Research Article

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A Tauberian theorem for the generalized Nörlund summability method

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Abstract: Let (p_n) and (q_n) be any two non-negative real sequences, with $R_n := \sum_{k=0}^n p_k q_{n-k} \neq 0$ ($n \in \mathbb{N}$). Let $\sum_{k=0}^{\infty} a_k$ be a series of real or complex numbers with partial sums (s_n) , and set $t_n^{p,q} := \frac{1}{R_n} \sum_{k=0}^n p_k q_{n-k} s_k$ for $n \in \mathbb{N}$. In this paper, we present the necessary and sufficient conditions under which the existence of the limit $\lim_{n \rightarrow \infty} s_n = L$ follows from that of $\lim_{n \rightarrow \infty} t_n^{p,q} = L$. These conditions are one-sided or two-sided if (s_n) is a sequence of real or complex numbers, respectively.

Keywords: Generalized Nörlund summability, one-sided and two-sided Tauberian conditions

MSC 2010: 40G15, 41A36

1 Introduction

In what follows, we give the concept of the summability method, known as the generalized Nörlund summability method (N, p, q) (see [1]). Given two non-negative sequences (p_n) and (q_n) , the convolution $(p \star q)$ is defined by

$$R_n := (p \star q)_n = \sum_{k=0}^n p_k q_{n-k} = \sum_{k=0}^n p_{n-k} q_k.$$

Let $\sum_{k=0}^{\infty} a_k$ be a series of real or complex numbers with partial sums (s_n) . When $(p \star q)_n \neq 0$ for all $n \in \mathbb{N}$, the generalized Nörlund transform of the series $\sum_{k=0}^{\infty} a_k$ is the sequence $(t_n^{p,q})$ defined by

$$t_n^{p,q} = \frac{1}{(p \star q)_n} \sum_{k=0}^n p_k q_{n-k} s_k.$$

We say that the series $\sum_{k=0}^{\infty} a_k$ is generalized Nörlund summable to L , determined by the sequences (p_n) and (q_n) or, briefly, (N, p, q) summable to L if

$$\lim_{n \rightarrow \infty} t_n^{p,q} = L. \quad (1.1)$$

We define the following sets:

$$A(n, t) := \{q_{\lambda_n - k} - q_{n-k} : k = 0, 1, 2, \dots, n, \lambda > 1\},$$

$$B(n, t) := \{q_{k - \lambda_n} - q_{n-k} : k = 0, 1, 2, \dots, t_n, 0 < \lambda < 1\},$$

where $\lambda_n := [\lambda n]$ denotes the integral part of λn .

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Throughout the paper we assume that the sequence $q = (q_n)$ satisfies the following conditions:

$$\begin{aligned} p_n &\leq q_n, & n &\in \mathbb{N}, & q_n &\geq 1, & n &\in \mathbb{N}, \\ \sup_n A(n, \lambda) &< \infty, & \sup_n B(n, \lambda) &< \infty. \end{aligned}$$

If

$$\lim_{n \rightarrow \infty} s_n = L \quad (1.2)$$

implies (1.1), then the (N, p, q) method is called regular. The necessary and sufficient condition for the (N, p, q) method to be regular [6] is

$$p_{n-k}q_k = o(R_n) \quad (n \rightarrow \infty, k \in \mathbb{N})$$

and

$$\sum_{k=0}^n |p_{n-k}q_k| = O(R_n) \quad (n \rightarrow \infty).$$

Remark 1.1. The (N, p, q) summability method reduces to the Cesàro summability method when $p_n = 1$ and $q_n = 1$ for all $n \in \mathbb{N}$, to the weighted mean method $(\bar{N}, p)$ when $q_n = 1$ for all $n \in \mathbb{N}$, to the method (C, α, β) (see [1]) when $p_n = \binom{n+\beta}{\beta}$, $q_n = \binom{n+\alpha-1}{\alpha}$ and to the Nörlund method (N, q) when $p_n = 1$ for all $n \in \mathbb{N}$ (see [6]).

