We then find

$$\begin{aligned} \theta' &= \sum_i C^i \vartheta_i \left(\frac{\dot{\lambda}_i}{\lambda_i} + 5|a_i|^{-2} \left\langle \frac{a_i}{\lambda_i}, \lambda_i \dot{a}_i \right\rangle \right) \\ &\leq \frac{r}{k} \sum_i C^i \vartheta_i (1 + o_\epsilon(1)) ((4 \cdot 7\gamma_2 - 5 \cdot 4\gamma_3) \frac{|a_i|^2}{\lambda_i^2} + b \sum_{i \neq j=1}^p \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + o_\epsilon(\sum_{i \neq j=1}^p \varepsilon_{i,j})) \end{aligned}$$

up to some

$$\begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

Due to $\frac{\gamma_3}{\gamma_2} = 3$, cf. the proof of Proposition 6.8 in [15], we have

$$4 \cdot 7\gamma_2 - 5 \cdot 4\gamma_3 = -32\gamma_2$$

and there holds, cf. (2.6) and arguing as for (2.4), for $i > j$

$$-\vartheta_i \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \geq c \vartheta_i \varepsilon_{i,j} \quad \text{and} \quad -\sum_{i \neq j} C^i \vartheta_i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \geq c \sum_{i \neq j} C^i \vartheta_i \varepsilon_{i,j}, \quad (3.3)$$

as we shall prove below. We thus obtain

$$\theta' \leq -c \sum_i C^i \vartheta_i \left(\frac{|a_i|^2}{\lambda_i^2} + \sum_{i \neq j=1}^p \varepsilon_{i,j} \right) + \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}. \quad (3.4)$$

As a consequences θ , hence all $\lambda_i |a_i|^5$ are bounded and

$$\forall 1 \leq i \leq p : \int_0^\infty \left(\frac{|a_i|^2}{\lambda_i^2} + \sum_{i \neq j=1}^p \varepsilon_{i,j} \right) \chi_{\{\lambda_i |a_i|^5 \geq 2\epsilon\}} < \infty. \quad (3.5)$$

On the other hand for all $1 \leq i \leq p$

$$\frac{\dot{\lambda}_i}{\lambda_i} \lesssim \frac{|a_i|^2}{\lambda_i^2} + \sum_{i \neq j=1}^p \varepsilon_{i,j} + \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases},$$

whence $\lambda_i \rightarrow \infty$ due to (3.5) necessitates, that for some $t_{k,i} \xrightarrow{k \rightarrow \infty} \infty$ at least

$$\lambda_i |a_i|^5 \leq 2\epsilon \quad \text{at } t = t_{k,i},$$

while arguing as before on $\{\lambda_i |a_i|^5 \geq 2\varepsilon\}$

$$\partial_t \ln(\lambda_i |a_i|^5) \lesssim \frac{|a_i|^2}{\lambda_i^2} + \sum_{i \neq j=1}^{\infty} \varepsilon_{i,j} + \begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

Hence we may assume, that eventually $\forall 1 \leq i \leq p : \lambda_i |a_i|^5 \leq 4\varepsilon$, thus

$$\begin{aligned} \varepsilon_{i,j} &\gtrsim \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{\frac{2-n}{2}} \gtrsim \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j (|a_i|^2 + |a_j|^2) \right)^{-\frac{3}{2}} \\ &\gtrsim \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \left(\frac{\lambda_i |a_i|^5}{\lambda_i} + \frac{\lambda_j |a_j|^5}{\lambda_j} \right)^{\frac{2}{5}} \right)^{-\frac{3}{2}} \gtrsim \varepsilon^{-\frac{3}{5}} (\lambda_i^{1-\frac{2}{5}} \lambda_j + \lambda_i \lambda_j^{1-\frac{2}{5}})^{-\frac{3}{2}} \end{aligned}$$

and likewise $\frac{|a_i|^2}{\lambda_i^2} \leq \frac{\varepsilon^{\frac{2}{5}}}{\lambda_i^{2+\frac{2}{5}}}$. Recalling (2.6) we therefore obtain for $\lambda_m = \max_i \lambda_i$

$$\begin{aligned} \frac{\dot{\lambda}_m}{\lambda_m} &\leq \tilde{\gamma}_2 \frac{|a_m|^2}{\lambda_m^2} - \tilde{\gamma}_4 \sum_{m \neq j=1}^p \varepsilon_{m,j} \lesssim \tilde{\gamma}_2 \frac{\varepsilon^{\frac{2}{5}}}{\lambda_m^{2+\frac{2}{5}}} - \tilde{\gamma}_4 \sum_{m \neq j=1}^p \frac{\varepsilon^{-\frac{3}{5}}}{(\lambda_j^{1-\frac{2}{5}} \lambda_m + \lambda_j \lambda_m^{1-\frac{2}{5}})^{\frac{3}{2}}} \\ &\lesssim \tilde{\gamma}_2 \frac{\varepsilon^{\frac{2}{5}}}{\lambda_m^{2+\frac{2}{5}}} - \tilde{\gamma}_4 \frac{\varepsilon^{-\frac{3}{5}}}{\lambda_m^{\frac{3}{2}(2-\frac{2}{5})}} = \frac{\tilde{\gamma}_2 \varepsilon^{\frac{2}{5}} - \tilde{\gamma}_4 \varepsilon^{-\frac{3}{5}}}{\lambda_m^{2+\frac{2}{5}}} \leq 0 \end{aligned}$$

up to some

$$\begin{cases} O(|\delta J(u)|^2) & \text{under (1.1)} \\ O(|\partial J(u)|^2) & \text{under (1.3)} \end{cases}.$$

So $\lambda_m \rightarrow \infty$ is impossible and we are left with proving (3.3). Recalling

$$\lambda_1 |a_1|^5 \leq \dots \leq \lambda_p |a_p|^5$$

we have for $i > j$

$$-\lambda_i \partial_{\lambda_i} \varepsilon_{i,j} = \frac{n-2}{2} \varepsilon_{i,j}^{\frac{n}{n-2}} \left(\frac{\lambda_i}{\lambda_j} - \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \gamma_n G^{\frac{2-n}{2-n}}(a_i, a_j) \right)$$

and hence $-\lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \geq \frac{n-2}{4} \varepsilon_{i,j}$ in either of the cases

$$\lambda_i \geq \lambda_j \text{ or } \lambda_i \lambda_j d_{g_0}^2(a_i, a_j) \geq \frac{\lambda_j}{\lambda_i}.$$

Hence we may assume $d_{g_0}(a_i, a_j) \leq \frac{1}{\lambda_i}$ and $\frac{\lambda_j}{\lambda_i} \gg 1$. Since for $i > j$ by assumption

$$\lambda_i |a_i|^5 \geq \lambda_j |a_j|^5,$$

we then have $|a_i| \gg |a_j|$ and hence $d_{g_0}(a_i, a_j) \simeq |a_i - a_j| \simeq |a_i|$. Therefore

$$\lambda_i |a_i|^5 \simeq \lambda_i d_{g_0}^5(a_i, a_j) \lesssim \frac{1}{\lambda_i^4}.$$

However $\vartheta_i = 0$ on $\{\lambda_i |a_i|^5 \leq \varepsilon\}$ and we conclude

$$-\vartheta_i \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \geq \vartheta_i \frac{n-2}{4} \varepsilon_{i,j}.$$

