Research Article

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An improvement of the constant in Videnskiĭ's inequality for Bernstein polynomials

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Abstract: In this paper, we deal with improvements on the constant $M_n(\gamma)$ in the so-called *Videnskiĭ inequality*

$$\left| (B_n f)(x) - f(x) - \frac{x(1-x)}{2n} f''(x) \right| \leq M_n(\gamma) \frac{x(1-x)}{n} \omega\left(f''; \sqrt{\frac{\gamma}{n}}\right)$$

for a fixed constant $\gamma \geq 1$ and $x \in [0, 1]$, where $B_n f$ is the Bernstein polynomial, $f \in C^2[0, 1]$ and ω is the first order modulus of continuity. Let $M(\gamma) = \sup_{n \in \mathbb{N}} M_n(\gamma)$. We prove the Videnskiĭ inequality for arbitrary $\gamma \geq 1$. In particular, we improve the constant $M(2) = 0.9$ (Gonska and Raşa [8], 2008) to $M(2) = 0.6875$. Finally, we consider $M_n(1)$ for small values of n .

Keywords: Approximation by positive operators, rate of convergence, degree of approximation

MSC 2010: 41A36, 41A25

1 Introduction

The classical Bernstein polynomials, defined for $f \in C[0, 1]$ by

$$(B_n f)(x) = \sum_{v=0}^n p_{n,v}(x) f\left(\frac{v}{n}\right) \quad (x \in [0, 1])$$

with

$$p_{n,v}(x) = \binom{n}{v} x^v (1-x)^{n-v} \quad (v = 0, \dots, n)$$

converge uniformly to f on $[0, 1]$. Popoviciu [11] gave the rate of convergence

$$\|B_n f - f\|_\infty \leq c_0 \cdot \omega(f; n^{-1/2}),$$

estimated by the ordinary modulus of continuity with the constant $c_0 = \frac{3}{2}$. An elementary proof with constant $c_0 = \frac{5}{4}$ can be found in Lorentz's book [10]. Sikkema [12] found the best (smallest) value of the constant c_0 . The asymptotic behaviour was described by Voronovskaja [17] in 1932 (see, e.g., [10, p. 22]). She showed that, for each bounded function f on $[0, 1]$ whose second derivative $f''(x)$ exists at a certain point $x \in [0, 1]$, one has

$$\lim_{n \rightarrow \infty} n[(B_n f)(x) - f(x)] = \frac{1}{2} x(1-x) f''(x),$$

the convergence being uniform, for $f \in C^2[0, 1]$ (see, e.g., [3, Chapter 10, Theorem 3.1] and cf. [5]). In that

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same year Bernstein [2] published a generalization. Gonska [6, Theorem 5.1] gave the following quantitative form:

$$\left| n[(B_n f)(x) - f(x)] - \frac{1}{2}x(1-x)f''(x) \right| \leq \frac{x(1-x)}{2} \tilde{\omega}\left(f''; \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}\right),$$

where $\tilde{\omega}$ is the least concave majorant of ω satisfying

$$\omega(f; \delta) \leq \tilde{\omega}(f; \delta) \leq 2\omega(f; \delta) \quad \text{for } \delta > 0.$$

In 1985, Videnskii [16] proved that, for all $x \in [0, 1]$,

$$\left| (B_n f)(x) - f(x) - \frac{x(1-x)}{2n} f''(x) \right| \leq M \frac{x(1-x)}{n} \omega\left(f''; \sqrt{\frac{2}{n}}\right) \quad (1.1)$$

with the constant $M = 1$ being independent of x . By application of a general theorem due to Sikkema and van der Meer [13], Gonska and Raşa [8, p. 94] recently proved that the inequality is valid with constant $M = 0.9$.