Notice that (1.1) may imply (1.2) under a certain condition, which is called a Tauberian condition. Any theorem which states that the convergence of a series follows from its (N, p, q) summability and some Tauberian condition is said to be a Tauberian theorem for the (N, p, q) summability method. The inclusion and Tauberian type theorems are proved in the papers [7, 8], and some inclusion, Tauberian and convexity type theorems for certain families of generalized Nörlund methods are obtained in [12]. Moreover, some theorems of Abelian and Tauberian type have been recently proved for the methods of power series, (C, α) and weighted mean in [2–5, 13].

In this paper, we present the necessary and sufficient conditions under which the existence of the limit $\lim_{n \rightarrow \infty} s_n = L$ follows from that of $\lim_{n \rightarrow \infty} t_n^{p,q} = L$. These conditions are one-sided or two-sided if (s_n) is a sequence of real or complex numbers, respectively. Furthermore, we have some classical Tauberian theorems for the Cesàro and weighted mean summability methods as a corollary.

2 Main results

In the following theorem we characterize the converse implication when the ordinary convergence follows from its (N, p, q) summability.

Theorem 2.1. Let (p_n) and (q_n) be any two non-negative real sequences such that

$$\liminf_{n \rightarrow \infty} \frac{R_{\lambda_n}}{R_n} > 1 \quad \text{for every } \lambda > 1, \quad (2.1)$$

where $\lambda_n := [\lambda n]$ denotes the integral part of λn for every $n \in \mathbb{N}$, and let $\sum_{k=0}^{\infty} a_k$ be a series of real numbers which is (N, p, q) summable to a finite number L . Then $\sum_{k=0}^{\infty} a_k$ is convergent to the same number L if and only if the following two conditions hold:

$$\lim_{\lambda \rightarrow 1^+} \liminf_{n \rightarrow \infty} \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} (s_k - s_n) \geq 0, \quad (2.2)$$

$$\lim_{\lambda \rightarrow 1^-} \liminf_{n \rightarrow \infty} \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^n p_k q_{n-k} (s_n - s_k) \geq 0. \quad (2.3)$$

Remark 2.2. Following Schmidt [11], we say that a real sequence (s_n) is slowly decreasing if

$$\lim_{\lambda \rightarrow 1^+} \liminf_{n \rightarrow \infty} \min_{n+1 \leq k \leq \lambda_n} (s_k - s_n) \geq 0 \quad (2.4)$$

or, equivalently,

$$\lim_{\lambda \rightarrow 1^-} \liminf_{n \rightarrow \infty} \min_{\lambda_n+1 \leq k \leq n} (s_n - s_k) \geq 0.$$

Conditions (2.2) and (2.3) are satisfied if (s_n) is slowly decreasing.

Remark 2.3. The classical one-sided Tauberian condition of Landau (see [9])

$$ka_k \geq -C \quad (k = 1, 2, \dots)$$

is sufficient for (2.4) to hold.

Remark 2.4. If we take $q_n = 1$ for all $n \in \mathbb{N}$ as a special case of Theorem 2.1, we obtain the Móricz and Rhoades conditions (see [10])

$$\lim_{\lambda \rightarrow 1^+} \liminf_{n \rightarrow \infty} \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k (s_k - s_n) \geq 0$$

and

$$\lim_{\lambda \rightarrow 1^-} \liminf_{n \rightarrow \infty} \frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^n p_k (s_n - s_k) \geq 0,$$

where $P_n := (p * 1)_n = \sum_{k=0}^n p_k$.

In the next result we will consider the case where $\sum_{k=0}^{\infty} a_k$ is a series of complex numbers.

Theorem 2.5. Let condition (2.1) be satisfied and let $\sum_{k=0}^{\infty} a_k$ be a series of complex numbers which is (N, p, q) summable to a finite number L . Then $\sum_{k=0}^{\infty} a_k$ is convergent to the same number L if and only if one of the following two conditions holds:

$$\lim_{\lambda \rightarrow 1^+} \limsup_{n \rightarrow \infty} \left| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k} (s_k - s_n) \right| = 0 \quad (2.5)$$

or

$$\lim_{\lambda \rightarrow 1^-} \limsup_{n \rightarrow \infty} \left| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^n p_k q_{n-k} (s_n - s_k) \right| = 0. \quad (2.6)$$