This shows the first statement of (3.3). We then compute

$$\begin{aligned} & - \sum_{i \neq j} C^i \vartheta_i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \\ &= - \sum_{i > j} [C^i \vartheta_i \frac{\alpha_j}{\alpha_i} - C^j \vartheta_j \frac{\alpha_i}{\alpha_j}] \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} - \sum_{i < j} C^i \vartheta_i \frac{\alpha_j}{\alpha_i} [\lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + \lambda_j \partial_{\lambda_j} \varepsilon_{i,j}] \end{aligned}$$

and observe, that the latter sum is non positive, whence

$$-\sum_{i \neq j} C^i \vartheta_i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \geq -\sum_{i > j} [C^i \frac{\alpha_j}{\alpha_i} - C^j \frac{\vartheta_j}{\vartheta_i} \frac{\alpha_i}{\alpha_j}] (-\vartheta_i \lambda_i \partial_{\lambda_i} \varepsilon_{i,j}).$$

Hence the statement follows for $C \gg 1$ sufficiently large, provided we may uniformly bound $\vartheta_j \lesssim \vartheta_i$ for $i > j$, which recalling (3.2) translates into

$$\exists \kappa \geq 1 \forall r < s : \vartheta(r) \leq \kappa \vartheta(s) \text{ for } \vartheta(t) = \eta(t) + \eta'(t)t \ln t, \quad (3.6)$$

i.e. monotonicity in case $\kappa = 1$. Recalling furthermore

$$\eta|_{(0,1)} = 0, \quad \eta|_{(2,\infty)} = 1 \text{ and } \eta'|_{(1,2)} > 0,$$

evidently (3.6) is satisfied, whenever $s > 1 + \delta$ for some $\delta > 0$ small, while we may assume $\eta'' \geq 0$ on $(0, 1 + \delta)$. Hence ϑ as a sum of products of non negative monotone functions on $(0, 1 + \delta)$ is monotone. $\square$

Together Lemmata 3.1, 3.2 and 3.3 show, that a non compact flow line u has to satisfy

$$u = \alpha \delta_{a,\lambda} + v \in V(1, \varepsilon) \text{ eventually}$$

and $a \rightarrow x_0 = \max_M K$.

3.2 Diverging flow lines

The only possibility left for a non compact flow line of (1.1) or (1.3) under Condition 1.2 is realised.

Lemma 3.4. *Let $n = 5$ and Condition 1.2 hold true. Then for every $\varepsilon > 0$ small there exists $0 < \varepsilon_0 < \varepsilon$ such, that every flow line u under (1.1) or (1.3) and starting with initial data*

$$u_0 = \alpha_0 \varphi_{a_0, \lambda_0} \in V(1, \varepsilon_0) \text{ with } |a_0| < \varepsilon_0 \text{ and } \lambda_0 |a_0|^2 > \varepsilon_0^{-1}$$

remains in $V(1, \varepsilon)$ for all times and

$$\lambda \rightarrow \infty \text{ and } |a| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Proof. We prove the statement under (1.1). The proof under (1.3) is then analogous replacing in particular the appearance of $|\delta J|$ by $|\partial J|$. In order to prove, that u remains in $V(1, \varepsilon)$ for all times let us define

$$T = \sup\{\tau > 0 : \forall 0 \leq t < \tau : u \in V(1, \varepsilon), |a| < \varepsilon, \lambda |a|^2 > \varepsilon^{-1}\}.$$

We then have to show $T = \infty$. We may clearly assume

$$\int_0^\infty |\delta J(u)|^2 \leq c < \infty. \quad (3.7)$$

According to Proposition 2.14 and using the principal lower bound on ∂J , cf. Definition 2.15, the relevant evolution equations are

$$\begin{aligned} \text{(i)} \quad -\frac{\dot{\lambda}}{\lambda} &= \frac{r}{k} (\gamma_1 \frac{H(a)}{\lambda^3} + \gamma_2 \frac{\Delta K(a)}{K(a)\lambda^2}) (1 + o_{\frac{1}{\lambda}}(1)) + o(\frac{1}{\lambda^3}) + O(|\delta J(u)|^2) \\ \text{(ii)} \quad \lambda \dot{a} &= \frac{r}{k} (\gamma_3 \frac{\nabla K(a)}{K(a)\lambda} + \gamma_4 \frac{\nabla \Delta K(a)}{K(a)\lambda^3}) (1 + o_{\frac{1}{\lambda}}(1)) + o(\frac{1}{\lambda^3}) + O(|\delta J(u)|^2), \end{aligned}$$

where due to $k = 1$ and hence $\frac{r}{k} = J(u)$ we have for some constant $\kappa > 0$ during $(0, T)$

$$\frac{r}{k} = \kappa(1 + o_\varepsilon(1)).$$

Moreover

$$\nabla K(a) = -4|a|^2 a, \quad \Delta K(a) = -4 \cdot 7|a|^2 \text{ and } \nabla \Delta K(a) = -8 \cdot 7a. \quad (3.8)$$

We obtain during $(0, T)$ the simplified evolution equations

$$\begin{aligned}
 \text{(i)} \quad & -\frac{\dot{\lambda}}{\lambda} = \kappa\gamma_2 \frac{\Delta K(a)}{\lambda^2} (1 + o_\varepsilon(1)) + O(|\delta J(u)|^2) \\
 \text{(ii)} \quad & \lambda \dot{a} = \kappa\gamma_3 \frac{\nabla K(a)}{\lambda} (1 + o_\varepsilon(1)) + O(|\delta J(u)|^2).
 \end{aligned} \tag{3.9}$$

First note, that during $(0, T)$

$$\partial_t |a|^2 = \frac{2}{\lambda} \langle a, \lambda \dot{a} \rangle = 2\kappa\gamma_3 \frac{\langle \nabla K(a), a \rangle}{\lambda^2} (1 + o_\varepsilon(1)) + O\left(\frac{|a||\delta J(u)|^2}{\lambda}\right), \tag{3.10}$$

whence $\partial_t \ln |a|^2 \leq O\left(\frac{|\delta J(u)|^2}{\lambda|a|}\right)$. But during $(0, T)$ by definition

$$\lambda|a| = \lambda^{\frac{1}{2}} (\lambda|a|^2)^{\frac{1}{2}} > c\varepsilon^{-1},$$

whence $|a|$ remains uniformly small, e.g. $|a| \leq C\varepsilon_0$. Secondly

$$\begin{aligned}
 (\lambda \Delta K(a))' &= \frac{\dot{\lambda}}{\lambda} \lambda \Delta K(a) + \langle \nabla \Delta K(a), \lambda \dot{a} \rangle \\
 &= -\kappa\gamma_2 \frac{|\Delta K(a)|^2}{\lambda} (1 + o_\varepsilon(1)) + \kappa\gamma_3 \frac{\langle \nabla \Delta K(a), \nabla K(a) \rangle}{\lambda} (1 + o_\varepsilon(1)) \\
 &\quad + O((|\lambda \Delta K(a)| + |\nabla \Delta K(a)|)|\delta J(u)|^2),
 \end{aligned}$$

and hence, since $|\lambda \Delta K(a)| = 4 \cdot 7\lambda|a|^2 \geq c\varepsilon^{-1}$ and $|\nabla \Delta K(a)| \leq C\varepsilon$ during $(0, T)$,

$$\frac{(\lambda \Delta K(a))'}{\kappa} = (-4^2 \cdot 7^2 \gamma_2 |a|^4 + 4 \cdot 8 \cdot 7\gamma_3 |a|^4) \frac{1 + o_\varepsilon(1)}{\lambda}$$

up to some

$$O(|\lambda \Delta K(a)||\delta J(u)|^2).$$

Due to $\frac{\gamma_3}{\gamma_2} = 3$, cf. the proof of Proposition 6.8 in [15], this shows

$$(\lambda \Delta K(a))' \leq O(|\lambda \Delta K(a)||\delta J(u)|^2)$$

and therefore $\partial_t \ln(-\lambda \Delta K(a)) \geq O(|\delta J(u)|^2)$. We conclude using (3.7), that

$$4 \cdot 7\lambda|a|^2 = -\lambda \Delta K(a) \geq 4 \cdot 7\lambda_0|a_0|^2 e^{-C \int_0^\infty |\delta J(u)|^2}$$

remains during $(0, T)$ uniformly large, say $\lambda|a|^2 \geq c\varepsilon_0^{-1}$. As a consequence

$$-\frac{\dot{\lambda}}{\lambda} = \kappa\gamma_2 \frac{\Delta K(a)}{\lambda^2} (1 + o_\varepsilon(1)) = -4 \cdot 7\kappa\gamma_2 \frac{|a|^2}{\lambda^2} (1 + o_\varepsilon(1)) \leq -\frac{4 \cdot 7\kappa\gamma_2 c}{\varepsilon_0 \lambda^3}$$

up to some $O(|\delta J(u)|^2)$, whence

$$\partial_t \lambda^3 + \lambda^3 O(|\delta J(u)|^2) \geq \frac{4 \cdot 7\kappa\gamma_2 c}{3\varepsilon_0} = C_0.$$