The purpose of the present note is to show by direct methods that the constant M can be diminished to the value

$$M = \frac{11}{16} = 0.6875. \quad (1.2)$$

In 2005, Floater [5] generalized Voronovskaja's theorem [17] to differentiated Bernstein polynomials. Quantitative results of Floater's theorem have been given by Gonska and Raşa [9] and Gonska, Heilmann and Raşa [7] in 2009 and 2010, respectively. Further inequalities for the left-hand side of (1.1) with the aid of Ditzian–Totik moduli of first order and average moduli of smoothness were given by Tachev [15] and Finta [4].

2 Main result

We define the sequence of linear operators

$$(L_n f)(x) := n \frac{(B_n f)(x) - f(x)}{x(1-x)} - \frac{1}{2} f''(x) \quad (x \in (0, 1)), \quad (2.1)$$

which allows to write (1.1) as

$$|(L_n f)(x)| \leq M \cdot \omega\left(f''; \sqrt{\frac{2}{n}}\right).$$

Our main result is the following theorem which states a slightly more general inequality.

Theorem 2.1. *Let $\gamma \geq 1$ and $f \in C^2[0, 1]$. For all $n \geq 1$ and all $x \in [0, 1]$,*

$$|(L_n f)(x)| \leq M(\gamma) \cdot \omega\left(f''; \sqrt{\frac{\gamma}{n}}\right),$$

where

$$M(\gamma) = \frac{1}{2} + \frac{3}{8\gamma}.$$

In the particular case $n = 1$, the constant $M(\gamma)$ can be taken equal to $\frac{1}{2}$.

Remark 2.2. Notice that for $\gamma = 2$ the theorem provides inequality (1.1) with the constant M announced in (1.2).

Remark 2.3. In the convenient case $\gamma = 1$, we have the constant $M(1) = \frac{7}{8} = 0.875$.

Proof of Theorem 2.1. Let $\gamma \geq 1$ and $f \in C^2[0, 1]$. By taking into account the endpoint interpolation property of the Bernstein polynomials, the assertion is obvious for $x \in \{0, 1\}$. Now let $x \in (0, 1)$. By Taylor's formula we have, for $0 \leq v \leq n$,

$$f\left(\frac{v}{n}\right) = f(x) + f'(x)\left(\frac{v}{n} - x\right) + \frac{1}{2}f''(x)\left(\frac{v}{n} - x\right)^2 + \frac{1}{2}(f''(\xi_{v/n,x}) - f''(x))\left(\frac{v}{n} - x\right)^2$$

with certain intermediate points $\xi_{v/n,x}$ between $\frac{v}{n}$ and x . Since Bernstein polynomials preserve linear functions and $B_n e_2 = e_2 + \frac{e_1}{n}(1 - e_1)$, where $e_k(x) = x^k$, we conclude that

$$(B_n f)(x) = f(x) + \frac{1}{2} f''(x) \frac{x(1-x)}{n} + \frac{1}{2} \sum_{v=0}^n p_{n,v}(x) \left(\frac{v}{n} - x \right)^2 (f''(\xi_{v/n,x}) - f''(x)).$$

This implies that

$$\frac{x(1-x)}{n} |(L_n f)(x)| \leq \frac{1}{2} \sum_{v=0}^n p_{n,v}(x) \left(\frac{v}{n} - x \right)^2 \omega(f''; |\xi_{v/n,x} - x|). \quad (2.2)$$

The case $n = 1$ will be considered separately. In this case the right-hand side of the latter inequality becomes

$$\frac{1}{2} x^2 (1-x) \omega(f''; |\xi_{0,x} - x|) + \frac{1}{2} x (1-x)^2 \omega(f''; |\xi_{1,x} - x|)$$

and we obtain

$$x(1-x) |(L_1 f)(x)| \leq \frac{1}{2} x(1-x) \omega(f''; 1).$$

Taking advantage of the well-known obvious inequality

$$\omega(g; \lambda h) \leq [\lambda] \omega(g; h), \quad (2.3)$$

which is valid for any function g and $\lambda, h > 0$, we obtain, for $\delta > 0$,

$$x(1-x) |(L_1 f)(x)| \leq \frac{1}{2} x(1-x) [\delta^{-1}] \omega(f''; \delta).$$

In particular, for $\delta = \sqrt{y} \geq 1$, it follows that

$$x(1-x) |(L_1 f)(x)| \leq \frac{1}{2} x(1-x) \omega(f''; \sqrt{y}),$$

which proves the assertion for $n = 1$.

