Research Article

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A note on the trace inequality for Riesz potentials

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Abstract: We establish a necessary and sufficient condition on a non-negative locally integrable function ν guaranteeing the (trace) inequality

$$\|I_\alpha f\|_{L^p_\nu(\mathbb{R}^n)} \leq C \|f\|_{L^{p,1}(\mathbb{R}^n)}$$

for the Riesz potential I_α , where $L^{p,1}(\mathbb{R}^n)$ is the Lorentz space. The same problem is studied for potentials defined on spaces of homogeneous type.

Keywords: Riesz potential, trace inequality, weight

MSC 2010: 26A33, 42B35

1 Introduction

Trace inequalities for Riesz potentials I_α deal with non-negative measures ν such that

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q d\nu(x) \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}. \quad (1.1)$$

Adams [1] proved that the necessary and sufficient condition on ν guaranteeing (1.1) for $1 < p < q < \infty$ and $0 < \alpha < \frac{n}{p}$ is that the measure ν satisfies the following condition: there is a positive constant C such that for all balls $B \subset \mathbb{R}^n$,

$$\nu(B) \leq C |B|^{(\alpha/n-1/p)q}.$$

The Riesz potential operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \quad x \in \mathbb{R}^n,$$

plays an important role in the investigation of PDEs. It is worth mentioning its role in the theory of Sobolev's embeddings (see, e.g., [14]).

An appropriate fractional maximal operator is given by the formula

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_B |f(y)| dy, \quad 0 \leq \alpha < n, \quad x \in \mathbb{R}^n.$$

Here, $M_0 f = Mf$ is the Hardy–Littlewood maximal function having much importance in Harmonic Analysis, in particular, in the theory of singular integrals (see, e.g., [18]).

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Let v be a non-negative locally integrable function on $\mathbb{R}^n$. We are interested in inequality (1.1) for $dv(x) = v(x) dx$, i.e.,

$$\|I_\alpha f\|_{L^q_v(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \quad (1.2)$$

In this case by the result of Adams [1], the condition

$$[v]_{p,q,\alpha} := \sup_B (v(B))^{1/q} |B|^{\alpha/n-1/p} < \infty, \quad (1.3)$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, is simultaneously necessary and sufficient whenever $1 < p < q < \infty$ and $0 < \alpha < \frac{n}{p}$. In the case $p = q$ the implication (1.2) $\implies$ (1.3) can be checked easily by considering the test functions χ_B . However, (1.3) $\implies$ (1.2) is not true (see the counterexamples in [2, 8] for a measure v , and [10] for a non-negative function v).

Our aim is to find a Lorentz space $L^{p,s}$ which is narrower than the class $L^p(\mathbb{R}^n)$ (i.e., $s < p$) and for which the inequality

$$\|I_\alpha f\|_{L^p_v(\mathbb{R}^n)} \leq C \|f\|_{L^{p,s}(\mathbb{R}^n)} \quad (1.4)$$

holds if and only if (1.3) is satisfied for $p = q$. In particular, we show that (1.4) is equivalent to condition (1.3) for $s = 1$. The question whether this holds for $1 < s < p$ remains open.

As is known, various criteria were obtained for (1.2) with $p = q$ (see [2, 8, 12, 13, 15]). For the solution of the two-weight problem for Riesz potential operators I_α , we refer to [5, 17] (see also the monograph [9]).

Inequality (1.2) for $p = q$ implies the estimate

$$\|f\|_{L^q_v(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad f \in C_0^\infty,$$

which follows from the estimate

$$|f(x)| \leq CI_1(|\nabla f|)(x).$$

The following Fefferman–Phong-type theorem [4] holds true.

Theorem A. *Let $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$. Then the inequality*

$$\|I_\alpha f\|_{L^p_v} \leq C [v]_{p,r,\alpha}^* \|f\|_{L^p}$$

is fulfilled for some $p < r$, where

$$[v]_{p,r,\alpha}^* := \sup_B |B|^{\alpha/n-1/r} \left(\int_B v^{r/p}(x) dx \right)^{1/r} < \infty. \quad (1.5)$$

Remark 1.1. It is easy to see that by Hölder's inequality we have that condition (1.5) is stronger than (1.3) for $p = q$, in particular, $[v]_{p,\alpha} \leq [v]_{p,r,\alpha}^*$ for $r > p$, where $[v]_{p,\alpha} = [v]_{p,p,\alpha}$.

2 Preliminaries

Let f be a measurable function on $\mathbb{R}^n$ and let $1 \leq p < \infty$, $1 \leq s \leq \infty$.

We say that f belongs to the Lorentz space $L^{p,s}$ if

$$\|f\|_{L^{p,s}} = \begin{cases} \left(s \int_0^\infty (|\{x \in \mathbb{R}^n : |f(x)| > \tau\}|)^{s/p} \tau^{s-1} d\tau \right)^{1/s} & \text{if } 1 \leq s < \infty, \\ \sup_{s>0} s (|\{x \in \mathbb{R}^n : |f(x)| > s\}|)^{1/p} & \text{if } s = \infty \end{cases}$$

is finite.

If $p = s$, then $L^{p,s}$ coincides with the Lebesgue space L^p .