Letting $\vartheta = \lambda^3$ this becomes $\dot{\vartheta} + \vartheta O(|\delta J(u)|^2) \geq C_0$. Thus there holds

$$\dot{\tau}(t) = (\dot{\vartheta} + \vartheta O(|\delta J(u)|^2))(t) e^{\int_0^t O(|\delta J(u)|^2)} \geq C_0 e^{\int_0^t O(|\delta J(u)|^2)}$$

for $\tau(t) = \vartheta(t) e^{\int_0^t O(|\delta J(u)|^2)}$ and therefore

$$\dot{\tau}(t) \geq C_0 e^{-C \int_0^\infty |\delta J(u)|^2}, \tag{3.11}$$

whence

$$\vartheta(0) = \tau(0) \leq \tau(t) = \vartheta(t) e^{\int_0^t O(|\delta J(u)|^2)} \leq \vartheta(t) e^{C \int_0^\infty |\delta J(u)|^2}, \tag{3.12}$$

so ϑ and thus λ remain uniformly large, say $\lambda \geq c\varepsilon_0^{-1}$. In summa we cannot escape from

$$|a| < C\varepsilon_0, \quad \lambda|a|^2 > c\varepsilon_0^{-1} \quad \text{and} \quad \lambda > c\varepsilon_0^{-1} \quad (3.13)$$

during $(0, T)$. Clearly $T = \infty$ follows, if and as we shall prove

$$u \in V(1, \frac{\varepsilon}{2}) \quad \text{during} \quad (0, T).$$

And by definition 2.5 and the remarks thereafter this is equivalent to showing

$$|1 - \frac{r\alpha^{\frac{4}{n-2}}K(a)}{4n(n-1)k}|, \|u - \alpha\varphi_{a,\lambda}\| = \|v\| < \frac{\varepsilon}{2},$$

To that end let us expand using $k = 1$ provided $\varepsilon_0 \ll \varepsilon$, cf. (3.13).

$$\begin{aligned} J(u) &= r = \int L_{g_0} u u = \int L_{g_0} (\alpha\varphi_{a,\lambda} + v) (\alpha\varphi_{a,\lambda} + v) \\ &= \alpha^2 \int L_{g_0} \varphi_{a,\lambda} \varphi_{a,\lambda} + 2\alpha \int L_{g_0} \varphi_{a,\lambda} v + \int L_{g_0} v v. \end{aligned}$$

Since $L_{g_0} \varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} + o_{\frac{1}{\lambda}}(1)$, we find by simple expansions

$$\begin{aligned} \frac{\int L_{g_0} \varphi_{a,\lambda} v}{4n(n-1)} &= \int \varphi_{a,\lambda}^{\frac{n+2}{n-2}} v = \int K \varphi_{a,\lambda}^{\frac{n+2}{n-2}} v = \alpha^{-\frac{4}{n-2}} \int K(u-v)^{\frac{4}{n-2}} \varphi_{a,\lambda} v \\ &= -\frac{4}{n-2} \alpha^{-\frac{4}{n-2}} \int K u^{\frac{6-n}{n-2}} \varphi_{a,\lambda} v^2 = -\frac{4}{n-2} \alpha^{-1} \int K \varphi_{a,\lambda}^{\frac{4}{n-2}} v^2 \end{aligned} \quad (3.14)$$

up to some $o_{\frac{1}{\lambda}+|a|}(1) + o(\|v\|^2)$, where we made use of the orthogonality

$$\int K u^{\frac{4}{n-2}} \varphi_{a,\lambda} v = 0$$

considered under (1.1). Hence and still up to some $o_{\frac{1}{\lambda}+|a|}(1) + o(\|v\|^2)$

$$J(u) = 4n(n-1)c_1\alpha^2 + \int L_{g_0} v v - \frac{32n(n-1)}{n-2} \int \varphi_{a,\lambda}^{\frac{4}{n-2}} v^2,$$

cf. Lemma 2.3. On the other hand we have up to some $o(\|v\|^2)$

$$\begin{aligned} 1 &= \int K u^{\frac{2n}{n-2}} = \int K (\alpha\varphi_{a,\lambda} + v)^{\frac{2n}{n-2}} \\ &= \alpha^{\frac{2n}{n-2}} \int K \varphi_{a,\lambda}^{\frac{2n}{n-2}} + \frac{2n}{n-2} \alpha^{\frac{n+2}{n-2}} \int K \varphi_{a,\lambda}^{\frac{n+2}{n-2}} v + \frac{n}{n-2} \frac{n+2}{n-2} \alpha^{\frac{4}{n-2}} \int K \varphi_{a,\lambda}^{\frac{4}{n-2}} v^2. \end{aligned}$$

Considering the second summand above we obtain using (3.14)

$$1 = \alpha^{\frac{2n}{n-2}} c_1 + \frac{n(n-6)}{(n-2)^2} \alpha^{\frac{4}{n-2}} \int \varphi_{a,\lambda}^{\frac{4}{n-2}} v^2 + o_{\frac{1}{\lambda}+|a|}(1) + o(\|v\|^2),$$

whence

$$\alpha = c_1^{-\frac{n-2}{2n}} + \frac{6-n}{2(n-2)} c_1^{-\frac{n+2}{2n}} \int \varphi_{a,\lambda}^{\frac{4}{n-2}} v^2 + o_{\frac{1}{\lambda}+|a|}(1) + o(\|v\|^2)$$

and therefore

$$c_1\alpha^2 = c_1^{\frac{2}{n}} + \frac{6-n}{n-2} \int \varphi_{a,\lambda}^{\frac{4}{n-2}} v^2 + o_{\frac{1}{\lambda}+|a|}(1) + o(\|v\|^2).$$

Consequently and up to some $o_{\frac{1}{\lambda}+|a|}(1) + o(\|v\|^2)$

$$J(u) = 4n(n-1)c_1^{\frac{2}{n}} + \int L_{g_0} v v - 4n(n-1) \frac{n+2}{n-2} \int \varphi_{a,\lambda}^{\frac{4}{n-2}} v^2$$

and, since the latter quadratic form in v corresponding to $\partial^2 J(\varphi_{a,\lambda})v^2$ is well known to be positive, we obtain with some uniform $c > 0$

$$J(u) \geq 4n(n-1)c_1^{\frac{2}{n}} + o_{\frac{1}{\lambda}+|a|}(1) + c\|v\|^2.$$

