Research Article

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On the solvability of the modified Cauchy problem for linear systems of generalized ordinary differential equations with singularities

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Abstract: Effective sufficient conditions are given for the unique solvability of the Cauchy problem for linear systems of generalized ordinary differential equations with singularities.

Keywords: Linear systems of generalized ordinary differential equations, Kurzweil integral, singularities, Cauchy problem, unique solvability, effective sufficient conditions, spectral condition

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1 Statement of the problem and basic notation

Let $I \subset \mathbb{R}$ be a non-degenerate interval at a point $t_0 \in \mathbb{R}$ and let

$$I_{t_0} = I \setminus \{t_0\}, \quad I_{t_0}^- =] - \infty, t_0[\cap I, \quad I_{t_0}^+ =]t_0, +\infty[\cap I.$$

Consider the linear system of generalized ordinary differential equations

$$dx = dA(t) \cdot x + df(t) \quad \text{for } t \in I_{t_0}, \quad (1.1)$$

where

$$A = (a_{ik})_{i,k=1}^n \in BV_{\text{loc}}(I_{t_0}; \mathbb{R}^{n \times n}) \quad \text{and} \quad f = (f_k)_{k=1}^n \in BV_{\text{loc}}(I_{t_0}; \mathbb{R}^n)$$

are matrix- and vector-functions, respectively.

Let $H = \text{diag}(h_1, \dots, h_n) : I_{t_0} \rightarrow \mathbb{R}^{n \times n}$ be an arbitrary diagonal matrix-function with continuous diagonal elements

$$h_k : I_{t_0} \rightarrow]0, +\infty[\quad (k = 1, \dots, n).$$

We consider the problem of finding a solution $x \in BV_{\text{loc}}(I_{t_0}, \mathbb{R}^n)$ of system (1.1), satisfying the condition

$$\lim_{t \rightarrow t_0^-} (H^{-1}(t)x(t)) = 0 \quad \text{and} \quad \lim_{t \rightarrow t_0^+} (H^{-1}(t)x(t)) = 0. \quad (1.2)$$

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Analogous problems for systems of ordinary differential equations with singularities

$$\frac{dx}{dt} = P(t)x + q(t) \quad \text{for } t \in I, \quad (1.3)$$

where $P \in L_{\text{loc}}(I_{t_0}; \mathbb{R}^{n \times n})$, $q \in L_{\text{loc}}(I_{t_0}; \mathbb{R}^n)$, have been investigated in the papers [6–8].

The singularity of system (1.3) is considered in the sense that the matrix-function P and the vector-function q are not, in general, integrable at the point t_0 . The solution of problem (1.3), (1.2) is not, in general, continuous at the point t_0 and, therefore, it is not a solution in the classical sense. But its restriction on every interval from I_{t_0} is a solution of system (1.3). To illustrate this we give the example from [8].

Let $\alpha > 0$ and $\varepsilon \in]0, \alpha[$. Then the problem

$$\frac{dx}{dt} = -\frac{\alpha x}{t} + \varepsilon |t|^{\varepsilon-1\alpha}, \quad \lim_{t \rightarrow 0} (t^\alpha x(t)) = 0$$

has the unique solution $x(t) = |t|^{\varepsilon-\alpha} \operatorname{sgn} t$. This function is not a solution of the equation on the set $I = \mathbb{R}$, but its restrictions on $] -\infty, 0[$ and $]0, +\infty[$ are solutions of the one.

The singularity of system (1.1) is considered in the sense that A or f may have non-bounded total variations at the point t_0 , i.e., on some closed interval $[a, b]$ from I such that $t_0 \in [a, b]$.

As we know, such a problem for the generalized differential system (1.1) has not been investigated. So, the present investigation is quite topical.

Some singular two-point boundary problems for the generalized differential system (1.1) are investigated in [3–5].

The interest in the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive differential and difference equations from a unified point of view (see [1–5, 10, 11] and the references therein).

In the present paper, we give the sufficient conditions for the unique solvability of problem (1.1), (1.2). Analogous results for the Cauchy problem for systems of ordinary differential equations with singularities belong to Kiguradze [7, 8].

Throughout the paper, use will be made of the following notation and definitions:

- $\mathbb{R} =] -\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, and $[a, b]$ and $]a, b[$ ($a, b \in \mathbb{R}$) are, respectively, closed and open intervals.
- $\mathbb{R}^{n \times m}$ is the space of all real $n \times m$ matrices $X = (x_{ik})_{i,k=1}^{n,m}$ with the norm

$$\|X\| = \max_{k=1, \dots, m} \sum_{i=1}^n |x_{ik}|.$$

- $O_{n \times m}$ (or O) is the zero $n \times m$ matrix.
- If $X = (x_{ik})_{i,k=1}^{n,m} \in \mathbb{R}^{n \times m}$, then

$$|X| = (|x_{ik}|)_{i,k=1}^{n,m}, \quad [X]_+ = \frac{|X| + X}{2}, \quad [X]_- = \frac{|X| - X}{2}.$$

- $\mathbb{R}_+^{n \times m} = \{(x_{ik})_{i,k=1}^{n,m} : x_{ik} \geq 0 \ (i = 1, \dots, n, k = 1, \dots, m)\}$.
- $\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the space of all real column n -vectors $x = (x_i)_{i=1}^n$, and $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$.
- If $X \in \mathbb{R}^{n \times n}$, then X^{-1} , $\det X$ and $r(X)$ are, respectively, the matrix inverse to X , the determinant of X and the spectral radius of X , and I_n is the identity $n \times n$ -matrix.
- The inequalities between the matrices are understood componentwise.
- A matrix-function is said to be continuous, integrable, nondecreasing etc., if each of its components is such.
- If $X : \mathbb{R} \rightarrow \mathbb{R}^{n \times m}$ is a matrix-function, then $\overset{b}{V}(X)$ is the sum of total variations on $[a, b]$ of its components x_{ik} ($i = 1, \dots, n, k = 1, \dots, m$). If $a > b$, then we assume

$$\overset{b}{V}(X) = -\overset{a}{V}(X).$$

- $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a, b] \rightarrow \mathbb{R}^{n \times m}$ at the point t ($X(a-) = X(a)$, $X(b+) = X(b)$),

$$d_1 X(t) = X(t) - X(t-), \quad d_2 X(t) = X(t+) - X(t).$$

- $BV([a, b], \mathbb{R}^{n \times m})$ is the set of all bounded variation matrix-functions $X : [a, b] \rightarrow \mathbb{R}^{n \times m}$, i.e., such that

$$\bigvee_a^b(X) < \infty.$$

- $BV_{\text{loc}}(J; D)$, where $J \subset \mathbb{R}$ is an interval and $D \subset \mathbb{R}^{n \times m}$, is the set of all $X : J \rightarrow D$ for which the restriction on $[a, b]$ belongs to $BV([a, b]; D)$ for every closed interval $[a, b]$ from J .
- $BV_{\text{loc}}(I_{t_0}; D)$ is the set of all $X : I \rightarrow D$ for which the restriction on $[a, b]$ belongs to $BV([a, b]; D)$ for every closed interval $[a, b]$ from I_{t_0} .
- Everywhere we assume that $a_1 \in I_{t_0}^-$ and $a_2 \in I_{t_0}^+$ are some fixed points.
- If $X \in BV_{\text{loc}}(I_{t_0}; \mathbb{R}^{n \times m})$, then

$$V(X)(t) = (v(x_{ik})(t))_{i,k=1}^{n,m} \quad \text{for } t \in I_{t_0},$$

where

$$v(x_{ik})(a_j) = 0, \quad v(x_{ik})(t) \equiv \bigvee_{a_j}^t(x_{ik}) \quad \text{for } (t - t_0)(a_j - t_0) > 0 \quad (j = 1, 2).$$

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- $[X(t)]_+^v \equiv \frac{V(X)(t) + X(t)}{2} \quad \text{and} \quad [X(t)]_-^v \equiv \frac{V(X)(t) - X(t)}{2}.$
- s_1, s_2, s_c and $\mathcal{J} : BV_{\text{loc}}(I_{t_0}; \mathbb{R}) \rightarrow BV_{\text{loc}}(I_{t_0}; \mathbb{R})$ are the operators defined, respectively, by

$$\begin{aligned} s_1(x)(a_j) &= s_2(x)(a_j) = 0, \\ s_c(x)(a_j) &= x(a_j) \quad (j = 1, 2), \\ s_1(x)(t) &= s_1(x)(s) + \sum_{s < \tau \leq t} d_1 x(\tau), \\ s_2(x)(t) &= s_2(x)(s) + \sum_{s \leq \tau < t} d_2 x(\tau), \\ s_c(x)(t) &= s_c(x)(s) + x(t) - x(s) - \sum_{j=1}^2 (s_j(x)(t) - s_j(x)(s)) \end{aligned}$$

for $s < t < t_0$ and for $t_0 < s < t$, and

$$\begin{aligned} \mathcal{J}(x)(a_j) &= x(a_j) \quad (j = 1, 2), \\ \mathcal{J}(x)(t) &= \mathcal{J}(x)(s) + s_0(x)(t) - s_0(x)(s) + \sum_{t < \tau \leq s} \ln|1 - d_1 x(\tau)| - \sum_{t \leq \tau < s} \ln|1 + d_2 x(\tau)| \quad \text{for } t < s < t_0, \\ \mathcal{J}(x)(t) &= \mathcal{J}(x)(s) + s_0(x)(t) - s_0(x)(s) - \sum_{s < \tau \leq t} \ln|1 - d_1 x(\tau)| + \sum_{s \leq \tau < t} \ln|1 + d_2 x(\tau)| \quad \text{for } t_0 < s < t. \end{aligned}$$

- If $X \in BV_{\text{loc}}(I_{t_0}; \mathbb{R}^{n \times n})$, $\det(I_n + (-1)^l d_l X(t)) \neq 0$ for $t \in I_{t_0}$ ($l = 1, 2$), and $Y \in BV_{\text{loc}}(I_{t_0}; \mathbb{R}^{n \times m})$, then

$$\begin{aligned} \mathcal{A}(X, Y)(a_j) &= O_{n \times m}, \\ \mathcal{A}(X, Y)(t) - \mathcal{A}(X, Y)(s) &= Y(t) - Y(s) + \sum_{s < \tau \leq t} d_1 X(\tau) \cdot (I_n - d_1 X(\tau))^{-1} d_1 Y(\tau) \\ &\quad - \sum_{s \leq \tau < t} d_2 X(\tau) \cdot (I_n + d_2 X(\tau))^{-1} d_2 Y(\tau) \end{aligned}$$

for $s < t < t_0$ if $a_j < t_0$, and for $t_0 < s < t < t_0$ if $a_j > t_0$ ($j = 1, 2$).