Remark 2.6. Following Schmidt [11], we say that a real sequence (s_n) is said to be slowly oscillating if

$$\lim_{\lambda \rightarrow 1^+} \liminf_{n \rightarrow \infty} \min_{n+1 \leq k \leq \lambda_n} |s_k - s_n| = 0$$

or, equivalently,

$$\lim_{\lambda \rightarrow 1^-} \liminf_{n \rightarrow \infty} \min_{\lambda_n+1 \leq k \leq n} |s_n - s_k| = 0.$$

Conditions (2.5) and (2.6) are satisfied if (s_n) is slowly oscillating.

Remark 2.7. The classical two-sided Tauberian condition

$$k|a_k| \leq C \quad (k = 1, 2, \dots)$$

is sufficient for (2.5) and (2.6) to hold.

Remark 2.8. If we take $q_n = 1$ for all $n \in \mathbb{N}$ as a special case of Theorem 2.1, we obtain the Móricz and Rhoades conditions (see [10])

$$\lim_{\lambda \rightarrow 1^+} \limsup_{n \rightarrow \infty} \left| \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k (s_k - s_n) \right| = 0$$

and

$$\lim_{\lambda \rightarrow 1^-} \limsup_{n \rightarrow \infty} \left| \frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^n p_k (s_n - s_k) \right| = 0,$$

where $P_n := (p * 1)_n = \sum_{k=0}^n p_k$.

3 Auxiliary results

In what follows, we list some auxiliary lemmas which are needful in the sequel.

Lemma 3.1. *The condition given by relation (2.1) is equivalent to the condition*

$$\liminf_{n \rightarrow \infty} \frac{R_n}{R_{\lambda_n}} > 1, \quad 0 < \lambda < 1. \quad (3.1)$$

Proof. Suppose that relation (2.1) is valid, $0 < \lambda < 1$ and $m = \lambda_n = [\lambda n]$, $n \in \mathbb{N}$. Then it follows that

$$\frac{1}{\lambda} > 1 \implies \frac{m}{\lambda} = \frac{[\lambda n]}{\lambda} \leq n.$$

From the above relation and the definition of the positive real sequences (p_n) and (q_n) , we obtain

$$\frac{R_n}{R_{\lambda_n}} \geq \frac{R_{[\frac{m}{\lambda}]}}{R_{\lambda_n}} \implies \liminf_{n \rightarrow \infty} \frac{R_n}{R_{\lambda_n}} \geq \liminf_{n \rightarrow \infty} \frac{R_{[\frac{m}{\lambda}]}}{R_{\lambda_n}} > 1.$$

Conversely, suppose that relation (3.1) is valid. Let $\lambda > 1$ be a given number and let λ_1 be chosen so that $1 < \lambda_1 < \lambda$. Set $m = \lambda_n = [\lambda n]$. From $0 < \frac{1}{\lambda} < \frac{1}{\lambda_1} < 1$, it follows that

$$n \leq \frac{\lambda n - 1}{\lambda_1} < \frac{[\lambda n]}{\lambda_1} = \frac{m}{\lambda_1},$$

provided that $\lambda_1 \leq \lambda - \frac{1}{n}$, which is the case when n is large enough. Under these conditions, we have

$$\frac{R_{\lambda_n}}{R_n} \geq \frac{R_{\lambda_n}}{R_{[\frac{m}{\lambda_1}]}} \implies \liminf_{n \rightarrow \infty} \frac{R_{\lambda_n}}{R_n} \geq \liminf_{n \rightarrow \infty} \frac{R_{\lambda_n}}{R_{[\frac{m}{\lambda_1}]}} > 1$$

Lemma 3.2. *Let condition (2.1) be satisfied and let $\sum_{k=0}^{\infty} a_k$ be a series of complex numbers which is generalized Nörlund summable to L . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k} s_k = L \quad \text{for } \lambda > 1 \quad (3.2)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^n p_k q_{n-k} s_k = L \quad \text{for } 0 < \lambda < 1. \quad (3.3)$$