But $J(u) \leq J(u_0) = 4n(n-1)c_1^{\frac{2}{n}} + o_{\frac{1}{\lambda_0}+|a_0|}(1)$ as $u_0 = \alpha_0 \varphi_{a_0, \lambda_0}$ and therefore

$$\|v\|^2 = o_{\frac{1}{\lambda}+\frac{1}{\lambda_0}+|a|+|a_0|}(1) \quad (3.15)$$

remains uniformly small during $(0, T)$, cf. (3.13). Finally note, that

$$\frac{r\alpha^{\frac{4}{n-2}}K(a)}{4n(n-1)k} = \frac{K(a) \int L_{g_0} \varphi_{a,\lambda} \varphi_{a,\lambda}}{4n(n-1) \int K \varphi_{a,\lambda}^{\frac{2n}{n-2}}} + o_{\|v\|}(1) = 1 + o_{\frac{1}{\lambda}+\|v\|}(1),$$

whence by virtue of (3.15)

$$\left| 1 - \frac{r\alpha^{\frac{4}{n-2}}K(a)}{4n(n-1)k} \right| = o_{\frac{1}{\lambda}+\frac{1}{\lambda_0}+|a|+|a_0|}(1)$$

and therefore remains uniformly small, cf. (3.13). This completes the proof of $T = \infty$. Then by (3.11)

$$\tau > ct,$$

whence $\vartheta = \lambda^3 \geq \tilde{c}t$ according to (3.12). This shows $\lambda \rightarrow \infty$. Finally by (3.8) and (3.10)

$$\partial_t |a|^2 \leq -c \frac{|a|^4}{\lambda^2} + O\left(\frac{|a||\delta J(u)|^2}{\lambda}\right) = c|a|^2 \left(-\frac{|a|^2}{\lambda^2} + O\left(\frac{|\delta J(u)|^2}{|a|\lambda}\right)\right) \text{ for some } c > 0.$$

Since $\lambda|a|^2$ and therefore $\lambda|a|$ as well remain large, cf. (3.13), we obtain

$$\partial_t \ln |a|^2 \leq -c \frac{|a|^2}{\lambda^2} + O(|\delta J(u)|^2),$$

whence due to (3.8) and 3.9 for some $\tilde{c} > 0$

$$\partial_t \ln |a|^2 \leq -\tilde{c} \frac{\dot{\lambda}}{\lambda} + O(|\delta J(u)|^2) = -\partial_t \ln \lambda^{\tilde{c}} + O(|\delta J(u)|^2).$$

Therefore $\lambda \rightarrow \infty$ implies $|a| \rightarrow 0$. □

3.3 Modifying the gradient flow

We finally discuss how to compactify (1.1) and (1.3) in the situation of Lemma 3.4. From Section 3.2 the only critical value for a non compact flow line is

$$J_\infty = J(\varphi_{x_0, \infty}) = \frac{c_0}{K^{\frac{n-2}{n}}(x_0)}, \quad c_0 > 0.$$

Hence it is sufficient to only modify (1.1) and (1.3) on

$$\mathcal{M}_\delta = \{J_\infty - \delta < J < J_\infty + \delta\}, \quad 0 < \delta \ll 1.$$

We then pass from (1.1) to (1.3) on $\mathcal{M}_\delta$ and are left with suitably compactifying (1.3) on $\mathcal{M}_\delta$. Clearly we may restrict ourselves to modifications on

$$\mathcal{N}_{a,\varepsilon} = V(1, \varepsilon) \cap \{d(a, x_0) < \varepsilon\} \subset \mathcal{M}_\delta$$

for sufficiently small $0 < \varepsilon \ll \delta$. To that end consider a cut-off function

$$\eta_1 \in C^\infty(\mathbb{R}_+, [0, 1]) \text{ with } \eta_1|_{(0,1)} = 1, \quad \eta_1|_{(2,\infty)} = 0 \text{ and } \eta'_1 \leq 0$$

and let for $0 < \epsilon \ll \varepsilon$

$$\eta_V = \eta\left(\frac{d(\cdot, V(1, \frac{\varepsilon}{2}))}{\epsilon}\right) \text{ on } X \text{ and } \eta_a = \eta_1\left(\frac{|a|}{\epsilon}\right) \text{ on } V(1, \epsilon),$$

where $|\cdot|$ denotes the euclidean distance from x_0 in conformal normal coordinates around x_0 . Moreover consider a second cut-off function

$$\eta_2 \in C^\infty(\mathbb{R}_+, [0, 1]) \text{ with } \eta_2|_{(0,1)} = 0, \eta_2|_{(2,\infty)} = 1 \text{ and } \eta_2' \geq 0$$

and let

$$\eta_{a,\lambda} = \eta_2\left(\frac{\lambda|a|^2}{\epsilon}\right) \text{ on } \mathcal{N}_{a,\epsilon}.$$

Hence $\eta_V \eta_a \eta_{a,\lambda}$ is well defined on X and

$$\text{supp}(\eta_V \eta_a \eta_{a,\lambda}) \subset \text{supp}(\eta_V \eta_a) \subset \mathcal{N}_{a,\epsilon} \subset \mathcal{M}_\delta.$$

We then consider for some $C \geq 1$

$$W = -\varepsilon \eta_V \eta_a \eta_{a,\lambda} \left(\frac{\alpha^{-1} \nabla K(a)}{|\nabla K(a)| \lambda} \frac{\nabla_a}{\lambda} \varphi_{a,\lambda} - C \frac{V}{\lambda} \right) \quad (3.16)$$

as a bounded, locally Lipschitz vectorfield on X , which is well defined due to

$$\nabla K(a) = -4|a|^2 a \neq 0 \text{ on } \text{supp}(\eta_{a,\lambda}),$$

and study the flow generated by

$$\partial_t u = -\frac{r}{2k} (\nabla J(u) + W + \frac{\int K u^{\frac{n+2}{n-2}} (\nabla J(u) + W)}{k} u). \quad (3.17)$$

Clearly $k = 1$ is preserved as is positivity

$$u > 0$$

along flow lines and consequently (3.17) induces a flow on X . Indeed

$$-W \geq \varepsilon \eta_V \eta_a \eta_{a,\lambda} \left(-C \frac{\varphi_{a,\lambda}}{\lambda} - C \frac{V}{\lambda} \right) \geq -\frac{C\varepsilon}{\lambda} \eta_V \eta_a \eta_{a,\lambda} u \geq -u$$

for $C \gg 1$ sufficiently large, whence we obtain in combination with (1.5)

$$\partial_t u \geq -\tilde{C}(1 + |\partial J(u)|)u$$

and therefore u exists positively for all times, provided we have uniform a priori bounds on $|\partial J(u)|$, which we derive from Proposition 1.1 using $k = 1$ and the boundedness of energy along a flow line. The latter boundedness follows from the subsequent Lemma 3.5.