- If $g : [a, b] \rightarrow \mathbb{R}$ is a nondecreasing function, $x : [a, b] \rightarrow \mathbb{R}$ and $a \leq s < t \leq b$, then

$$\int_s^t x(\tau) dg(\tau) = \int_{]s, t[} x(\tau) ds_0(g)(\tau) + \sum_{s < \tau \leq t} x(\tau) d_1 g(\tau) + \sum_{s \leq \tau < t} x(\tau) d_2 g(\tau),$$

where $\int_{]s, t[} x(\tau) ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s, t[$ with respect to the measure $\mu_0(s_c(g_c))$ corresponding to the function $s_0(g)$. So $\int_s^t x(\tau) dg(\tau)$ is the Kurzweil integral (see [9–11]).

- If $a = b$, then we assume

$$\int_a^b x(t) dg(t) = 0,$$

and if $a > b$, then we assume

$$\int_a^b x(t) dg(t) = - \int_b^a x(t) dg(t).$$

- Moreover, we put

$$\int_s^{t+} x(\tau) dg(\tau) = \lim_{\delta \rightarrow 0+} \int_s^{t+\delta} x(\tau) dg(\tau), \quad \int_s^{t-} x(\tau) dg(\tau) = \lim_{\delta \rightarrow 0+} \int_s^{t-\delta} x(\tau) dg(\tau).$$

- If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau) dg(\tau) = \int_s^t x(\tau) dg_1(\tau) - \int_s^t x(\tau) dg_2(\tau) \quad \text{for } s, t \in \mathbb{R}.$$

- If $G = (g_{ik})_{i,k=1}^{l,n} : [a, b] \rightarrow \mathbb{R}^{l \times n}$ is a nondecreasing matrix-function and $X = (x_{kj})_{k,j=1}^{n,m} : [a, b] \rightarrow \mathbb{R}^{n \times m}$, then

$$\begin{aligned} \int_s^t dG(\tau) \cdot X(\tau) &= \left(\sum_{k=1}^n \int_s^t x_{kj}(\tau) dg_{ik}(\tau) \right)_{i,j=1}^{l,m} \quad \text{for } a \leq s \leq t \leq b, \\ S_j(G)(t) &\equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n} \quad (j = 1, 2), \quad S_c(G)(t) \equiv (s_c(g_{ik})(t))_{i,k=1}^{l,n}. \end{aligned}$$

- If $G_j : [a, b] \rightarrow \mathbb{R}^{l \times n}$ ($j = 1, 2$) are nondecreasing matrix-functions, $G = G_1 - G_2$ and $X : [a, b] \rightarrow \mathbb{R}^{n \times m}$, then

$$\begin{aligned} \int_s^t dG(\tau) \cdot X(\tau) &= \int_s^t dG_1(\tau) \cdot X(\tau) - \int_s^t dG_2(\tau) \cdot X(\tau) \quad \text{for } s, t \in \mathbb{R}, \\ S_k(G) &= S_k(G_1) - S_k(G_2) \quad (k = 1, 2), \quad S_c(G) = S_c(G_1) - S_c(G_2). \end{aligned}$$

- Here use will be made of the following formulas:

$$\int_a^b f(t) d\left(\int_a^t h(s) dg(s)\right) = \int_a^b f(t)h(t) dg(t) \quad (\text{substitution formula}),$$

$$\begin{aligned} \int_a^b f(t) dg(t) + \int_a^b g(t) df(t) &= f(b)g(b) - f(a)g(a) + \sum_{a < t \leq b} d_1 f(t) \cdot d_1 g(t) - \sum_{a \leq t < b} d_2 f(t) \cdot d_2 g(t) \\ &(\text{integration by parts formula}), \end{aligned}$$

$$\begin{aligned} \int_a^b h(t) d(f(t)g(t)) &= \int_a^b h(t)f(t) dg(t) + \int_a^b h(t)g(t) df(t) - \sum_{a < t \leq b} h(t)d_1 f(t) \cdot d_1 g(t) + \sum_{a \leq t < b} h(t)d_2 f(t) \cdot d_2 g(t) \\ &(\text{general integration by parts formula}) \end{aligned}$$

and

$$d_j \left(\int_a^t f(s) dg(s) \right) = f(t) d_j g(t) \quad \text{for } t \in [a, b] \quad (j = 1, 2),$$

where $f, g, h \in \text{BV}([a, b], \mathbb{R})$ (see [11, Theorems I.4.25 and I.4.33]). In the sequel, we use these formulas without making this reference.

- A vector-function $x : I_{t_0} \rightarrow \mathbb{R}^n$ is said to be a solution of system (1.1) if $x \in \text{BV}([a, b], \mathbb{R}^n)$ for every closed interval $[a, b]$ from I_{t_0} and

$$x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s) \quad \text{for } a \leq s < t \leq b.$$

- We assume that

$$\det(I_n + (-1)^j d_j A(t)) \neq 0 \quad \text{for } t \in I_{t_0} \ (j = 1, 2).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems (see [9–11]), i.e., for the case where $A \in \text{BV}_{\text{loc}}(I, \mathbb{R}^{n \times n})$ and $f \in \text{BV}_{\text{loc}}(I, \mathbb{R}^n)$.

Let $A_0 \in \text{BV}_{\text{loc}}(I_{t_0}; \mathbb{R}^{n \times n})$. Then a matrix-function $C_0 : I_{t_0} \times I_{t_0} \rightarrow \mathbb{R}^{n \times n}$ is said to be the Cauchy matrix of the generalized homogeneous differential system

$$dx = dA_0(t) \cdot x \tag{1.4}$$

if, for every interval $J \subset I$ and $\tau \in J$, the restriction of the matrix-function $C_0(\cdot, \tau) : I_{t_0} \rightarrow \mathbb{R}^{n \times n}$ on J is the fundamental matrix of system (1.4), satisfying the condition

$$C_0(\tau, \tau) = I_n.$$

Therefore, C_0 is the Cauchy matrix of system (1.4) if and only if the restriction of C_0 on $J \times J$ for every interval $J \subset I_{t_0}$ is the Cauchy matrix of the system in the sense of the definition given in [11].

We assume

$$I_{t_0}^-(\delta) = [t_0 - \delta, t_0[\cap I_{t_0}, \quad I_{t_0}^+(\delta) =]t_0, t_0 + \delta] \cap I_{t_0}, \quad I_{t_0}(\delta) = I_{t_0}^-(\delta) \cup I_{t_0}^+(\delta)$$

for every $\delta > 0$.

Theorem 1.1. *Let there exist a matrix-function $A_0 \in \text{BV}_{\text{loc}}(I_{t_0}, \mathbb{R}^{n \times n})$ and constant matrices B_0 and B from $\mathbb{R}_+^{n \times n}$ such that*

$$\det(I_n + (-1)^j d_j A_0(t)) \neq 0 \quad \text{for } t \in I_{t_0} \ (j = 1, 2), \tag{1.5}$$

$$r(B) < 1, \tag{1.6}$$

and the estimates

$$|C_0(t, \tau)| \leq H(t)B_0H^{-1}(\tau) \quad \text{for } t \in I_{t_0}(\delta), \ (t - t_0)(\tau - t_0) > 0, \ |\tau - t_0| \leq |t - t_0| \tag{1.7}$$

and

$$V(C_H)(t, \cdot)(t) - V(C_H)(t, \cdot)(t_0 \pm) \leq H(t)B \quad \text{for } t \in I_{t_0}^+(\delta) \text{ and } t \in I_{t_0}^-(\delta), \text{ respectively,} \tag{1.8}$$

hold for some $\delta > 0$, where C_0 is the Cauchy matrix of system (1.4), and

$$C_H(t, \tau) = \int_{\tau}^t C_0(t, s) dA(A_0, A - A_0)(s) \cdot H(s) \quad \text{for } (t - t_0)(\tau - t_0) > 0, \ |\tau - t_0| \leq |t - t_0|. \tag{1.9}$$

Let, moreover, respectively,

$$\lim_{t \rightarrow t_0 \pm} \left\| \int_{t_0 \pm}^t H^{-1}(t) C_0(t, \tau) dA(A_0, f)(\tau) \right\| = 0. \tag{1.10}$$

Then problem (1.1), (1.2) has a unique solution.