Proof. Consider the case where $\lambda > 1$. Then we obtain

$$\begin{aligned} \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k} (s_k - L) &= \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} \frac{1}{R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n-k} (s_k - L) - \frac{R_n}{R_{\lambda_n} - R_n} \frac{1}{R_n} \sum_{k=0}^n p_k q_{\lambda_n-k} (s_k - L) \\ &= \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} \frac{1}{R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n-k} (s_k - L) - \frac{R_n}{R_{\lambda_n} - R_n} \frac{1}{R_n} \sum_{k=0}^n p_k q_{n-k} (s_k - L) \\ &\quad - \frac{R_n}{R_{\lambda_n} - R_n} \frac{1}{R_n} \sum_{k=0}^n p_k (q_{\lambda_n-k} - q_{n-k}) (s_k - L), \end{aligned} \quad (3.4)$$

provided that $R_{\lambda_n} > R_n$.

Since, by (2.1),

$$\limsup_{n \rightarrow \infty} \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} = 1 + \left\{ -1 + \liminf_{n \rightarrow \infty} \frac{R_{\lambda_n}}{R_n} \right\}^{-1} < \infty, \quad (3.5)$$

(3.2) follows from (3.4), (3.5) and the definition of the sequence (q_n) .

Consider the case where $0 < \lambda < 1$. Then we obtain

$$\begin{aligned} \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_{n+1}}^n p_k q_{n-k}(s_k - L) &= \frac{R_n}{R_n - R_{\lambda_n}} \frac{1}{R_n} \sum_{k=0}^n p_k q_{n-k}(s_k - L) - \frac{R_{\lambda_n}}{R_n - R_{\lambda_n}} \frac{1}{R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{n-k}(s_k - L) \\ &= \frac{R_n}{R_n - R_{\lambda_n}} \frac{1}{R_n} \sum_{k=0}^n p_k q_{n-k}(s_k - L) - \frac{R_{\lambda_n}}{R_n - R_{\lambda_n}} \frac{1}{R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{n-k}(s_k - L) \\ &\quad - \frac{R_{\lambda_n}}{R_n - R_{\lambda_n}} \frac{1}{R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k (q_{n-k} - q_{\lambda_n-k})(s_k - L), \end{aligned} \quad (3.6)$$

provided that $R_n > R_{\lambda_n}$.

Since, by (2.1),

$$\limsup_{n \rightarrow \infty} \frac{R_{\lambda_n}}{R_n - R_{\lambda_n}} = \left\{ -1 + \liminf_{n \rightarrow \infty} \frac{R_n}{R_{\lambda_n}} \right\}^{-1} < \infty, \quad (3.7)$$

(3.3) follows from (3.6), (3.7) and the definition of the sequence (q_n) . $\square$

4 Proofs of the theorems

Proof of Theorem 2.1. Necessity. Suppose that $\lim_{n \rightarrow \infty} s_n = L$, and (1.1) holds. From Lemma 3.2, we have

$$\lim_{n \rightarrow \infty} \frac{1}{R_{\lambda_n} - R_n} \sum_{k=\lambda_{n+1}}^{\lambda_n} p_k q_{\lambda_n-k}(s_k - s_n) = \lim_{n \rightarrow \infty} \left\{ \left(\frac{1}{R_{\lambda_n} - R_n} \sum_{k=\lambda_{n+1}}^{\lambda_n} p_k q_{\lambda_n-k} s_k \right) - s_n \right\} = 0$$

for every $\lambda > 1$. In the case where $0 < \lambda < 1$, we find that

$$\lim_{n \rightarrow \infty} \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_{n+1}}^n p_k q_{n-k}(s_n - s_k) = \lim_{n \rightarrow \infty} \left\{ \left(\frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_{n+1}}^n p_k q_{n-k} s_k \right) - s_n \right\} = 0.$$