Lemma 3.5. *Along a flow line there holds $\partial_t J(u) \lesssim -|\partial J(u)|^2$.*

Proof. Since $\partial J(u)u = 0$ by scaling invariance, we clearly have

$$\partial_t J(u) = -\frac{r}{2k} (\|\nabla J(u)\|^2 + \partial J(u)W).$$

Then Proposition 2.12 and the principal lower bound on ∂J yield

$$\partial J(u) \frac{V}{\lambda} = O\left(\frac{|\partial J(u)|^2}{\lambda}\right),$$

cf. Definition 2.15, whence

$$\partial_t J(u) = -\frac{r}{2k} (|\partial J(u)|^2 (1 + o(1)) - \varepsilon \eta_V \eta_a \eta_{a,\lambda} \frac{\alpha^{-1} \nabla K(a)}{|\nabla K(a)| \lambda} \partial J(u) \frac{\nabla_a}{\lambda} \varphi_{a,\lambda}).$$

From Proposition 2.13 and (2.1) we then find

$$\partial J(u) \frac{\nabla_a}{\lambda} \varphi_{a,\lambda} = -\sigma_{3,\cdot} = -4n(n-1)e_3 \alpha \frac{\nabla K(a)}{K(a)\lambda} + o_\varepsilon\left(\frac{1}{\lambda^3}\right) + O(|\partial J(u)|^2)$$

using again Proposition 2.12 and the principal lower bound on ∂J . Therefore

$$\partial_t J(u) = -\frac{r}{2K}(|\partial J(u)|^2(1+o(1)) + 4n(n-1)e_3 \varepsilon \eta_V \eta_a \eta_{a,\lambda} \left(\frac{|\nabla K(a)|}{K(a)\lambda^2} + o_\varepsilon\left(\frac{1}{\lambda^4}\right)\right)).$$

Note, that on $\text{supp}(\eta_{a,\lambda})$ we have $\lambda|a|^2 \geq \varepsilon$ and hence $\frac{|\nabla K(a)|}{\lambda^2} \gg \frac{1}{\lambda^4}$. □

In particular the flow generated by (3.17) decreases energy and we have

$$\int_0^\infty |\partial J(u)|^2 < \infty$$

just like under (1.3). Since (3.17) coincides with (1.3) outside $V(1, \varepsilon)$, whereupon the flow generated by (1.3) is compact, cf. Section 3.1, every non compact flow line u for (3.17) has to enter $V(1, \varepsilon)$ for at least a sequence in time. If we suppose, that u does not remain in $V(1, 2\varepsilon)$ eventually, then there exists

$$s_1 \leq s'_1 \leq \dots \leq s_k \leq s'_k \leq \dots \quad \text{with } s_k, s'_k \xrightarrow{k \rightarrow \infty} \infty$$

such, that

$$u_{s_k} \in \partial V(1, \varepsilon), \quad u_{s'_k} \in \partial V(1, 2\varepsilon) \quad \text{and } u \in V(1, 2\varepsilon) \setminus V(1, \varepsilon) \quad \text{during } (s_k, s'_k).$$

However, since $\|\partial_t u\| \leq C$ under (3.17), as

$$\|\nabla J(u)\| = |\partial J(u)|$$

is uniformly bounded along a flow line, and

$$d(\partial V(1, 2\varepsilon), \partial V(1, \varepsilon)) \geq \tilde{\varepsilon},$$

we find $|s'_k - s_k| \geq \frac{\tilde{\varepsilon}}{C}$. Moreover there holds

$$|\partial J| \geq \bar{\varepsilon} \quad \text{on } V(1, 2\varepsilon) \setminus V(1, \varepsilon)$$

by combining Proposition 2.12 and (i) from Proposition 4.1 with the principal lower bound on ∂J , cf. Definition 2.15. Therefore we infer from Lemma 3.5

$$J(u_{s'_k}) - J(u_{s_k}) = \int_{s_k}^{s'_k} \partial_t J(u) \leq -c \int_{s_k}^{s'_k} |\partial J(u)|^2 \leq -\frac{c\bar{\varepsilon}^2 \tilde{\varepsilon}}{C}$$

and hence iteratively

$$\begin{aligned} J(u_{s'_k}) &= J(u_{s'_k}) - J(u_{s_k}) + J(u_{s_k}) \leq J(u_{s'_k}) - J(u_{s_k}) + J(u_{s'_{k-1}}) \\ &\leq J(u_{s_1}) + \sum_{i=1}^k \left(J(u_{s'_i}) - J(u_{s_i}) \right), \end{aligned}$$

which necessitates $J(u_{s_k}) \rightarrow -\infty$, a contradiction. Hence we may assume

$$u \in V(1, 2\varepsilon) \quad \text{eventually.}$$

On the other hand, since by Lemma 3.5 every flow line up to a sequence in time is a Palais-Smale, cf. (1.8), we may assume, that u is precompact in some $V(\omega, p, \delta)$ for every $\delta > 0$. Since

$$d(V(\omega, p, \delta), V(1, 2\delta)) > \tilde{\delta} \quad \text{in case } \omega \neq 0 \quad \text{or } p \neq 1$$

for all $\delta > 0$ sufficiently small, the same energy consumption argument as before would lead to the same contradiction. Hence necessarily

$$u = \alpha \varphi_{a,\lambda} + v \in V(1, \delta) \text{ for every } \delta > 0 \text{ eventually.} \quad (3.18)$$

In particular we may assume $\eta_V = 1$ eventually for a non compact flow line.

So let us analyse the impact on the shadow flow, when passing from (1.3) to (3.17), in particular on the evolution equations for a and λ . Comparing to Section 2.1 we find in the present one bubble scenario

- (i) $\dot{\xi}_k = (\frac{\dot{\alpha}}{\alpha}, -\frac{\dot{\lambda}}{\lambda}, \lambda \dot{a})$ and $\phi_l = (\varphi_{a,\lambda}, -\lambda \partial_\lambda \varphi_{a,\lambda}, \frac{\nabla_a}{\lambda} \varphi_{a,\lambda})$
- (ii) $\Xi_{k,l} = 4n(n-1)\alpha c_k \delta_{k,l} + O(\frac{1}{\lambda^2} + |\partial J(u)|)$
- (iii) $\Xi_{k,l} \dot{\xi}_k = \langle \partial_t u, \phi_l \rangle$.

To achieve the simple form of Ξ in (ii) above, we applied Proposition 2.12 and the principal lower bound on ∂J , cf. Definition 2.15, to (2.2). Note, that due to $k = 1$, cf. Proposition 1.1,

$$\begin{aligned} \frac{r}{k} \int K u^{\frac{n+2}{n-2}} (\nabla J(u) + W) &= -\partial J(u) (\nabla J(u) + W) + \int L_{g_0} u (\nabla J(u) + W) \\ &= -|\partial J(u)|^2 + \partial J(u) u - \partial J(u) W + \int L_{g_0} u W, \end{aligned}$$

where $\partial J(u)u = 0$ by scaling invariance, $\partial J(u)W = O(\frac{|\partial J(u)|}{\lambda})$ by (3.16) and

$$\begin{aligned} \int L_{g_0} u W &= -\varepsilon \eta_V \eta_a \eta_{a,\lambda} \left(\frac{\alpha^{-1} \nabla K(a)}{|\nabla K(a)| \lambda} \int L_{g_0} u \frac{\nabla_a}{\lambda} \varphi_{a,\lambda} - C \int L_{g_0} u \frac{v}{\lambda} \right) \\ &= O\left(\frac{1}{\lambda^2} + \|v\|^2\right) \end{aligned}$$

by orthogonalities $\langle v, \phi_k \rangle = 0$ and $\int L_{g_0} \varphi_{a,\lambda} \frac{\nabla_a}{\lambda} \varphi_{a,\lambda} = O(\frac{1}{\lambda^2})$. Hence

$$\int K u^{\frac{n+2}{n-2}} (\nabla J(u) + W) = O\left(\frac{1}{\lambda^2} + |\partial J(u)|^2\right)$$

absorbing $\|v\|^2$ by Proposition 2.12 and the principal lower bound on ∂J . We therefore have for (3.17), cf. Proposition 2.13,

$$\begin{aligned} \langle \partial_t u, \phi_l \rangle &= -\frac{r}{2k} (\langle \nabla J(u), \phi_l \rangle_{L_{g_0}} + \langle W, \phi_l \rangle_{L_{g_0}} + O(\frac{1}{\lambda^2} + |\partial J(u)|^2) \langle u, \phi_l \rangle_{L_{g_0}}) \\ &= \frac{r}{k} \sigma_l + \varepsilon \eta_V \eta_a \eta_{a,\lambda} \frac{r}{2k} \frac{\alpha^{-1} \nabla K(a)}{|\nabla K(a)| \lambda} \langle \frac{\nabla_a}{\lambda} \varphi_{a,\lambda}, \phi_l \rangle_{L_{g_0}} \\ &\quad + O\left(\frac{1}{\lambda^2} + |\partial J(u)|^2\right) \frac{r}{k} \langle \varphi_{a,\lambda}, \phi_l \rangle_{L_{g_0}} \end{aligned}$$