We continue with the case $n \geq 2$. A consequence of (2.3) is

$$\omega(f''; |\xi_{v/n,x} - x|) \leq \omega\left(f''; \left| \frac{v}{n} - x \right| \right) \leq [\delta^{-1} \left| \frac{v}{n} - x \right|] \omega(f''; \delta) \leq \left(1 + \delta^{-1} \left| \frac{v}{n} - x \right| \right) \omega(f''; \delta).$$

Hence, by inequality (2.2),

$$\frac{2x(1-x)}{n} |(L_n f)(x)| \leq \sum_{\substack{0 \leq v \leq n, \\ |\frac{v}{n} - x| < \delta}} p_{n,v}(x) \left(\frac{v}{n} - x \right)^2 \omega(f''; \delta) + \sum_{\substack{0 \leq v \leq n, \\ |\frac{v}{n} - x| \geq \delta}} p_{n,v}(x) \left(\frac{v}{n} - x \right)^2 \left(1 + \delta^{-1} \left| \frac{v}{n} - x \right| \right) \omega(f''; \delta).$$

If $|\frac{v}{n} - x| \geq \delta$, then $\delta^{-1} |\frac{v}{n} - x| \leq \delta^{-2} |\frac{v}{n} - x|^2$, so we conclude that

$$\begin{aligned} \frac{2x(1-x)}{n} |(L_n f)(x)| &\leq \sum_{v=0}^n p_{n,v}(x) \left(\frac{v}{n} - x \right)^2 \omega(f''; \delta) + \delta^{-2} \sum_{\substack{0 \leq v \leq n, \\ |\frac{v}{n} - x| \geq \delta}} p_{n,v}(x) \left(\frac{v}{n} - x \right)^4 \omega(f''; \delta) \\ &\leq [(B_n \psi_x^2)(x) + \delta^{-2} (B_n \psi_x^4)(x)] \omega(f''; \delta), \end{aligned}$$

where the function ψ_x is defined by $\psi_x(t) = t - x$. Using $(B_n \psi_x^2)(x) = \frac{x}{n}(1-x)$ and

$$(B_n \psi_x^4)(x) = \frac{x(1-x)}{n^3} (1 - 6x(1-x) + 3nx(1-x))$$

(see, e.g., [10, p. 14]), we have

$$|(L_n f)(x)| \leq \left(\frac{1}{2} + \frac{1 - 6x(1-x) + 3nx(1-x)}{2\delta^2 n^2} \right) \omega(f''; \delta).$$

For $n \geq 2$, the nominator $1 + (3n - 6)x(1-x)$ can be estimated by $\frac{1}{4}(3n - 2) < \frac{3}{4}n$, since $3n - 2$ is nonnegative and $0 \leq x(1-x) \leq \frac{1}{4}$ on $[0, 1]$. Hence, we obtain, for each $n \geq 2$,

$$|(L_n f)(x)| \leq \left(\frac{1}{2} + \frac{3}{8\delta^2 n} \right) \omega(f''; \delta).$$

Inserting the special value $\delta = \sqrt{\frac{y}{n}}$ proves Theorem 2.1. □

3 Best constants in the case $\delta = \sqrt{\frac{1}{n}}$ for some small n

Let us now consider the special choice $\delta = \sqrt{\frac{1}{n}}$ in the inequality

$$\sup_{x \in (0,1)} \frac{|(L_n f)(x)|}{\omega(f''; \delta)} \leq M_n \quad (n \in \mathbb{N}).$$

We are interested in the best (smallest) value $M_n^{[\text{opt}]}$ of the constant M_n , i.e.,

$$M_n^{[\text{opt}]} := \sup_{f \in X_n} \sup_{x \in (0,1)} |(L_n f)(x)|,$$

where X_n consists of all functions $f \in C^2[0, 1]$ with $\omega(f''; \sqrt{\frac{1}{n}}) = 1$. By the interpolation property of B_n we have to consider only the case $x \in (0, 1)$.