Theorem 1.2. *Let there exist a constant matrix $B = (b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n \times n}$ such that conditions (1.6) and*

$$\begin{cases} [d_1 a_{ii}(t)]_+ < 1 & \text{for } t < t_0 \ (i = 1, \dots, n), \\ [d_2 a_{ii}(t)]_- < 1 & \text{for } t > t_0 \ (i = 1, \dots, n) \end{cases} \tag{1.11}$$

hold, and the estimates

$$\begin{cases} c_i(t, \tau) \leq b_0 \frac{h_i(t)}{h_i(\tau)} & \text{for } t \in I_{t_0}(\delta), \\ (t - t_0)(\tau - t_0) > 0, & |\tau - t_0| \leq |t - t_0| \quad (i = 1, \dots, n), \end{cases} \quad (1.12)$$

and

$$\begin{cases} \left| \int_{t_0 \pm}^t c_i(t, \tau) h_i(\tau) d[a_{ii}(\tau) \operatorname{sgn}(\tau - t_0)]_+^v \right| \leq b_{ii} h_i(t) \\ \text{for } t \in I_{t_0}^+(\delta) \text{ and } t \in I_{t_0}^-(\delta), \text{ respectively } (i = 1, \dots, n), \\ \left| \int_{t_0 \pm}^t c_i(t, \tau) h_k(\tau) dV(\mathcal{A}(a_{0ii}, a_{ik}))(\tau) \right| \leq b_{ik} h_i(t) \\ \text{for } t \in I_{t_0}^+(\delta) \text{ and } t \in I_{t_0}^-(\delta), \text{ respectively } (i \neq k, i, k = 1, \dots, n), \end{cases} \quad (1.13)$$

hold for some $b_0 > 0$ and $\delta > 0$. Let, moreover, respectively,

$$\lim_{t \rightarrow t_0 \pm} \int_{t_0 \pm}^t \frac{c_i(t, \tau)}{h_i(t)} dV(\mathcal{A}(a_{0ii}, f_i))(\tau) = 0 \quad (i = 1, \dots, n), \quad (1.14)$$

where $a_{0ii}(t) \equiv -[a_{ii}(t) \operatorname{sgn}(t - t_0)]_+^v \operatorname{sgn}(t - t_0)$ ($i = 1, \dots, n$), and c_i is the Cauchy function of the equation $dx = x da_{0ii}(t)$ for $i \in \{1, \dots, n\}$. Then problem (1.1), (1.2) has the unique solution.

Remark 1.3. For $t, \tau \in I_{t_0}^-$ and $t, \tau \in I_{t_0}^+$, the Cauchy functions $c_i(t, \tau)$ ($i = 1, \dots, n$) in Theorem 1.2 have the form

$$c_i(t, s) = \begin{cases} \exp(s_c(a_{0ii})(t) - s_c(a_{0ii})(s)) \prod_{s < \tau \leq t} (1 - d_1 a_{0ii}(\tau))^{-1} \times \prod_{s \leq \tau < t} (1 + d_2 a_{0ii}(\tau)) & \text{for } t > s, \\ \exp(s_c(a_{0ii})(t) - s_c(a_{0ii})(s)) \prod_{t < \tau \leq s} (1 - d_1 a_{0ii}(\tau)) \times \prod_{t \leq \tau < s} (1 + d_2 a_{0ii}(\tau))^{-1} & \text{for } t < s, \\ 1 & \text{for } t = s. \end{cases} \quad (1.15)$$

Corollary 1.4. Let there exist a constant matrix $B = (b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n \times n}$ such that conditions (1.6) and (1.11) hold, and the estimates

$$\begin{cases} \mathcal{J}(a_{0ii})(t) - \mathcal{J}(a_{0ii})(\tau) \leq \mu_i \ln \frac{t - t_0}{\tau - t_0} & \text{for } t, \tau \in I_{t_0}, \\ (t - t_0)(\tau - t_0) > 0, & |\tau - t_0| \leq |t - t_0| \quad (i = 1, \dots, n), \end{cases} \quad (1.16)$$

and

$$\begin{cases} \lim_{\tau \rightarrow t_0 \pm} |[a_{ii}(t) \operatorname{sgn}(t - t_0)]_+^v - [a_{ii}(\tau) \operatorname{sgn}(\tau - t_0)]_+^v| \leq b_{ii} \\ \text{for } t \in I_{t_0}^+(\delta) \text{ and } t \in I_{t_0}^-(\delta), \text{ respectively } (i = 1, \dots, n), \\ \lim_{\tau \rightarrow t_0 \pm} |V(\mathcal{A}(a_{0ii}, a_{ik}))(t) - V(\mathcal{A}(a_{0ii}, a_{ik}))(\tau)| \leq b_{ik} \\ \text{for } t \in I_{t_0}^+(\delta) \text{ and } t \in I_{t_0}^-(\delta), \text{ respectively } (i \neq k, i, k = 1, \dots, n) \end{cases} \quad (1.17)$$

hold for some $\mu_i \geq 0$ ($i = 1, \dots, n$) and $\delta > 0$, where $a_{0ii}(t) \equiv -[a_{ii}(t) \operatorname{sgn}(t - t_0)]_+^v \operatorname{sgn}(t - t_0)$ ($i = 1, \dots, n$). Let, moreover, respectively,

$$\lim_{t \rightarrow t_0 \pm} \int_{t_0 \pm}^t \frac{1}{|\tau - t_0|^{\mu_i}} dV(\mathcal{A}(a_{0ii}, f_i))(\tau) = 0 \quad (i = 1, \dots, n).$$

Then system (1.1) has a unique solution satisfying the initial condition

$$\lim_{t \rightarrow t_0 \pm} \frac{x_i(t)}{|t - t_0|^{\mu_i}} = 0 \quad (i = 1, \dots, n). \quad (1.18)$$

Remark 1.5. In the results given above, the strong inequality (1.6) cannot be replaced by the non-strong one. We give the corresponding example from [8] for the ordinary differential case, i.e., when $A(t) \equiv \int_a^t P(\tau) d\tau$ and $f(t) \equiv \int_a^t q(\tau) d\tau$.

On the interval $] - 1, 0[$ consider the problem

$$\frac{dx}{dt} = \frac{x}{t} + \frac{1}{|\ln|t||}, \quad \lim_{t \rightarrow 0} \frac{x(t)}{t} = 0.$$

Every solution of the above equation has the form

$$x(t) = ct + t \ln|\ln|t|| \quad (c \in \mathbb{R}).$$

So, this problem is not solvable. On the other hand, the Cauchy function $c(t, \tau) = t\tau^{-1}$ for $t \leq \tau < 0$, and the conditions of Theorem 1.1, except of condition (1.6), are fulfilled (on $] - 1, 0[$) for $n = 1$, $P(t) \equiv t^{-1}$, $q(t) \equiv 0$ and $h(t) \equiv t$ only for the case where $B \geq 1$, i.e., when $r(B) \geq 1$.

Remark 1.6. Let, in addition to the conditions of Corollary 1.4, the condition

$$\limsup_{t \rightarrow t_0 \pm} \xi_{ji}(t) < +\infty \quad (j = 1, 2, i = 1, \dots, n) \quad (1.19)$$

hold, where

$$\xi_{ji}(t) = \sum_{\tau \in I_{ij}} \left(\sum_{k=1}^n |\tau - t_0|^{\mu_k} |d_j a_{ik}(\tau)| + |d_j f_i(\tau)| \right) \quad \text{for } t \in I_{t_0} \cap]a_1, a_2[\quad (j = 1, 2, i = 1, \dots, n), \quad (1.20)$$

$I_{t_1} =]a_1, t]$ and $I_{t_2} = [a_1, t[$ for $a_1 < t < t_0$, $I_{t_1} =]t, a_2]$ and $I_{t_2} = [t, a_2[$ for $t_0 < t < a_2$. Then the solution of problem (1.1), (1.18) belongs to $BV_{\text{loc}}(I; \mathbb{R}^n)$.

Corollary 1.7. Let there exist a constant matrix $B = (b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n \times n}$ such that conditions (1.6) and (1.11) hold, and estimates (1.16) for $\mu_i = 1$ ($i = 1, \dots, n$),

$$\left\{ \begin{array}{l} \left| \int_{t_0 \pm}^t |\tau - t_0| d[a_{ii}(\tau) \operatorname{sgn}(\tau - t_0)]_+^v \right| \leq b_{ii} |t - t_0| \\ \text{for } t \in I_{t_0}^+(\delta) \text{ and } t \in I_{t_0}^-(\delta), \text{ respectively } (i = 1, \dots, n), \\ \left| \int_{t_0 \pm}^t |\tau - t_0| dV(\mathcal{A}(a_{0ii}, a_{ik}))(\tau) \right| \leq b_{ik} |t - t_0| \\ \text{for } t \in I_{t_0}^+(\delta) \text{ and } t \in I_{t_0}^-(\delta), \text{ respectively } (i \neq k, i, k = 1, \dots, n) \end{array} \right. \quad (1.21)$$

hold for some $\delta > 0$, where $a_{0ii}(t) \equiv -[a_{ii}(t) \operatorname{sgn}(t - t_0)]_-^v \operatorname{sgn}(t - t_0)$ ($i = 1, \dots, n$). Let, moreover, respectively,

$$\lim_{t \rightarrow t_0 \pm} \frac{1}{|t - t_0|} \left| \int_{t_0 \pm}^t V(\mathcal{A}(a_{0ii}, f_i))(\tau) \right| = 0 \quad (i = 1, \dots, n). \quad (1.22)$$

Then system (1.1) has a unique solution satisfying the initial condition

$$\lim_{t \rightarrow t_0 \pm} \frac{\|x(t)\|}{|t - t_0|} = 0. \quad (1.23)$$

Remark 1.8. Let, in addition to the conditions of Corollary 1.7, condition (1.19) hold, where the functions ξ_{ji} ($j = 1, 2, i = 1, \dots, n$) are defined by (1.20), where $\mu_i = 1$ ($i = 1, \dots, n$), and the intervals I_{tj} ($j = 1, 2$) are defined as in Remark 1.6. Then the solution of problem (1.1), (1.23) belongs to $BV_{\text{loc}}(I; \mathbb{R}^n)$.