and obtain using $\int L_{g_0} \phi_k \phi_l = c_k \delta_{k,l} + O(\frac{1}{\lambda^2})$

$$\begin{pmatrix} \langle \partial_t u, \phi_1 \rangle \\ \langle \partial_t u, \phi_2 \rangle \\ \langle \partial_t u, \phi_3 \rangle \end{pmatrix} = \frac{r}{k} \begin{pmatrix} \sigma_1 + O(\frac{1}{\lambda^2}) \\ \sigma_2 \\ \sigma_3 + \frac{\varepsilon}{2} c_3 \eta_V \eta_a \eta_{a,\lambda} \frac{\alpha^{-1}}{\lambda} \frac{\nabla K(a)}{|\nabla K(a)|} \end{pmatrix} + o_\varepsilon\left(\frac{1}{\lambda^3}\right) + O(|\partial J(u)|^2)$$

and hence by matrix inversion

$$\dot{\xi}_k = \Xi_k^l \langle \partial_t u, \phi_l \rangle = \frac{(1 + O(\frac{1}{\lambda^2} + |\partial J(u)|))r}{4n(n-1)\alpha k} \begin{pmatrix} \frac{\sigma_1}{c_1} + O(\frac{1}{\lambda^2}) \\ \frac{\sigma_2}{c_2} \\ \frac{\sigma_3}{c_3} + \frac{\varepsilon}{2} \eta_V \eta_a \eta_{a,\lambda} \frac{\alpha^{-1}}{\lambda} \frac{\nabla K(a)}{|\nabla K(a)|} \end{pmatrix}$$

up to some $o_\varepsilon(\frac{1}{\lambda^3}) + O(|\partial J(u)|^2)$. Recalling $\sigma_k = O(|\partial J(u)|)$ we may simplify to

$$\dot{\xi}_k = \frac{(1 + o_{\frac{1}{\lambda}}(1))r}{4n(n-1)\alpha k} \begin{pmatrix} \frac{\sigma_1}{c_1} \\ \frac{\sigma_2}{c_2} \\ \frac{\sigma_3}{c_3} + \frac{\varepsilon}{2}(1 + o_\varepsilon(1))\eta_V \eta_a \eta_{a,\lambda} \frac{\alpha^{-1}}{\lambda} \frac{\nabla K(a)}{|\nabla K(a)|} \end{pmatrix} + o_\varepsilon(\frac{1}{\lambda^3}) + O(|\partial J(u)|^2).$$

From Proposition 2.13 we thus obtain using (2.1), Proposition 2.12 and the principal lower bound on ∂J

Lemma 3.6. *Along (3.17) there holds on $V(1, \varepsilon)$*

$$\begin{aligned} (i) \quad -\frac{\dot{\lambda}}{\lambda} &= \frac{r}{k} \left(\frac{d_2}{c_2} \frac{H(a)}{\lambda^{n-2}} + \frac{e_2}{c_2} \frac{\Delta K(a)}{K(a)\lambda^2} \right) (1 + o_{\frac{1}{\lambda}}(1)) \\ (ii) \quad \lambda \dot{a} &= \frac{r}{k} \left[\frac{e_3}{c_3} \frac{\nabla K(a)}{K(a)\lambda} + \frac{e_4}{c_3} \frac{\nabla \Delta K(a)}{K(a)\lambda^3} \right] (1 + o_{\frac{1}{\lambda}}(1)) + \frac{\varepsilon r}{8n(n-1)\alpha^2 k} \eta_V \eta_a \eta_{a,\lambda} \frac{\nabla K(a)}{|\nabla K(a)|\lambda} (1 + o_\varepsilon(1)) \end{aligned}$$

up to some $o_\varepsilon(\frac{1}{\lambda^3}) + O(|\partial J(u)|^2)$ and for $d(a, x_0) \ll 1$ up to the same error

$$\begin{aligned} (i) \quad -\frac{\dot{\lambda}}{\lambda} &= \frac{r}{k} \left(\gamma_1 \frac{H(a)}{\lambda^3} - \gamma_2 \frac{|a|^2}{\lambda^2} \right) (1 + o_{\frac{1}{\lambda}+|a|}(1)) \\ (ii) \quad \lambda \dot{a} &= -\frac{r}{k} \left(\gamma_3 \frac{|a|^2 a}{\lambda} (1 + o_{\frac{1}{\lambda}+|a|}(1)) - \gamma_4 \varepsilon \eta_V \eta_a \eta_{a,\lambda} \frac{e_a}{\lambda} (1 + o_\varepsilon(1)) \right) \end{aligned}$$

with $e_a = \frac{a}{|a|}$, $\gamma_1, \dots, \gamma_4 > 0$.

Clearly the latter version for $d(a, x_0) \ll 1$ follows from (3.8). Comparing to Proposition 2.14 we observe, that by passing from (1.3) to (3.17) we have simply added the term

$$\frac{\varepsilon(1 + o_\varepsilon(1))r}{8n(n-1)\alpha^2 k} \eta_V \eta_a \eta_{a,\lambda} \frac{\nabla K(a)}{|\nabla K(a)|\lambda}$$

to the evolution equation of a , hence moving a faster towards x_0 .

3.4 Excluding diverging flow lines

As we had seen, cf. (3.18), the only possibility for a diverging flow line under (3.17) is

$$u = \alpha \varphi_{a,\lambda} + v \in V(1, \varepsilon) \text{ eventually for every } \varepsilon > 0$$

with corresponding modified shadow flow given by Lemma 3.6, from which

$$\begin{aligned} \left(\frac{\ln \frac{1}{\lambda}}{K(a)} \right)' &= \frac{\ln \lambda}{K^2(a)} \left\langle \frac{\nabla K(a)}{\lambda}, \lambda \dot{a} \right\rangle - \frac{1}{K(a)} \frac{\dot{\lambda}}{\lambda} \\ &\geq c \left[\ln \lambda \left(\frac{|\nabla K(a)|^2}{\lambda^2} + \varepsilon \eta_V \eta_a \eta_{a,\lambda} \frac{|\nabla K(a)|}{\lambda^2} \right) + \frac{H(a)}{\lambda^3} \right] \\ &\quad + \frac{r}{k} \frac{e_2}{c_2} \frac{\Delta K(a)}{K^2(a)\lambda^2} (1 + o_{\frac{1}{\lambda}}(1)) + O(|\partial J(u)|^2), \end{aligned}$$

as an easy computation shows. Hence $\lambda \rightarrow \infty$ necessitates

$$a \rightarrow x_0 = \{\nabla K = 0\} \cap \mathcal{M}_\delta$$

at least for a sequence in times, while on the other hand

$$\begin{aligned} \partial_t |\nabla K(a)|^2 &= 2 \nabla^2 K(a) \frac{\nabla K(a)}{\lambda} \lambda \dot{a} \\ &= \frac{\varepsilon(1 + o_\varepsilon(1))r}{4n(n-1)c_3 \alpha^2 k} \eta_V \eta_a \eta_{a,\lambda} \frac{\nabla^2 K(a) \nabla K(a) \nabla K(a)}{|\nabla K(a)|\lambda^2} \\ &\quad + O\left(\frac{1}{\lambda^6} + \frac{|\nabla K(a)|^2}{\lambda^2} + |\partial J(u)|^2\right) \leq O(|\partial J(u)|^2) \end{aligned}$$