3.1 The case $n = 1$

We prove that $M_1^{[\text{opt}]} = \frac{1}{2}$. For $n = 1$, by Theorem 2.1 we know that $M_1^{[\text{opt}]} \leq \frac{1}{2}$. Put $b_k = 1 - \frac{1}{k}$ ($k \in \mathbb{N}$) and define a sequence of functions $f_k(x) = \frac{k}{6}(x - b_k)_+^3$, where $u_+ = u$ if $u \geq 0$ and $u_+ = 0$ if $u < 0$. Obviously, $f_k \in C^2[0, 1]$ and $f_k''(x) = k(x - b_k)_+$ and we see that $\omega(f_k''; 1) = f_k''(1) - f_k''(b_k) = 1$. We show that

$$\lim_{k \rightarrow \infty} \sup_{x \in (0,1)} |(L_1 f_k)(x)| = \frac{1}{2}.$$

Since

$$x(1-x)(L_1 f)(x) = (1-x)f(0) + xf(1) - f(x) - \frac{1}{2}x(1-x)f''(x)$$

and $f_k(x) = \frac{k}{6}(x - b_k)_+^3$, we have

$$(L_1 f_k)(x) = \frac{f_k(1)}{1-x} = \frac{1}{6(1-x)k^2} \leq \frac{1}{6k} \quad (0 < x < b_k).$$

Now let $x \in (b_k, 1]$. We obtain

$$(L_1 f_k)(x) = \frac{f(1) - f(x)}{x(1-x)} - \frac{f_k(1)}{x} - \frac{1}{2}f_k''(x).$$

By the mean value theorem we conclude that

$$\left| \frac{f(1) - f(x)}{x(1-x)} \right| = x^{-1} |f'_k(\xi)| \leq b_k^{-1} \frac{k}{2} (1 - b_k)^2 = \frac{1}{2(k-1)}.$$

Furthermore,

$$0 \leq \frac{f_k(1)}{x} \leq \frac{1}{6kb_k} = \frac{1}{6(k-1)}$$

and

$$0 \leq \frac{1}{2}f_k''(x) = \frac{1}{2}k(x - b_k) \leq \frac{1}{2}.$$

Hence, $\lim_{k \rightarrow \infty} (L_1 f_k)(1) = -\frac{1}{2}$ and the claim follows by continuity.

3.2 The cases $n \geq 2$

For the considerations we use the following auxiliary results.

Lemma 3.1 (Stancu [14, equation (11.5)]). *For each function $f : [0, 1] \rightarrow \mathbb{R}$ there holds*

$$(B_n f)(x) - f(x) = \frac{x(1-x)}{n} \sum_{v=0}^{n-1} p_{n-1,v}(x) \left[x, \frac{v}{n}, \frac{v+1}{n}; f \right] \quad (3.1)$$

provided that $x \in [0, 1] \setminus \{\frac{v}{n} \mid v = 0, \dots, n\}$.

A new interesting proof of Stancu's formula is given in [5, p. 132]. In case when x coincides with $\frac{\nu}{n}$ or $\frac{1}{n}(\nu + 1)$, an extension of Stancu's formula to the whole interval $[0, 1]$ can be obtained using a limit process if f possesses a derivative on $[0, 1]$. For instance, the divided difference on multiple knots $[\frac{\nu}{n}, \frac{\nu}{n}, \frac{\nu+1}{n}; f]$ is equal to $n^2(f(\frac{\nu+1}{n}) - f(\frac{\nu}{n})) - nf'(\frac{\nu}{n})$.