Corollary 1.9. Let $a_{ii} \in BV_{\text{loc}}(I; \mathbb{R}^n)$ ($i = 1, \dots, n$) and let conditions (1.11) and

$$\left\{ \begin{array}{l} \mathcal{J}(a_{0ii})(t) - \mathcal{J}(a_{0ii})(s) \leq -\lambda_i \ln \frac{t - t_0}{s - t_0} + a_{ii}^*(t) - a_{ii}^*(s) \\ \text{for } t, s \in I_{t_0}, (t - t_0)(s - t_0) > 0, |s - t_0| \leq |t - t_0| \quad (i = 1, \dots, n) \end{array} \right. \quad (1.24)$$

hold, where $a_{0ii}(t) \equiv -[a_{ii}(t) \operatorname{sgn}(t - t_0)]_-^V \operatorname{sgn}(t - t_0)$ ($i = 1, \dots, n$), $\lambda_i \geq 0$ ($i = 1, \dots, n$), and $a_{ii}^*(t) \operatorname{sgn}(t - t_0)$ ($i = 1, \dots, n$) are the nondecreasing functions on the interval I . Let, moreover,

$$\left\{ \left| \int_{t_0 \pm}^t |\tau - t_0|^{\lambda_i - \lambda_k} dV(\mathcal{A}(a_{0ii}, a_{ik}))(\tau) \right| < +\infty \right. \\ \left. \text{for } t \in I_{t_0}^+ \text{ and } t \in I_{t_0}^-, \text{ respectively } (i \neq k, i, k = 1, \dots, n), \right. \quad (1.25)$$

and

$$\left| \int_{t_0 \pm}^t |\tau - t_0|^{\lambda_i} dV(\mathcal{A}(a_{0ii}, f_i))(\tau) \right| < +\infty \quad \text{for } t \in I_{t_0}^+ \text{ and } t \in I_{t_0}^-, \text{ respectively } (i = 1, \dots, n). \quad (1.26)$$

Then system (1.1) has a unique solution satisfying the initial condition

$$\lim_{t \rightarrow t_0 \pm} (|t - t_0|^{\lambda_i} x_i(t)) = 0 \quad (i = 1, \dots, n). \quad (1.27)$$

Remark 1.10. Let the conditions of Corollary 1.9 hold, where $\lambda_i = 0$ ($i = 1, \dots, n$). Let further condition (1.19) hold, where the functions ξ_{ji} ($j = 1, 2$; $i = 1, \dots, n$) are defined by (1.20), where $\mu_i = 0$ ($i = 1, \dots, n$), and the intervals I_{tj} ($j = 1, 2$) are defined as in Remark 1.6. Then the solution of problem (1.1), (1.27) belongs to $BV_{\text{loc}}(I, \mathbb{R}^n)$.

Remark 1.11. In Remarks 1.6–1.10, condition (1.19) is essential, i.e., if the condition is violated, then the conclusion of the above remarks is not true. Below we present an example. Let $I = [0, 1]$, $n = 1$, $t_0 = 0$, $\tau_l = 1/\sqrt{l}$ ($l = 1, 2, \dots$), and let the function $A(t) \equiv a(t)$ be defined by

$$\begin{aligned} a(0) &= 0, \\ a(1) &= -2 \ln 2, \\ a(t) &= \ln \left(k_l(t - \tau_l) + \frac{1}{l} \right) + \left(\ln 2 - \frac{1}{2} \right) l \quad \text{for } \tau_l \leq t < \tau_{l-1} \quad (l = 2, 3, \dots), \end{aligned}$$

where $k_l = (l - 2)(2l(l - 1)(\tau_l - \tau_{l-1}))^{-1}$ ($l = 2, 3, \dots$).

It is not difficult to verify that the modified Cauchy problem

$$dx = x da(t), \quad \lim_{t \rightarrow 0} \frac{|x(t)|}{t} = 0$$

has the unique solution x defined by the equalities

$$x(t) = k_l(t - \tau_l) + \frac{1}{l} \quad \text{for } \tau_l \leq t < \tau_{l-1} \quad (l = 2, 3, \dots), \quad x(1) = -2 \ln 2.$$

Indeed, we have $x'(t) = k_l$ and $a'(t)x(t) = k_l/(k_l(t - \tau_l) + l^{-1}) \cdot x(t) = k_l$ for $\tau_l \leq t < \tau_{l-1}$ ($l = 2, 3, \dots$). Moreover, $d_2 x(t) = d_2 a(t) \equiv 0$, $x(\tau_{l-1}-) = k_l(\tau_{l-1} - \tau_l) + l^{-1} = (2(l - 1))^{-1}$ ($l = 2, 3, \dots$) and $d_1 x(\tau_l) = (2l)^{-1}$ ($l = 1, 2, \dots$). On the other hand, $a(\tau_{l-1}-) = -\ln(2(l - 1)) + (\ln 2 - \frac{1}{2})l$, $d_1 a(\tau_l) = \frac{1}{2}$ and $x(\tau_l)d_1 a(\tau_l) = (2l)^{-1}$ ($l = 1, 2, \dots$). Therefore, $d_1 a(\tau_l) = x(\tau_l)d_1 a(\tau_l)$ ($l = 1, 2, \dots$).

As for the limit condition, it is evident.

So, we conclude that $x \in BV_{\text{loc}}(I_{t_0}; \mathbb{R})$, but $x \notin BV_{\text{loc}}(I; \mathbb{R})$. Besides, taking into account that the function $a(t)$ is nonincreasing on the intervals $\tau_l \leq t < \tau_{l-1}$ ($l = 2, 3, \dots$), we conclude that $[a(t)]_+^V = 0$ on the same intervals. Therefore, due to the equalities $d_2 a(t) \equiv 0$ and $d_1 a(\tau_l) = \frac{1}{2}$ ($l = 1, 2, \dots$), all conditions of Remarks 1.6–1.10 are fulfilled except of (1.19).

2 Auxiliary propositions

We use the lemma (see Lemma 2.2 below) on the a priori estimate of solutions of system (1.1). To prove the lemma, we use a new type of Cauchy formula for the representation of solutions differing from the previous earlier ones [11].

Lemma 2.1. Let $A_* \in BV_{\text{loc}}(I, \mathbb{R}^{n \times n})$ be such that

$$\det(I_n + (-1)^j d_j A_*(t)) \neq 0 \quad \text{for } t \in I \ (j = 1, 2), \quad (2.1)$$

and $f_* \in BV_{\text{loc}}(I, \mathbb{R}^n)$. Then every solution $x \in BV_{\text{loc}}(I, \mathbb{R}^n)$ of the system

$$dx = dA_*(t) \cdot x + df_*(t) \quad \text{for } t \in I \quad (2.2)$$

admits the representation

$$x(t) = C_*(t, s)x(s) + \int_s^t C_*(t, \tau) dA(A_*, f_*)(\tau) \quad \text{for } s, t \in I, \quad (2.3)$$

where C_* is the Cauchy matrix of system (2.2), and A is the operator defined above.

Proof. By (2.1), the integration by parts formula and the equalities

$$d_{j(\tau)} C_*(t, \tau) \equiv -C_*(t, \tau) d_j A_*(\tau) \cdot (I_n + (-1)^j d_j A_*(\tau))^{-1} \quad (j = 1, 2),$$

it follows from the variation of constant formula (see [11]) that

$$\begin{aligned} x(t) &= C_*(t, \tau)x(s) + f_*(t) - f_*(s) - \int_s^t dC_*(t, \tau) \cdot (f_*(\tau) - f_*(s)) \\ &= C_*(t, \tau)x(s) + \int_s^t C_*(t, \tau) \cdot df_*(\tau) - \sum_{s < \tau \leq t} d_1 C_*(t, \tau) \cdot d_1 f_*(\tau) + \sum_{s \leq \tau < t} d_2 C_*(t, \tau) \cdot d_2 f_*(\tau) \\ &= C_*(t, \tau)x(s) + \int_s^t C_*(t, \tau) \cdot df_*(\tau) + \sum_{s < \tau \leq t} C_*(t, \tau) d_1 A_*(\tau) \cdot (I_n - d_1 A_*(\tau))^{-1} \cdot d_1 f_*(\tau) \\ &\quad - \sum_{s \leq \tau < t} C_*(t, \tau) d_2 A_*(\tau) \cdot (I_n + d_2 A_*(\tau))^{-1} \cdot d_2 f_*(\tau) \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

Therefore, due to the definition of the operator A , equality (2.3) holds. $\square$

Lemma 2.2. Let the matrix-function $A_0 \in BV_{\text{loc}}(I; \mathbb{R}^{n \times n})$ and constant matrices B_0 and B from $\mathbb{R}_+^{n \times n}$ be such that conditions (1.5)–(1.8) hold for some $\delta > 0$, where C_0 is the Cauchy matrix of system (1.4). Let, moreover,

$$\begin{cases} \gamma(t) = \sup_{t_0 \pm} \left\{ \left\| \int_{t_0}^s H^{-1}(s) C_0(s, \tau) dA(A_0, f)(\tau) \right\| : (s - t_0)(t - t_0) > 0, |s - t_0| \leq |t - t_0| \right\} < +\infty \\ \text{for } t \in I_{t_0}(\delta). \end{cases} \quad (2.4)$$

Then every solution $x \in BV_{\text{loc}}(J; \mathbb{R}^n)$ of system (1.1) admits the estimate

$$\begin{cases} \|H^{-1}(t)x(t)\| \leq \rho(\|B_0\| \cdot \|H^{-1}(s_0)x(s_0)\| + \gamma(t)) \\ \text{for } t \in J, (s_0 - t_0)(t - t_0) \geq 0, |s_0 - t_0| \leq |t - t_0|, \end{cases} \quad (2.5)$$

where $\rho = \|(I_n - B)^{-1}\|$, and $J \subset I_{t_0}(\delta)$ and $s_0 \in J$ are an arbitrary interval and point, respectively.