due to the principal lower bound on ∂J , cf. Definition 2.15, and

$$\nabla^2 K \leq 0 \text{ close to } x_0 = \{K = \max K\},$$

i.e. on $\text{supp}(\eta_a)$. Therefore and by $\int_0^\infty |\partial J(u)|^2 < \infty$ we find, that necessarily

$$\lambda \longrightarrow \infty \implies a \longrightarrow x_0.$$

In particular we may assume $\eta_V = \eta_a = 1$ from now on. Then on

$$\{\eta_{a,\lambda} = 1\} = \{\lambda|a|^2 \geq 2\varepsilon\}$$

we find from Lemma 3.6 in its refined version for a close to x_0

$$\begin{aligned} (\lambda|a|^2)' &= \lambda|a|^2 \frac{\dot{\lambda}}{\lambda} + 2\langle a, \lambda \dot{a} \rangle \\ &\leq -\lambda|a|^2 \left(\tilde{\gamma}_1 \frac{H(a)}{\lambda^3} - \tilde{\gamma}_2 \frac{|a|^2}{\lambda^2} + O(|\partial J(u)|^2) \right) - (\tilde{\gamma}_4 \varepsilon \frac{|a|}{\lambda} + O(|a||\partial J(u)|^2)) \\ &\leq -\tilde{\gamma}_4 \varepsilon \frac{|a|}{\lambda} + C\lambda|a|^2 |\partial J(u)|^2 \leq C\lambda|a|^2 |\partial J(u)|^2. \end{aligned}$$

Consequently $\lambda|a|^2$ is bounded and considering $\psi = \max\{2\varepsilon, \lambda|a|^2\}$ there necessarily holds

$$\int_0^\infty \frac{|a|}{\lambda} \chi_{\{\lambda|a|^2 \geq 2\varepsilon\}} < \infty.$$

But then

$$\begin{aligned} \partial_t \ln \lambda &\leq -\tilde{\gamma}_1 \frac{H(a)}{\lambda^3} + \tilde{\gamma}_2 \frac{|a|^2}{\lambda^2} + O(|\partial J(u)|^2) \\ &\leq -\tilde{\gamma}_1 \frac{H(a)}{\lambda^3} + \tilde{\gamma}_2 \frac{2\varepsilon}{\lambda^3} \chi_{\{\lambda|a|^2 < 2\varepsilon\}} + \tilde{\gamma}_2 \frac{|a|^2}{\lambda^2} \chi_{\{\lambda|a|^2 \geq 2\varepsilon\}} + O(|\partial J(u)|^2) \end{aligned}$$

by Lemma 3.6 and, since $|a| \ll 1 \ll \lambda$, we obtain for $\varepsilon > 0$ sufficiently small

$$\partial_t \ln \lambda \leq \tilde{\gamma}_2 \frac{|a|}{\lambda} \chi_{\{\lambda|a|^2 \geq 2\varepsilon\}} + O(|\partial J(u)|^2)$$

and the right hand side is integrable in time. Hence $\lambda \longrightarrow \infty$ is impossible.

4 Appendix

We first recall some testings of the derivative $\partial J(u)$ with $\phi_{k,i}$ from [12], where we had worked with the representation of $u \in V(\omega, p, \varepsilon)$ based on minimising

$$\int Ku^{\frac{4}{n-2}} |u - u_{\tilde{\alpha}, \tilde{\beta}} - \tilde{\alpha}^i \varphi_{\tilde{\alpha}_i, \tilde{\lambda}_i}|^2$$

leading to the orthogonalities $v \in \langle u_{\alpha, \beta}, \phi_{k,i} \rangle^{\perp_{Ku^{\frac{4}{n-2}}}}$. By Lemma 2.9 we may carry over these testings to the representation induced by minimising

$$\|u - u_{\tilde{\alpha}, \tilde{\beta}} - \tilde{\alpha}^i \varphi_{\tilde{\alpha}_i, \tilde{\lambda}_i}\|_{L_{g_0}}^2.$$

Proposition 4.1. For $u \in V(p, \varepsilon)$ and

$$\sigma_{k,i} = - \int (L_{g_0} u - r \bar{K} u^{\frac{n+2}{n-2}}) \phi_{k,i}, \quad i = 1, \dots, p, \quad k = 1, 2, 3$$

we have with constants $b_1, \dots, e_4 > 0$

$$\begin{aligned}
(i) \quad \sigma_{1,i} &= \alpha_i \left[\frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} - 4n(n-1) \right] \int \varphi_i^{\frac{2n}{n-2}} \\
&\quad + \sum_{i \neq j=1}^p \alpha_j \left[\frac{r\alpha_j^{\frac{4}{n-2}} K_j}{k} - 4n(n-1) \right] b_1 \varepsilon_{i,j} \\
&\quad + d_1 \alpha_i \frac{H_i}{\lambda_i^{n-2}} + e_1 \frac{r\alpha_i^{\frac{n+2}{n-2}} \Delta K_i}{k \lambda_i^2} + b_1 \frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} \sum_{i \neq j=1}^p \alpha_j \varepsilon_{i,j} \\
(ii) \quad \sigma_{2,i} &= -\alpha_i \left[\frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} - 4n(n-1) \right] \int \varphi_i^{\frac{n+2}{n-2}} \lambda_i \partial_{\lambda_i} \varphi_i \\
&\quad - b_2 \sum_{i \neq j=1}^p \alpha_j \left[\frac{r\alpha_j^{\frac{4}{n-2}} K_j}{k} - 4n(n-1) \right] \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + d_2 \alpha_i \frac{H_i}{\lambda_i^{n-2}} \\
&\quad + e_2 \frac{r\alpha_i^{\frac{n+2}{n-2}} \Delta K_i}{k \lambda_i^2} - b_2 \frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} \sum_{i \neq j=1}^p \alpha_j \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \\
(iii) \quad \sigma_{3,i} &= \alpha_i \left[\frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} - 4n(n-1) \right] \int \varphi_i^{\frac{n+2}{n-2}} \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i \\
&\quad + b_3 \sum_{i \neq j=1}^p \alpha_j \left[\frac{r\alpha_j^{\frac{4}{n-2}} K_j}{k} - 4n(n-1) \right] \frac{1}{\lambda_i} \nabla_{a_i} \varepsilon_{i,j} \\
&\quad + \frac{r\alpha_i^{\frac{n+2}{n-2}}}{k} \left[e_3 \frac{\nabla K_i}{\lambda_i} + e_4 \frac{\nabla \Delta K_i}{\lambda_i^3} \right] + b_3 \frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} \sum_{i \neq j=1}^p \frac{\alpha_j}{\lambda_i} \nabla_{a_i} \varepsilon_{i,j}
\end{aligned}$$

up to some $o_\varepsilon(\lambda_i^{2-n} + \sum_{i \neq j=1}^p \varepsilon_{i,j}) + O(\sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \varepsilon_{r,s}^2 + \|v\|^2)$.

Proof. This follows from Proposition 4.2 from [15] in case

$$\langle v, \phi_{k,i} \rangle_{Ku^{\frac{4}{n-2}}} = 0.$$

In case $\langle v, \phi_{k,i} \rangle_{L_{g_0}} = 0$ we have from Lemma 2.9

$$\Pi_{\langle \phi_{k,i} \rangle}^{\top Ku^{\frac{4}{n-2}}} v = O\left(\left(\sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \varepsilon_{r,s} + \|v\|\right)\|v\|\right)$$

and may consequently reduce the latter case to the former one. $\square$

Likewise we may carry over Proposition 5.2 from [15] for the case $u \in V(\omega, p, \varepsilon)$ and $\omega > 0$. We next analyse the gradient orthogonally.