Lemma 3.2 (see, e.g., [10, Chapter 4, Section 7, equation (7.16), p. 123]). *Let $m \in \mathbb{N}$ and $f \in C^m[a, b]$. If all points $x_0, \dots, x_m \in [a, b]$ are distinct, there holds*

$$[x_0, \dots, x_m; f] = \frac{1}{(m-1)!} \int_a^b f^{(m)}(t) [x_0, \dots, x_m; (\cdot - t)_+^{m-1}] dt.$$

Because of $\sum_{\nu=0}^{n-1} p_{n-1,\nu}(x) = 1$, Stancu's formula (3.1) implies

$$(B_n f)(x) - f(x) - \frac{x(1-x)}{2n} f''(x) = \frac{x(1-x)}{n} \sum_{\nu=0}^{n-1} p_{n-1,\nu}(x) \left(\left[x, \frac{\nu}{n}, \frac{\nu+1}{n}; f \right] - \frac{1}{2} f''(x) \right).$$

By Lemma 3.2,

$$\left[x, \frac{\nu}{n}, \frac{\nu+1}{n}; f \right] = \int_0^1 f''(t) \left[x, \frac{\nu}{n}, \frac{\nu+1}{n}; (\cdot - t)_+ \right] dt \quad (x \in (0, 1)).$$

In the particular case $f(x) = x^2$, we see that

$$1 = \left[x, \frac{\nu}{n}, \frac{\nu+1}{n}; f \right] = 2 \int_0^1 \left[x, \frac{\nu}{n}, \frac{\nu+1}{n}; (\cdot - t)_+ \right] dt \quad (3.2)$$

and therefore, by (2.1),

$$(L_n f)(x) = \sum_{\nu=0}^{n-1} p_{n-1,\nu}(x) \int_0^1 (f''(t) - f''(x)) \left[x, \frac{\nu}{n}, \frac{\nu+1}{n}; (\cdot - t)_+ \right] dt. \quad (3.3)$$

Note that $[x_0, x_1, x_2; (\cdot - t)_+] \geq 0$ for all pairwise different points x_0, x_1, x_2 with $0 \leq x_0 < x_1 < x_2 \leq 1$, and that $[x_0, x_1, x_2; (\cdot - t)_+] = 0$ if $t < x_0$ or $t > x_2$.

Taking advantage of inequality (2.3), we obtain, for $\delta > 0$,

$$|f''(t) - f''(x)| \leq \omega(f''; |t - x|) \leq \lceil \delta^{-1} |t - x| \rceil \cdot \omega(f''; \delta).$$

By (3.2) and (3.3), we conclude that, for $\delta > 0$,

$$|(L_n f)(x)| \leq c_n(\delta, x) \cdot \omega(f''; \delta),$$

where

$$c_n(\delta, x) = \sum_{\nu=0}^{n-1} p_{n-1,\nu}(x) \int_0^1 \lceil \delta^{-1} |t - x| \rceil \left[x, \frac{\nu}{n}, \frac{\nu+1}{n}; (\cdot - t)_+ \right] dt. \quad (3.4)$$

We define

$$C_n := \sup_{x \in (0,1)} c_n\left(\frac{1}{\sqrt{n}}, x\right).$$

A list giving the exact values of C_n for small n can be found in [1, p. 31]:

$$\begin{aligned} C_1 &= \frac{1}{2} = 0.500000, \\ C_2 &= \frac{1}{8} \left(\sqrt{97 + 140\sqrt{2}} - 10\sqrt{2} + 1 \right) \approx 0.504142, \\ C_3 &= \frac{1}{162} \left(4\sqrt{5340 + 6678\sqrt{3}} - 196\sqrt{3} - 99 \right) \approx 0.503828, \\ C_4 &= \frac{1}{32} (78\sqrt{3} - 119) \approx 0.503124. \end{aligned}$$

We conjecture that $M_n^{[\text{opt}]} = C_n$ ($n \in \mathbb{N}$). Since the proofs are rather long and tedious, we omit the calculations for $n = 3$ and $n = 4$. In order to illustrate the method, we restrict ourselves in proving that

$$C_2 = \frac{1}{8} \left(\sqrt{97 + 140\sqrt{2}} - 10\sqrt{2} + 1 \right)$$

by estimating $c_2(\frac{1}{\sqrt{2}}, x)$.