Proof. First consider the case where $J \subset (t_0 + \delta, \sup I)$. Let $x = (x_i)_{i=1}^n$ be an arbitrary solution of system (1.1) on J . Then x satisfies the system

$$dx = dA_0(t) \cdot x + d(A(t) - A_0(t)) \cdot x + df(t) \quad \text{for } t \in J,$$

i.e., the system

$$dx = dA_0(t) \cdot x + d(f(t) + f_0(t)) \quad \text{for } t \in J,$$

where

$$f_0(t) \equiv \int_{s_0}^t d(A(\tau) - A_0(\tau)) \cdot x(\tau).$$

Let the vector-function $z(t) = (z_i(t))_{i=1}^n$ be defined by $z(t) \equiv H^{-1}(t)x(t)$. According to Lemma 2.1, we have

$$z(t) \equiv H^{-1}(t)C_0(t, s_0)x(s_0) + \int_{s_0}^t H^{-1}(t)dC_0(t, \tau) \cdot \mathcal{A}(A_0, f + f_0)(\tau). \quad (2.6)$$

From (2.6), by the definitions of the operator $\mathcal{A}$ and the vector-function f_0 , we obtain

$$\begin{aligned} z(t) &= H^{-1}(t)C_0(t, s_0)x(s_0) + \int_{s_0}^t H^{-1}(t)dC_0(t, \tau) \cdot \mathcal{A}(A_0, f)(\tau) \\ &\quad + \int_{s_0}^t H^{-1}(t)C_0(t, \tau)d(A(\tau) - A_0(\tau)) \cdot H(\tau)z(\tau) \\ &\quad - \sum_{s_0 < \tau \leq t} H^{-1}(t)d_1 C_0(t, \tau) \cdot d_1(A(\tau) - A_0(\tau)) \cdot H(\tau)z(\tau) \\ &\quad + \sum_{s_0 \leq \tau < t} H^{-1}(t)d_2 C_0(t, \tau) \cdot d_2(A(\tau) - A_0(\tau)) \cdot H(\tau)z(\tau) \\ &= H^{-1}(t)C_0(t, s_0)x(s_0) + \int_{s_0}^t H^{-1}(t)dC_0(t, \tau) \cdot \mathcal{A}(A_0, f)(\tau) \\ &\quad + \int_{s_0}^t H^{-1}(t)C_0(t, \tau)dA_H(\tau) \cdot z(\tau) - \sum_{s_0 < \tau \leq t} H^{-1}(t)d_1 C_0(t, \tau) \cdot d_1 A_H(\tau) \cdot z(\tau) \\ &\quad + \sum_{s_0 \leq \tau < t} H^{-1}(t)d_2 C_0(t, \tau) \cdot d_2 A_H(\tau) \cdot z(\tau) \quad \text{for } t > s_0, \end{aligned}$$

and so, using the general integration by parts formula, we get

$$\begin{aligned} z(t) &= H^{-1}(t)C_0(t, s_0)x(s_0) + \int_{s_0}^t H^{-1}(t)dC_0(t, \tau) \cdot \mathcal{A}(A_0, f)(\tau) \\ &\quad + H^{-1}(t) \int_{s_0}^t d(C_0(t, \tau)A_H(\tau)) \cdot z(\tau) - H^{-1}(t) \int_{s_0}^t dC_0(t, \tau) \cdot A_H(\tau)z(\tau) \quad \text{for } t \geq s_0, \end{aligned}$$

where

$$A_H(t) \equiv \int_{s_0}^t d(A(\tau) - A_0(\tau)) \cdot H(\tau).$$

Analogously, we show the last inequality for $t < s_0$. Therefore,

$$\begin{aligned} z(t) &= H^{-1}(t)C_0(t, s_0)x(s_0) + H^{-1}(t) \int_{s_0}^t dC_0(t, \tau) \cdot \mathcal{A}(A_0, f)(\tau) \\ &\quad + H^{-1}(t) \int_{s_0}^t dC^*(t, \tau) \cdot z(\tau) \quad \text{for } t \in J, \end{aligned} \quad (2.7)$$

where

$$C^*(t, \tau) \equiv C_0(t, \tau)A_H(\tau) - \int_{s_0}^{\tau} dC_0(t, s) \cdot A_H(s).$$

In addition, using the integration by parts formula and taking into account the equality

$$d_{j(\tau)}C_0(t, \tau) \equiv -C_0(t, \tau)d_j A_0(\tau) \cdot (I_n + (-1)^j d_j A_0(\tau))^{-1} \quad (j = 1, 2),$$

we have $C^*(t, \tau) \equiv C_H(t, \tau)$, where $C_H(t, \tau)$ is defined by (1.9). Therefore, due to (2.7), we have

$$\begin{aligned} z(t) &= H^{-1}(t)C_0(t, s_0)x(s_0) + H^{-1}(t) \int_{s_0}^t dC_H(t, \tau) \cdot z(\tau) \\ &\quad + H^{-1}(t) \int_{s_0}^t dC_0(t, \tau) \cdot \mathcal{A}(A_0, f)(\tau) \quad \text{for } t \in J. \end{aligned} \quad (2.8)$$

Let the components of the vector-function $y(t) = (y_i(t))_{i=1}^n$ be defined as follows:

$$\begin{aligned} y_i(t) &= \sup\{h_i^{-1}(s)|x_i(s)| : t \leq s \leq s_0\} \quad \text{if } s_0 < t_0, \\ y_i(t) &= \sup\{h_i^{-1}(s)|x_i(s)| : s_0 \leq s \leq t\} \quad \text{if } s_0 > t_0. \end{aligned}$$

In view of (2.8), by taking into account (1.8), it is not difficult to verify that

$$\begin{aligned} y(t) &\leq H^{-1}(t)|C_0(t, s_0)x(s_0)| + H^{-1}(t)(V(C_H)(t, \cdot)(t) - V(C_H)(t, \cdot)(s_0))y(t) + H^{-1}(t) \int_{s_0}^t dC_0(t, \tau) \cdot \mathcal{A}(A_0, f)(\tau) \\ &\leq B_0|H^{-1}(s_0)x(s_0)| + By(t) + g(t) \quad \text{for } s_0 \leq t, \end{aligned}$$

where

$$g(t) \equiv H^{-1}(t) \left| \int_{s_0}^t C_0(t, \tau) d\mathcal{A}(A_0, f)(\tau) \right|.$$

Therefore, thanks to (1.6) and the nonnegativity of the matrix B_0 , we have

$$(I_n - B)y(t) \leq B_0|H^{-1}(s_0)x(s_0)| + g(t) \quad \text{for } s_0 \leq t,$$

and

$$y(t) \leq (I_n - B)^{-1}(B_0|H^{-1}(s_0)x(s_0)| + g(t)) \quad \text{for } s_0 \leq t.$$

Hence estimate (2.5) holds.

Analogously, we can prove the estimate in other cases. $\square$

Lemma 2.3. Let $g \in \text{BV}([a, b]; \mathbb{R})$. Then

$$\int_a^b \text{sgn } g(t) dg(t) = |g(b)| - |g(a)| + \sum_{a < t \leq b} (|g(t-)| - g(t-) \text{sgn } g(t)) - \sum_{a \leq t < b} (|g(t+)| - g(t+) \text{sgn } g(t)) \quad (2.9)$$

and

$$\int_a^b \text{sgn } g(t) d|g(t)| = g(b) - g(a) + \sum_{a < t \leq b} (g(t-) - |g(t-)| \text{sgn } g(t)) - \sum_{a \leq t < b} (g(t+) - |g(t+)| \text{sgn } g(t)). \quad (2.10)$$

Proof. Due to the integration by parts formula and [11, Lemma I.4.23], we find

$$\begin{aligned} \int_a^b \text{sgn } g(t) dg(t) &= g(b) \text{sgn } g(b) - g(a) \text{sgn } g(a) - \int_a^b g(t) d(\text{sgn } g(t)) \\ &\quad + \sum_{a < t \leq b} d_1 g(t) \cdot d_1(\text{sgn } g(t)) - \sum_{a \leq t < b} d_2 g(t) \cdot d_2(\text{sgn } g(t)) \\ &= |g(b)| - |g(a)| - \sum_{a < t \leq b} g(t) \cdot d_1(\text{sgn } g(t)) - \sum_{a \leq t < b} g(t) \cdot d_2(\text{sgn } g(t)) \\ &\quad + \sum_{a < t \leq b} d_1 g(t) \cdot d_1(\text{sgn } g(t)) - \sum_{a \leq t < b} d_2 g(t) \cdot d_2(\text{sgn } g(t)) \\ &= |g(b)| - |g(a)| - \sum_{a < t \leq b} g(t-)(\text{sgn } g(t) - \text{sgn } g(t-)) - \sum_{a \leq t < b} g(t+)(\text{sgn } g(t+) - \text{sgn } g(t)). \end{aligned}$$

So equality (2.9) holds.

In the same way we establish equality (2.10). $\square$

Lemma 2.4. Let the restriction on the set I_{t_0} of a function $g : I \rightarrow \mathbb{R}$ belong to the set $BV_{\text{loc}}(I_{t_0}, \mathbb{R})$,

$$w(t) - w(\tau) \leq v(t) - v(\tau) \quad \text{for } t, \tau \in I, \quad t_0 < \tau < t \quad (t < \tau < t_0), \quad (2.11)$$

and let there exist one-sided limits $w(t_0-)$ and $w(t_0+)$, where

$$w(t) = \int_a^t \sigma(s) \operatorname{sgn} g(s) dg(s) \quad \text{for } t \in I_{t_0}^+ \quad (t \in I_{t_0}^-),$$

$a \in I_{t_0}^+$ ($a \in I_{t_0}^-$) is some fixed point,

$$v \in BV_{\text{loc}}(I \setminus I_{t_0}^+; \mathbb{R}) \quad (v \in BV_{\text{loc}}(I \setminus I_{t_0}^-; \mathbb{R})), \quad (2.12)$$

and $\sigma : I_{t_0} \rightarrow \{-1, 1\}$ is a continuous function. Then

$$g \in BV_{\text{loc}}(I \setminus I_{t_0}^-; \mathbb{R}) \quad (g \in BV_{\text{loc}}(I \setminus I_{t_0}^+; \mathbb{R})). \quad (2.13)$$

Proof. First consider the case where $t \in I_{t_0}^+$ and $\sigma(t) \equiv 1$. Let us introduce the function

$$u(t) = v(t) - w(t).$$

Then there exists a one-sided limit $u(t_0+)$. By (2.12), we have $u \in BV_{\text{loc}}(I \setminus I_{t_0}^-, \mathbb{R})$ because, due to (2.11), the function u is nondecreasing on the set $I_{t_0}^+$. Therefore, $w \in BV_{\text{loc}}(I \setminus I_{t_0}^-, \mathbb{R})$ as well.