Proposition 4.2. *Let $u = \alpha^i \varphi_i + v \in V(p, \varepsilon)$ and*

$$h_1, h_2 \in H = H_u(p, \varepsilon).$$

Then

- (i) $|\partial J(\alpha^i \varphi_i)|_H| = O(\sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{|\Delta K_r|}{\lambda_r^2} + \lambda_r^{2-n} + \varepsilon_{r,s} + \|v\|^2 + |\partial J(u)|)$
- (ii) *and up to some $o_\varepsilon(\|h_1\| \|h_2\|)$ we have*

$$\frac{1}{2} \partial^2 J(\alpha^i \varphi_i) h_1 h_2 = k_{\alpha^i \varphi_i}^{\frac{2-n}{n}} \left[\int L_{g_0} h_1 h_2 - c_n n(n+2) \sum_i \int \varphi_i^{\frac{4}{n-2}} h_1 h_2 \right]$$

Proof. Cf. Proposition 4.4 from [15] in case $H_u(p, \varepsilon) = \langle \phi_{k,i} \rangle^{\perp_{Ku \frac{4}{n-2}}}$. In case

$$H_u(p, \varepsilon) = \langle \phi_{k,i} \rangle^{\perp_{L_{g_0}}}$$

statement (ii) still holds true by virtue of Lemma 2.9. Also note, that for

$$h \in \langle \phi_{k,i} \rangle^{\perp_{L_{g_0}}} \text{ with } \|h\| = 1$$

we have again by Lemma 2.9

$$\tilde{h} = \Pi_{\langle \phi_{k,i} \rangle}^{\perp_{Ku \frac{4}{n-2}}} h = O\left(\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n-2}} + \sum_{j \neq i} \varepsilon_{i,j} + \|v\|\right)$$

and hence, since $\partial J(u) = \partial J(\alpha^i \varphi_i) + O(\|v\|)$,

$$\begin{aligned} \partial J(\alpha^i \varphi_i) \tilde{h} &= (\partial J(u) + O(\|v\|)) \tilde{h} \\ &= O\left(\frac{|\nabla K_i|^2}{\lambda_i^2} + \frac{1}{\lambda_i^4} + \frac{1}{\lambda_i^{2(n-2)}} + \sum_{j \neq i} \varepsilon_{i,j}^2 + \|v\|^2 + |\partial J(u)|^2\right). \end{aligned}$$

Hence the Proposition follows. $\square$

Proposition 4.3. Let $u = u_{\alpha,\beta} + \alpha^i \varphi_i + v \in V(\omega, p, \varepsilon)$ and

$$h_1, h_2 \in H = H_u(\omega, p, \varepsilon).$$

Then

- (i) $|\partial J(u_{\alpha,\beta} + \alpha^i \varphi_i)|_H = o_\varepsilon(\|v\|) + O(\sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \lambda_r^{\frac{2-n}{2}} + \varepsilon_{r,s} + |\partial J(u)|)$
- (ii) and up to some $o_\varepsilon(\|h_1\| \|h_2\|)$ we have

$$\begin{aligned} &\frac{1}{2} \partial^2 J(u_{\alpha,\beta} + \alpha^i \varphi_i) h_1 h_2 \\ &= k_{u_{\alpha,\beta} + \alpha^i \varphi_i}^{\frac{2-n}{n}} \left[\int L_{g_0} h_1 h_2 - c_n n(n+2) \int \left(\frac{K \omega^{\frac{4}{n-2}}}{4n(n-1)} + \sum_i \varphi_i^{\frac{4}{n-2}} \right) h_1 h_2 \right]. \end{aligned}$$

Proof. Cf. Proposition 5.4 in [15] in case $H_u(p, \varepsilon) = \langle \phi_{k,i} \rangle^{\perp_{Ku \frac{4}{n-2}}}$. In case

$$H_u(\omega, p, \varepsilon) = \langle u_{\alpha,\beta}, \phi_{k,i} \rangle^{\perp_{L_{g_0}}}$$

the statement follow from the former case via Lemma 2.9 arguing as in the proof of Proposition 4.2. $\square$

Proof of Lemma 2.9. Let us just show the case

$$v_1 \in H_u(p, \varepsilon) = \langle \phi_{k,i} \rangle^{\perp_{Ku \frac{4}{n-2}}},$$

as the other cases follow analogously. We may write with suitable $\beta^{k,i} = O(1)$ and arbitrary $\alpha \in \mathbb{R}$

$$\begin{aligned} \Pi_{\langle \phi_{k,i} \rangle}^{\perp_{L_{g_0}}} v_1 &= \beta^{k,i} \langle v_1, \phi_{k,i} \rangle_{L_{g_0}} \phi_{k,i} = \beta^{k,i} \langle v_1, L_{g_0} \phi_{k,i} \rangle_{L_{g_0}^2} \phi_{k,i} \\ &= \beta^{k,i} \langle v_1, (L_{g_0} - \alpha Ku^{\frac{4}{n-2}}) \phi_{k,i} \rangle_{L_{g_0}^2} \phi_{k,i}. \end{aligned}$$

From Lemma 2.3 we then find via expansion and Hölder's inequality

$$\begin{aligned} \int Ku^{\frac{4}{n-2}} \phi_{k,i} v_1 &= K_i \int (\alpha^j \varphi_j + v)^{\frac{4}{n-2}} \phi_{k,i} v_1 + O\left(\left(\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^2}\right) \|v_1\|\right) \\ &= K_i \int (\alpha^j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v_1 + O\left(\left(\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^2} + \|v\|\right) \|v_1\|\right). \end{aligned}$$

Decomposing

$$M = \{\varphi_i \geq \sum_{i \neq j=1}^p \varphi_j\} + \{\varphi_i < \sum_{i \neq j=1}^p \varphi_j\}$$

and applying again Lemma 2.3 then show via expansion and Hölder inequality

$$\begin{aligned} \int K u^{\frac{4}{n-2}} \phi_{k,i} v_1 &= K_i \alpha_i^{\frac{4}{n-2}} \int \varphi_i^{\frac{4}{n-2}} \phi_{k,i} v_1 \\ &+ O\left(\left(\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{i,j} + \|v\|\right) \|v_1\|\right), \end{aligned}$$

where we made use of $n = 3, 4, 5$. Consequently

$$\begin{aligned} \Pi_{\langle \phi_{k,i} \rangle}^{\top_{L_{g_0}}} v_1 &= \beta^{k,i} \langle v_1, (L_{g_0} - \alpha K_i \alpha_i^{\frac{4}{n-2}} \varphi_i^{\frac{4}{n-2}}) \phi_{k,i} \rangle_{L_{g_0}^2} \phi_{k,i} \\ &+ O\left(\left(\sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \varepsilon_{r,s} + \|v\|\right) \|v_1\|\right). \end{aligned}$$

Note, that $L_{g_0} \phi_{k,i} = c_k \varphi_i^{\frac{4}{n-2}} \phi_{k,i}$ on $\mathbb{R}^n$ for suitable constants c_k , while

$$\|L_{g_0} \phi_{k,i} - c_k \varphi_i^{\frac{4}{n-2}} \phi_{k,i}\|_{L^{\frac{2n}{n+2}}} = O\left(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{\frac{n}{n-2}}}\right)$$

generally, cf. Lemma 2.1 in [12]. Hence choosing α suitably, we derive

$$\Pi_{\langle \phi_{k,i} \rangle}^{\top_{L_{g_0}}} v_1 = O\left(\left(\sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \frac{1}{\lambda_r^{\frac{n}{n-2}}} + \varepsilon_{r,s} + \|v\|\right) \|v_1\|\right),$$

what had to be shown. □

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