Sufficiency. Assume that conditions (2.2) and (2.3) are satisfied. In what follows, we will prove that $\lim_{n \rightarrow \infty} s_n = L$. Given any $\epsilon > 0$, by relation (2.2), we can choose $\lambda_1 > 0$ so that

$$\liminf_{n \rightarrow \infty} \frac{1}{R_{\lambda_{n_1}} - R_n} \sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1}-k}(s_k - s_n) \geq -\epsilon, \quad (4.1)$$

where $\lambda_{n_1} = [\lambda_1 n]$. By the assumed summability (N, p, q) of (x_n) and Lemma 3.2, we have

$$\liminf_{n \rightarrow \infty} \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k} s_k = L \quad (4.2)$$

for any $\lambda > 1$. Taking (4.1) and (4.2) into account, we obtain

$$\limsup_{n \rightarrow \infty} s_n \leq L + \epsilon. \quad (4.3)$$

On the other hand, if $0 < \lambda < 1$, then, for every $\epsilon > 0$, we can choose $0 < \lambda_2 < 1$ so that

$$\liminf_{n \rightarrow \infty} \frac{1}{R_n - R_{\lambda_{n_2}}} \sum_{k=\lambda_{n_2}+1}^n p_k q_{n-k}(s_n - s_k) \geq -\epsilon, \quad (4.4)$$

where $\lambda_{n_2} = [\lambda_2 n]$. By the assumed summability (N, p, q) of (x_n) and Lemma 3.2, we have

$$\liminf_{n \rightarrow \infty} \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_{n_2}+1}^n p_k q_{n-k} s_k = L \quad (4.5)$$

for any $0 < \lambda < 1$.

Taking (4.4) and (4.5) into account, we obtain

$$\liminf_{n \rightarrow \infty} s_n \geq L - \epsilon. \quad (4.6)$$

Since $\epsilon > 0$ is arbitrary, combining relations (4.3) and (4.6), we obtain

$$\lim_{n \rightarrow \infty} s_n = L. \quad \square$$

[Correction added on 12 September 2018 after online publication: Mistakes within the proof of Theorem 2.1, part “Sufficiency”, have been corrected.]

Proof of Theorem 2.5. Necessity. If both (1.1) and (1.2) hold, then Lemma 3.2 yields (2.5) for every $\lambda > 1$ and (2.6) for every $0 < \lambda < 1$.

Sufficiency. Suppose that (1.1), (2.1) and one of conditions (2.5) or (2.6) is satisfied. For any given $\epsilon > 0$, there exists some $\lambda_1 > 1$ such that

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{R_{\lambda_{n_1}} - R_n} \sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1}-k} (s_k - s_n) \right| \leq \epsilon,$$

where $\lambda_{n_1} = [\lambda_1 n]$. Taking into account the fact that $\sum_{k=0}^{\infty} a_k$ is (N, p, q) summable to L , we get the following estimation:

$$\limsup_{n \rightarrow \infty} |L - s_n| \leq \limsup_{n \rightarrow \infty} \left| L - \frac{1}{R_{\lambda_{n_1}} - R_n} \sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1}-k} s_k \right| + \limsup_{n \rightarrow \infty} \left| \frac{1}{R_{\lambda_{n_1}} - R_n} \sum_{k=n+1}^{\lambda_{n_1}} p_k q_{\lambda_{n_1}-k} (s_k - s_n) \right| \leq \epsilon.$$

For any given $\epsilon > 0$, there exists some $0 < \lambda_2 < 1$ such that

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{R_n - R_{\lambda_{n_2}}} \sum_{k=\lambda_{n_2}+1}^n p_k q_{n-k} (s_k - s_n) \right| \leq \epsilon,$$

where $\lambda_{n_2} = [\lambda_2 n]$. Taking into account the fact that $\sum_{k=0}^{\infty} a_k$ is (N, p, q) summable to L , we get the following estimation:

$$\limsup_{n \rightarrow \infty} |L - s_n| \leq \limsup_n \left| L - \frac{1}{R_n - R_{\lambda_{n_2}}} \sum_{k=\lambda_{n_2}+1}^n p_k q_{n-k} s_k \right| + \limsup_{n \rightarrow \infty} \left| \frac{1}{R_n - R_{\lambda_{n_2}}} \sum_{k=\lambda_{n_2}+1}^n p_k q_{n-k} (s_k - s_n) \right| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, in either case, we get $\lim_{n \rightarrow \infty} s_n = L$. □

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