By (2.9), we find

$$\begin{aligned} |g(t)| - |g(\tau)| &= w(t) - w(\tau) - \sum_{\tau < s \leq t} (|g(s-)| - g(s-) \operatorname{sgn} g(s)) + \sum_{\tau \leq s < t} (|g(s+)| - g(s+) \operatorname{sgn} g(s)) \\ &= w(t) - w(\tau) + \sum_{\tau < s \leq t} (|g(s)| - |g(s-)|) - \sum_{\tau < s \leq t} (|g(s)| - g(s-) \operatorname{sgn} g(s)) \\ &\quad + \sum_{\tau \leq s < t} (|g(s+)| - g(s+) \operatorname{sgn} g(s)) \\ &= w(t) - w(\tau) + \sum_{\tau < s \leq t} (|g(s)| - |g(s-)|) - \sum_{\tau < s \leq t} (|g(s)| - g(s-) \operatorname{sgn} g(s)) \\ &\quad + \sum_{\tau \leq s < t} (|g(s+)| - |g(s)|) + \sum_{\tau \leq s < t} (|g(s)| - g(s+) \operatorname{sgn} g(s)) \\ &= w(t) - w(\tau) - s_1(w)(t) + s_1(w)(\tau) - s_2(w)(t) + s_2(w)(\tau) \\ &\quad + s_1(|g|)(t) - s_1(|g|)(\tau) + s_2(|g|)(t) - s_1(|g|)(\tau) \quad \text{for } t_0 < \tau < t. \end{aligned}$$

Hence

$$s_0(|g|)(t) - s_0(|g|)(\tau) = s_0(w)(t) - s_0(w)(\tau) \quad \text{for } t_0 < \tau < t,$$

and

$$s_0(|g|) \in BV(I \setminus I_{t_0}^-, \mathbb{R}). \quad (2.14)$$

Using now the estimates

$$|d_j(|g(t)|)| \leq |d_j g(t)| = |d_j w(t)| \quad \text{for } t > t_0 \quad (j = 1, 2)$$

we conclude that

$$v(s_j(|g|))(t) - v(s_j(|g|))(t_0) \leq v(s_j(w))(t) - v(s_j(w))(t_0) < +\infty \quad \text{for } t > t_0 \quad (j = 1, 2)$$

and, therefore,

$$s_1(g), s_2(g) \in BV(I \setminus I_{t_0}^-; \mathbb{R}) \quad (2.15)$$

and

$$s_1(|g|), s_2(|g|) \in BV(I \setminus I_{t_0}^-; \mathbb{R}).$$

By the last inclusions and (2.14), we get

$$|g| \in \text{BV}(I \setminus I_{t_0}^-; \mathbb{R}). \quad (2.16)$$

Because of (2.16), we have

$$\overline{w} \in \text{BV}(I \setminus I_{t_0}^-; \mathbb{R}),$$

where

$$\overline{w}(t) = \int_a^t \text{sgn } g(s) d|g(s)| \quad \text{for } t \geq t_0.$$

As above, we conclude that

$$s_0(g)(t) - s_0(g)(\tau) = s_0(\overline{w})(t) - s_0(\overline{w})(\tau) \quad \text{for } t_0 < \tau < t,$$

and

$$s_0(g) \in \text{BV}(I \setminus I_{t_0}^-; \mathbb{R}). \quad (2.17)$$

So inclusion (2.13) follows from (2.15) and (2.17). $\square$

Lemma 2.5. Let $X \in \text{BV}_{\text{loc}}(I; \mathbb{R}^{n \times n})$, $\det(I_n + (-1)^j d_j X(t)) \neq 0$ for $t \in I$ ($j = 1, 2$), and $Y \in \text{BV}_{\text{loc}}(I; \mathbb{R}^{n \times m})$. Then

$$\begin{aligned} Y(t) - Y(s) &= \mathcal{A}(X, Y)(t) - \mathcal{A}(X, Y)(s) - \sum_{s < \tau \leq t} d_1 X(\tau) \cdot d_1 \mathcal{A}(X, Y)(\tau) \\ &\quad + \sum_{s \leq \tau < t} d_2 X(\tau) \cdot d_2 \mathcal{A}(X, Y)(\tau) \quad \text{for } s, t \in I, s < t, \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} \mathbb{V}_a^b(Y) &\leq \mathbb{V}_a^b(\mathcal{A}(X, Y)) + \max_{a < t \leq b} \{\|d_1 X(t)\|\} \sum_{a < t \leq b} \|d_1 \mathcal{A}(X, Y)(t)\| \\ &\quad + \max_{a \leq t < b} \{\|d_2 X(t)\|\} \sum_{a \leq t < b} \|d_2 \mathcal{A}(X, Y)(t)\| \quad \text{for } a, b \in I, a < b. \end{aligned} \quad (2.19)$$

Proof. By the definition of the matrix-function $\mathcal{A}(X, Y)(t)$, we have

$$d_j \mathcal{A}(X, Y)(\tau) = d_j Y(\tau) - (-1)^j d_j X(\tau) \cdot (I_n + (-1)^j d_j X(\tau))^{-1} d_j Y(\tau) \quad \text{for } \tau \in I \ (j = 1, 2).$$

So

$$d_j \mathcal{A}(X, Y)(\tau) = (I_n + (-1)^j d_j X(\tau))^{-1} d_j Y(\tau) \quad \text{for } \tau \in I \ (j = 1, 2).$$

From this and the definition of the matrix-function $\mathcal{A}(X, Y)(t)$ there immediately follows equality (2.18). On the other hand, (2.18) guarantees equality (2.19). $\square$

3 Proofs of the results

Proof of Theorem 1.1. First consider the case where $\sup I_{t_0} > t_0$ and prove the existence and uniqueness of a solution of problem (1.1), (1.2) on the interval $I_{t_0}^+ = \{t \in I_{t_0} : t > t_0\}$.

First, note that by (1.10) estimate (2.4) holds and

$$\lim_{t \rightarrow t_0^+} \gamma(t) = 0, \quad (3.1)$$

where the function $\gamma(t)$ is defined by (2.4) as in Lemma 2.2.

Moreover, we suppose that $t_0 + \delta \in I_{t_0}^+$.

Let $t_k \in]t_0, t_0 + \delta[$ ($k = 1, 2, \dots$) be some decreasing sequence such that

$$\lim_{k \rightarrow +\infty} t_k = t_0. \quad (3.2)$$

According to [11, Theorem III.1.4], for every natural k , system (1.1) has a unique solution x_k defined on the interval $I_{t_0}^+$ and satisfying the condition $x_k(t_k) = 0$.

Moreover, due to Lemma 2.2 we have the estimates

$$\|H^{-1}(t)x_k(t)\| \leq \rho\gamma(t) \quad \text{for } t_k \leq t \leq t_1 \quad (k = 1, 2, \dots), \quad (3.3)$$

where $\rho = \|(I_n - B)^{-1}\|\gamma(t_1)$. In particular, from (3.3) it follows that

$$\|x_k(t_1)\| \leq \rho_0 \quad (k = 1, 2, \dots),$$

where $\rho_0 = \rho\|H(t_1)\|\gamma(t_1)$. So without loss of generality we can assume that the sequence $x_k(t_1)$ ($k = 1, 2, \dots$) converges. Let

$$\lim_{k \rightarrow +\infty} x_k(t_1) = c_0.$$

By the theorems on the well-posedness of the Cauchy problem (see [2]), we conclude

$$\lim_{k \rightarrow +\infty} x_k(t) = x(t)$$

uniformly on every closed interval from $I_{t_0}^+$, where x is a solution of system (1.1) under the condition

$$x(t_1) = c_0.$$

On the other hand, from (3.2) and (3.3) it follows that

$$\|H^{-1}(t)x(t)\| \leq \rho\gamma(t) \quad \text{for } t_0 < t \leq t_1.$$

From this, by (3.1) we conclude that the vector-function x is a solution of problem (1.1), (1.2) on the interval $I_{t_0}^+$.

Now, let us show that problem (1.1), (1.2) has a unique solution on the interval $I_{t_0}^+$. Let $\bar{x}$ be an arbitrary solution of the problem. Then the vector-function $x_0(t) = x(t) - \bar{x}(t)$ is a solution of the homogeneous system

$$dx = dA(t) \cdot x$$

under the condition

$$\lim_{t \rightarrow t_0+} (H^{-1}(t)x_0(t)) = 0. \quad (3.4)$$

In view of Lemma 2.2, we find

$$\|x_0(t_0 + \delta)\| \leq \rho_0 \|H(t_0 + \delta)\| \cdot \|H^{-1}(s_0)x_0(s_0)\| \quad \text{for } t_0 < s_0 \leq t_0 + \delta,$$

where $\rho_0 = \|(I_n - B)^{-1}\| \cdot \|B_0\|$. Passing to the limit as $s_0 \rightarrow t_0$ in the last estimate and taking into account (3.4), we get

$$x(t_0 + \delta) = 0.$$

Since $A \in BV_{\text{loc}}(I_{t_0}^+; \mathbb{R}^{n \times n})$ and $t_0 + \delta \in I_{t_0}^+$, i.e., we have the regular case, by the above-mentioned theorem from [11] we conclude that $x_0(t) \equiv 0$ and, therefore, $x(t) = \bar{x}(t)$ for $t \in I_{t_0}^+$.