3.3 The case $n = 2$

Put $\delta = \delta_2 = \frac{1}{\sqrt{2}} \approx 0.707$. By (3.4), we have $c_2(\delta_2, x) = (1-x)I_1 + xI_2$ with

$$I_1 = \int_0^1 [\delta^{-1}|t-x|] \left[x, 0, \frac{1}{2}; (\cdot - t)_+ \right] dt,$$

$$I_2 = \int_0^1 [\delta^{-1}|t-x|] \left[x, \frac{1}{2}, 1; (\cdot - t)_+ \right] dt.$$

If $1 - \delta_2 \leq x \leq \delta_2$, we have $|t-x| \leq \delta_2$ for all $t \in [0, 1]$, which implies $I_1 = I_2 = \frac{1}{2}$ such that $c_2(\delta_2, x) = \frac{1}{2}$. Now we consider the case $0 < x < 1 - \delta_2 \approx 0.293$. Since $[x, 0, \frac{1}{2}; (\cdot - t)_+] = 0$ for $t \notin [0, \frac{1}{2}]$,

$$I_1 = \int_0^{\frac{1}{2}} \left[x, 0, \frac{1}{2}; (\cdot - t)_+ \right] dt = \int_0^{\frac{1}{2}} \left[x, 0, \frac{1}{2}; (\cdot - t)_+ \right] dt = \frac{1}{2}.$$

Since $[x, \frac{1}{2}, 1; (\cdot - t)_+] = 0$ for $t \notin [x, 1]$,

$$I_2 = \int_x^{x+\delta_2} \left[x, \frac{1}{2}, 1; (\cdot - t)_+ \right] dt + 2 \int_{x+\delta_2}^1 \left[x, \frac{1}{2}, 1; (\cdot - t)_+ \right] dt$$

$$= \int_x^1 \left[x, \frac{1}{2}, 1; (\cdot - t)_+ \right] dt + \int_{x+\delta_2}^1 \left[x, \frac{1}{2}, 1; (\cdot - t)_+ \right] dt.$$

By (3.2), the first integral is equal to $\frac{1}{2}$. Furthermore, $[x, \frac{1}{2}, 1; (\cdot - t)_+] = \frac{(1-t)_+}{(1-x)(1-\frac{1}{2})}$ if $x + \delta_2 \leq t \leq 1$. Hence,

$$I_2 = \frac{1}{2} + \frac{1}{1-x} (1-x-\delta_2)^2.$$

Finally,

$$c_2(\delta_2, x) = \frac{x^3 - (2 - \sqrt{2})x^2 - (\sqrt{2} - 1)x + \frac{1}{2}}{1-x}.$$

The critical points of $c_2(\delta_2, x)$ are given by

$$x_1 = \frac{1}{4} \left(3 - \sqrt{1 + 4\sqrt{2}} \right) \approx 0.104978,$$

$$x_2 = \frac{1}{2} \left(2 - \sqrt{2} \right) \approx 0.292893,$$

$$x_3 = \frac{1}{4} \left(3 + \sqrt{1 + 4\sqrt{2}} \right) \approx 1.39502.$$

Note that $c_2(\delta_2, 0) = c_2(\delta_2, 1 - \delta_2) = \frac{1}{2}$. Since $x_2, x_3 \notin (0, 1 - \delta_2)$, we conclude that

$$c_2(\delta_2, x) \leq c_2(\delta_2, x_1) = \frac{1}{8} \left(\sqrt{97 + 140\sqrt{2}} - 10\sqrt{2} + 1 \right) \approx 0.504142$$

for all $x \in (0, 1 - \delta_2)$. The symmetry $c_2(\delta_2, x) = c_2(\delta_2, 1 - x)$ implies this estimate also for $x \in (\delta_2, 1)$.

This completes the proof of

$$C_2 = \frac{1}{8} \left(\sqrt{97 + 140\sqrt{2}} - 10\sqrt{2} + 1 \right).$$

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