Analogously, we show that problem (1.1), (1.2) has a unique solution y on the interval $I_{t_0}^- = \{t \in I_{t_0} : t < t_0\}$ as well if $\inf I_{t_0} < t_0$. And if $\inf I_{t_0} < t_0 < \sup I_{t_0}$, it is evident that the vector-function

$$z(t) = \begin{cases} y(t) & \text{if } t \in I_{t_0}^-, \\ x(t) & \text{if } t \in I_{t_0}^+, \end{cases}$$

will be a unique solution of problem (1.1), (1.2) on the interval I_{t_0} . □

Proof of Theorem 1.2. Let us assume

$$\begin{aligned} A_0(t) &= \text{diag}(a_{011}(t), \dots, a_{0nn}(t)) & \text{for } t \in I_{t_0}(\delta), \\ \mathcal{A}(A_0, A - A_0)(t) &= (\tilde{a}_{ik}(t))_{i,k=1}^n & \text{for } t \in I_{t_0}. \end{aligned}$$

By the definition of the operator $\mathcal{A}$, we find

$$\begin{aligned}\tilde{a}_{ik}(t) - \tilde{a}_{ik}(s) &= (a_{ii}(t) - a_{0ik}(t)) - (a_{ii}(s) - a_{0ik}(s)) \\ &\quad + \sum_{s_0 < \tau \leq t} d_1 a_{0ii}(\tau) \cdot (1 - d_1 a_{0ii}(\tau))^{-1} d_1 (a_{ii}(\tau) - a_{0ik}(\tau)) \\ &\quad - \sum_{s_0 \leq \tau < t} d_2 a_{0ii}(\tau) \cdot (1 + d_2 a_{0ii}(\tau))^{-1} d_2 (a_{ii}(\tau) - a_{0ik}(\tau)) \\ &\quad \text{for } s < t, (s - t_0)(t - t_0) > 0 \ (i, k = 1, \dots, n),\end{aligned}\quad (3.5)$$

where $a_{0ik}(t) = 0$ if $i \neq k$ ($i, k = 1, \dots, n$).

Consider the case where $i = k$ ($i = 1, \dots, n$). It is not difficult to verify that

$$\begin{aligned}a_{ii}(t) - a_{0ii}(t) &= -[a_{ii}(t)]_-^\vee \quad \text{for } t < t_0 \ (i = 1, \dots, n), \\ a_{ii}(t) - a_{0ii}(t) &= [a_{ii}(t)]_+^\vee \quad \text{for } t > t_0 \ (i = 1, \dots, n).\end{aligned}$$

Therefore,

$$d_j a_{0ii}(t) = [d_j a_{ii}(t)]_+ \quad \text{and } d_j(a_{ii}(t) - a_{0ii}(t)) = -[d_j a_{ii}(t)]_- \quad \text{for } t < t_0 \ (j = 1, 2, i = 1, \dots, n), \quad (3.6)$$

$$d_j a_{0ii}(t) = -[d_j a_{ii}(t)]_- \quad \text{and } d_j(a_{ii}(t) - a_{0ii}(t)) = [d_j a_{ii}(t)]_+ \quad \text{for } t > t_0 \ (j = 1, 2, i = 1, \dots, n). \quad (3.7)$$

Hence, due to (3.6) and (3.7),

$$d_j a_{0ii}(t) \cdot d_j(a_{ii}(t) - a_{0ii}(t)) = 0 \quad \text{for } t \in I_{t_0}(\delta) \ (j = 1, 2, i = 1, \dots, n).$$

Thus from (3.5) we have

$$\begin{aligned}\tilde{a}_{ii}(t) &= -[a_{ii}(t)]_-^\vee \quad \text{for } t < t_0 \ (i = 1, \dots, n), \\ \tilde{a}_{ii}(t) &= [a_{ii}(t)]_+^\vee \quad \text{for } t > t_0 \ (i = 1, \dots, n).\end{aligned}$$

On the other hand, it is evident that

$$\tilde{a}_{ik}(t) \equiv \mathcal{A}(a_{0ii}, a_{ik})(t) \quad \text{for } i \neq k \ (i, k = 1, \dots, n).$$

The Cauchy matrix of system (1.4) has the form $C(t, \tau) \equiv \text{diag}(c_1(t, \tau), \dots, c_n(t, \tau))$, and so

$$C_H(t, \tau) \equiv \left(\int_{\tau}^t c_i(t, \tau) h_k(\tau) d\tilde{a}_{i,k} \right)_{i,k=1}^n,$$

where $C_H(t, \tau)$ is the matrix-function defined by (1.9).

In addition, note that due to (1.11), (1.15), (3.6) and (3.7), conditions (1.5) and

$$c_i(t, \tau) > 0 \quad \text{for } (t - t_0)(\tau - t_0) > 0 \ (i = 1, \dots, n)$$

hold. By these results and (1.12)–(1.14), we conclude that conditions (1.7), (1.8) and (1.10) of Theorem 1.1 are valid. Hence, the theorem immediately follows from Theorem 1.1. $\square$

Proof of Corollary 1.4. In view of (1.11), (1.15), (1.16), (3.6) and (3.7), we have

$$0 < c_i(t, \tau) \leq \left| \frac{t - t_0}{\tau - t_0} \right|^{\mu_i} \quad \text{for } (t - t_0)(\tau - t_0) > 0 \ (i = 1, \dots, n), \quad (3.8)$$

since

$$c_i(t, \tau) = \exp(\mathcal{J}(a_{0ii})(t) - \mathcal{J}(a_{0ii})(\tau)) \quad (i = 1, \dots, n).$$

Thus it is clear that the functions

$$h_i(t) = |t - t_i|^{\mu_i} \quad (i = 1, \dots, n) \quad (3.9)$$

satisfy inequalities (1.12), where $b_0 = 1$.

On the other hand, with regard to (3.8) and (3.9), from (1.21) and (1.22) there follow conditions (1.13), (1.22) and (1.14). Therefore, according to Theorem 1.2, problem (1.1), (1.23) has the unique solution $x = (x_i)_{i=1}^n$. $\square$

Let us show that the statement of Remark 1.6 is valid.

Proof of Remark 1.6. Let $x = (x_i)_{i=1}^n$ be the solution of problem (1.1), (1.18). Let $y(t) = (y_i(t))_{i=1}^n$ for $t \in I_{t_0}$, where $y_i(t) = |t - t_0|^{-\mu_i} x_i(t)$ ($i = 1, \dots, n$). Then by condition (1.18) there exists $y(t_0 \pm) = 0$. On defining the continuity y at the point t_0 , i.e., $y(t_0) = 0$, we conclude that the vector-function y is bounded on the set I . Let a number $r > 0$ be such that

$$\|y(t)\| \leq r \quad \text{for } t \in I. \quad (3.10)$$

First consider the case where $t > t_0$.

By condition (1.19), there exist positive numbers δ and r_0 such that the functions ξ_1 and ξ_2 are bounded in the right δ neighborhood of the point t_0 , i.e.,

$$|\xi_{ji}(t)| < r_0 \quad \text{for } t \in I_{t_0}^+(\delta) \quad (j = 1, 2, i = 1, \dots, n).$$

Let

$$w_i(t) = \int_{a_2}^t \operatorname{sgn} x_i(s) dx_i(s) \quad \text{for } t > t_0 \quad (i = 1, \dots, n).$$

Due to the definition of solutions of system (1.1), using the substitution formula, we have

$$\begin{aligned} w_i(t) - w_i(\tau) &= \int_{\tau}^t |x_i(s)| da_{ii}(s) + \sum_{k=1, k \neq i}^n \int_{\tau}^t \operatorname{sgn} x_i(s) x_k(s) da_{ik}(s) + f_i(t) - f_i(\tau) \\ &\leq \int_{\tau}^t |y_i(s)| d\left(\int_{\tau}^s |\zeta - t_0|^{\mu_i} d[a_{ii}(\zeta)]_+^v\right) + \sum_{k=1, k \neq i}^n \left| \int_{\tau}^t |y_k(s)| d\left(\int_{\tau}^s |\zeta - t_0|^{\mu_i} dv(a_{ik})(\zeta)\right) \right| + |f_i(t) - f_i(\tau)| \\ &\quad \text{for } t_0 + \delta < \tau < t \quad (i = 1, \dots, n), \end{aligned} \quad (3.11)$$

because $a_{ii}(s) \equiv [a_{ii}(s)]_+^v - [a_{ii}(s)]_-^v$, and the function $[a_{ii}(s)]_-^v$ is nondecreasing.

It is evident that there exist positive numbers r_1 and δ such that

$$|d_j a_{ii}(t)| < r_1 \quad \text{for } t > t_0 + \delta \quad (j = 1, 2, i = 1, \dots, n). \quad (3.12)$$

Therefore, by Lemma 2.5 (see (2.19)) and (3.12), we get

$$v(a_{ik})(\zeta) - v(a_{ik})(\xi) \leq (1 + r_1)(V(\mathcal{A}(a_{0ii}, a_{ik})(\zeta) - V(\mathcal{A}(a_{0ii}, a_{ik})(\xi)))$$

and

$$v(f_i)(\zeta) - v(f_i)(\xi) \leq (1 + r_1)(V(\mathcal{A}(a_{0ii}, f_i)(\zeta) - V(\mathcal{A}(a_{0ii}, f_i)(\xi))) \quad \text{for } t_0 + \delta < \xi < \zeta \quad (i, k = 1, \dots, n).$$

Thanks to the above estimate, from (3.11) we conclude that

$$w_i(t) - w_i(\tau) \leq v_i(t) - v_i(\tau) \quad \text{for } t_0 + \delta < \tau < t \quad (i = 1, \dots, n), \quad (3.13)$$

where

$$\begin{aligned} v_i(t) &= b_{ii} \int_{a_2}^t |y_i(s)| ds + (1 + r_1) \sum_{k=1, k \neq i}^n b_{ik} \int_{a_2}^t |y_k(s)| ds + (1 + r_1)(V(\mathcal{A}(a_{0ii}, f_i)(t) - V(\mathcal{A}(a_{0ii}, f_i)(a_2))) \\ &\quad \text{for } t > t_0 + \delta \quad (i = 1, \dots, n). \end{aligned}$$

Now consider the case where $t \in I_{t_0}^+(\delta)$.

By the definition of a solution of system (1.1), we find

$$\begin{aligned} |d_j x_i(t)| &\leq \sum_{k=1}^n |x_k(t)| |d_j a_{ik}(t)| + |d_j f_i(t)| = \sum_{k=1}^n |t - t_0|^{\mu_i} |y_k(t)| |d_j a_{ik}(t)| + |d_j f_i(t)| \\ &\quad \text{for } t \in I_{t_0}^+(\delta) \quad (j = 1, 2, i = 1, \dots, n). \end{aligned}$$

Hence, due to (3.10) and (3.13), we get

$$\sigma_{ji}(t) \leq (1+r)\xi_j(t) < (1+r)r_0 \quad \text{for } t \in I_{t_0}^+(\delta) \quad (j = 1, 2, i = 1, \dots, n),$$

where

$$\sigma_{ji}(t) = \sum_{\tau \in I_{ij}} |d_j x_i(\tau)| \quad \text{for } t \in I_{t_0}^+(\delta) \quad (j = 1, 2, i = 1, \dots, n).$$

So, there exist the nonnegative finite limits

$$\lim_{t \rightarrow t_0^+} \sigma_{ji}(t) = c_{ji} \quad (j = 1, 2, i = 1, \dots, n). \quad (3.14)$$

On the other hand, it is not difficult to verify the equality

$$w_i(t) - w_i(\tau) = s_0(|x_i|)(t) - s_0(|x_i|)(\tau) + \sum_{j=1}^2 \int_{\tau}^t \operatorname{sgn} x_i(s) ds_j(x_i)(s) \quad \text{for } \tau, t \in I_{t_0}^+ \quad (i = 1, \dots, n),$$

and thus to obtain

$$w_i(t) - w_i(\tau) \leq s_0(|x_i|)(t) - s_0(|x_i|)(\tau) + \sum_{j=1}^2 (\sigma_{ji}(\tau) - \sigma_{ji}(t)) \quad \text{for } \tau < t, \tau, t \in I_{t_0}^+ \quad (i = 1, \dots, n),$$

and

$$|w_i(t) - w_i(\tau)| \leq |s_0(|x_i|)(t) - s_0(|x_i|)(\tau)| + \sum_{j=1}^2 |\sigma_{ji}(\tau) - \sigma_{ji}(t)| \quad \text{for } \tau, t \in I_{t_0}^+ \quad (i = 1, \dots, n). \quad (3.15)$$

Moreover, by condition (3.14), the equalities

$$s_0(|x_i|)(t) - s_0(|x_i|)(\tau) = |x_i(t)| - |x_i(\tau)| - \sum_{j=1}^2 (s_j(|x_i|)(t) - s_j(|x_i|)(\tau))$$

$$\text{for } \tau < t, \tau, t \in I_{t_0}^+ \quad (i = 1, \dots, n), \quad \lim_{t \rightarrow t_0^+} x_i(t) = 0 \quad (i = 1, \dots, n),$$

and the inequalities

$$|d_j|x_i|(t)| \leq |d_j x_i(t)| \quad \text{for } t \in I_{t_0}^+ \quad (j = 1, 2, i = 1, \dots, n),$$

we conclude that there exist the finite limits

$$\lim_{t \rightarrow t_0^+} s_0(|x_i|)(t) = c_{0i} \quad (i = 1, \dots, n). \quad (3.16)$$

Let now

$$v_i(t) = s_0(|x_i|)(t) - \sum_{j=1}^2 \sigma_{ji}(t) \quad \text{for } t \in I_{t_0}^+(\delta) \quad (i = 1, \dots, n).$$

Then by (3.14) and (3.16) there exist the finite limits

$$\lim_{t \rightarrow t_0^+} v_i(t) = c_{0i} - c_{j1} - c_{j2} \quad (i = 1, \dots, n).$$

Consequently, by (3.15) there exist the finite one-sided limits $w_i(t_0^+)$ ($i = 1, \dots, n$) as well. So the conditions of Lemma 2.4 are fulfilled on $I_{t_0}^+$. Analogously, we show the validity of conditions on the set I_{t_0} as well. The statement of the remark follows from the lemma. $\square$

Proof of Corollary 1.7. The proof of the corollary is analogous to that of Corollary 1.4. We only note that the estimate

$$0 < c_i(t, \tau) \leq 1 \quad \text{for } (t - t_0)(\tau - t_0) > 0 \quad (i = 1, \dots, n)$$

is fulfilled instead of (3.8), and that conditions (1.21) and (1.22) guarantee the fulfillment of (1.16) and (1.17). $\square$

The proof of the statement of Remark 1.8 is analogous to the proof of Remark 1.6 if we assume $\mu_i = 1$ ($i = 1, \dots, n$).

Proof of Corollary 1.9. First, consider the case where $t > t_0$. Since a_{ii} ($i = 1, \dots, n$) are the functions with bounded local variations on I , we conclude that

$$v(\alpha_i)(t) - v(\alpha_i)(\tau) \leq v(a_{ii})(t) - v(a_{ii})(\tau) \quad \text{for } t_0 \leq \tau < t \quad (i = 1, \dots, n), \quad (3.17)$$

where

$$\alpha_i(t) \equiv [a_{ii}(t)]_+^v \quad (i = 1, \dots, n).$$

Moreover, it is evident that

$$\lim_{t \rightarrow t_0+} v(a_{ii})(t) = v(a_{ii})(t_0+) \quad (i = 1, \dots, n).$$

From this, by (3.17) and (1.25) we conclude that there exists a small positive number δ such that

$$b_0(v(\alpha_i)(t) - v(\alpha_i)(t_0+)) \leq \frac{1}{2n^2} \quad \text{for } t \in I_{t_0}^+(\delta) \quad (i = 1, \dots, n), \quad (3.18)$$

and

$$b_0 \left| \int_{t_0+}^t |\tau - t_0|^{\lambda_i - \lambda_k} dV(\mathcal{A}(a_{0ii}, a_{ik}))(\tau) \right| \leq \frac{1}{2n^2} \quad \text{for } t \in I_{t_0}^+(\delta) \quad (i \neq k, i, k = 1, \dots, n), \quad (3.19)$$

where

$$b_0 = \sup\{\exp(v(a_{ii}^*)(t) - v(a_{ii}^*)(t_0+)) : t \in I_{t_0}^+(\delta)\} \quad (i = 1, \dots, n).$$

On the other hand, taking into account (1.11) and (1.24), from (1.15) it follows that

$$0 < c_i(t, \tau) \leq b_0 |t - t_0|^{-\lambda_i} |\tau - t_0|^{\lambda_i} \quad \text{for } t, \tau \in I_{t_0}^+(\delta), \tau < t \quad (i = 1, \dots, n). \quad (3.20)$$

Therefore, estimate (1.12) is valid, where

$$h_i(t) = |t - t_0|^{-\lambda_i} \quad (i = 1, \dots, n). \quad (3.21)$$

Thanks to (3.20) and (3.21), we see that conditions (1.26), (3.18) and (3.19) guarantee the validity of conditions (1.13) and (1.14), where $b_{ik}(t) = (2n^2)^{-1}$ ($i, k = 1, \dots, n$). Hence the matrix $B = (b_{ik})_{i,k=1}^n$ satisfies condition (1.6).

Analogously, we show the validity of the corresponding conditions of Theorem 1.2 for the case $t < t_0$.

So all conditions of Theorem 1.2 hold. The solvability of problem (1.1), (1.2) immediately follows from Theorem 1.2. $\square$

Remark 1.10 can be verified in the same way as the above considered remarks.

References

- [1] M. Ashordia, On the general and multipoint boundary value problems for linear systems of generalized ordinary differential equations, linear impulse and linear difference systems, *Mem. Differ. Equ. Math. Phys.* **36** (2005), 1–80.
- [2] M. Ashordia, The initial problem for linear systems of generalized ordinary differential equations, linear impulsive and ordinary differential systems. Numerical solvability, *Mem. Differ. Equ. Math. Phys.* **78** (2019), 1–162.
- [3] M. T. Ashordia, On boundary value problems for systems of linear generalized ordinary differential equations with singularities (in Russian), *Differ. Uravn.* **42** (2006), no. 3, 291–301; translation in *Differ. Equ.* **42** (2006), no. 3, 307–319.
- [4] M. T. Ashordia, On some boundary value problems for linear generalized differential systems with singularities (in Russian), *Differ. Uravn.* **46** (2010), no. 2, 163–177; translation in *Differ. Equ.* **46** (2010), no. 2, 167–181.
- [5] M. Ashordia, On two-point singular boundary value problems for systems of linear generalized ordinary differential equations, *Mem. Differ. Equ. Math. Phys.* **56** (2012), 9–35.
- [6] V. A. Čechik, Investigation of systems of ordinary differential equations with a singularity (in Russian), *Trudy Moskov. Mat. Obšč.* **8** (1959), 155–198.

- [7] I. Kiguradze, *Some Singular Boundary Value Problems for Ordinary Differential Equations* (in Russian), Izdat. Tbilis. University, Tbilisi, 1975.
- [8] I. Kiguradze, *The Initial Value Problem and Boundary Value Problems for Systems of Ordinary Differential Equations. Vol. I. Linear Theory* (in Russian), Metsniereba, Tbilisi, 1997.
- [9] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter (in Russian), *Czechoslovak Math. J.* **7(82)** (1957), 418–449.
- [10] Š. Schwabik, *Generalized Ordinary Differential Equations*, Ser. Real Anal. 5, World Scientific, River Edge, 1992.
- [11] Š. Schwabik, M. Tvrdý and O. Vejvoda, *Differential and Integral Equations. Boundary Value Problems and Adjoints*, D. Reidel, Dordrecht, 1979.