Denote by f^* a non-increasing rearrangement of f . Then, by integration by parts, it can be checked that (see also [7])

$$\|f\|_{L^{p,s}} = \begin{cases} \left(\frac{s}{p} \int_0^\infty (t^{1/p} f^*(t))^s \frac{dt}{t} \right)^{1/s} & \text{if } 1 \leq s < \infty, \\ \sup_{t>0} \{t^{1/p} f^*(t)\} & \text{if } s = \infty. \end{cases}$$

Now we list some useful properties of Lorentz spaces (see, e.g., [7]):

- (i) $\|\chi_E\|_{L^{p,s}} = |E|^{1/p}$.
- (ii) If $1 \leq p < \infty$ and $s_2 \leq s_1$, then $L^{p,s_2} \hookrightarrow L^{p,s_1}$ with the embedding constant C_{p,s_1,s_2} depending only on p , s_1 and s_2 .
- (iii) There is a positive constant $C_{p,s}$ such that

$$C_{p,s}^{-1} \|f\|_{L^{p,s}} \leq \sup_{\|h\|_{L^{p',s'}} \leq 1} \left| \int_X f(x) h(x) d\mu(x) \right| \leq C_{p,s} \|f\|_{L^{p,s}}$$

for every $f \in L^{p,s}_w$, where $p' = p/(p-1)$, $s' = s/(s-1)$.

- (iv) (Hölder's inequality) Let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$. Then

$$\|f_1 f_2\|_{L^{p,s}} \leq C \|f_1\|_{L^{p_1,s_1}} \|f_2\|_{L^{p_2,s_2}}$$

for all $f_1 \in L^{p_1,s_1}$ and $f_2 \in L^{p_2,s_2}$, where $C = C_{p,s,p_1,p_2,s_1,s_2}$.

In the sequel, for a non-negative function v and a set E the symbol $v(E)$ will denote the integral

$$v(E) := \int_E v(x) dx.$$

By the symbol $A \approx B$ we mean that there exists a positive constant c such that

$$c^{-1} A \leq B \leq c A.$$

3 Main result

The main result of this note reads as follows.

Theorem 3.1. Let $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$. Suppose that v is a non-negative locally integrable function on $\mathbb{R}^n$. Then the following statements are equivalent:

- (i) There is a positive constant C such that for all $f \in L^{p,1}(\mathbb{R}^n)$,

$$\|I_\alpha f\|_{L^p_v(\mathbb{R}^n)} \leq C \|f\|_{L^{p,1}(\mathbb{R}^n)}. \quad (3.1)$$

- (ii) There is a positive constant c such that for all $f \in L^{p,1}(\mathbb{R}^n)$,

$$\|M_\alpha f\|_{L^p_v(\mathbb{R}^n)} \leq c \|f\|_{L^{p,1}(\mathbb{R}^n)}. \quad (3.2)$$

- (iii)

$$[v]_{p,\alpha} = \sup_B (v(B))^{1/p} |B|^{\alpha/n-1/p} < \infty.$$

Moreover, if C and c are best constants in (3.1) and (3.2), respectively, then

$$C \approx c \approx [v]_{p,\alpha}.$$

To prove this theorem we need some lemmas.

For a non-negative a.e. finite function w we set $E_w := \{x \in \mathbb{R}^n : w(x) \neq 0\}$.

We also set $p' = \frac{p}{p-1}$ for $1 \leq p \leq \infty$.

Lemma 3.1. Let $1 < p < \infty$ and $0 < \alpha < n/p'$. Then the condition

$$[w]_{p,\alpha}' \equiv \sup_B \left(\int_B w^{1-p'}(x) \chi_{E_w}(x) dx \right)^{\frac{1}{p'}} |B|^{\frac{\alpha}{n}-\frac{1}{p'}} < \infty$$

guarantees that there is a positive constant $C = C_{\alpha,p}$ such that for all $f \in L^p_w(E_w)$,

$$\|M_\alpha(f\chi_{E_w})\|_{L^{p,\infty}(\mathbb{R}^n)} \leq C\|f\chi_{E_w}\|_{L^p_w(\mathbb{R}^n)}.$$

Proof. Observe that By Hölder's inequality we find that

$$\begin{aligned} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)|\chi_{E_w}(y) dy &= \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)|\chi_{E_w}(y) w^{\frac{1}{p}}(y) w^{-\frac{1}{p}}(y) dy \\ &\leq \frac{1}{|B|^{1-\frac{\alpha}{n}}} \left(\int_B |f(y)|^p w(y) \chi_{E_w}(y) dy \right)^{\frac{1}{p}} \left(\int_B w^{1-p'}(y) \chi_{E_w}(y) dy \right)^{\frac{1}{p'}} \\ &\leq [w]_{p,\alpha}' \left(\frac{1}{|B|} \int_B |f(y)|^p \chi_{E_w} w(y) dy \right)^{\frac{1}{p}} \\ &= [w]_{p,\alpha}' (M(|f|^p \chi_{E_w} w))^{\frac{1}{p}}, \end{aligned}$$

where M is the Hardy–Littlewood maximal operator.

Hence

$$M_\alpha(f\chi_{E_w})(x) \leq [w]_{p,\alpha}' (M(|f|^p \chi_{E_w} w))^{\frac{1}{p}}.$$

Consequently,

$$\begin{aligned} |\{x \in \mathbb{R}^n : M_\alpha(f\chi_{E_w})(x) > \lambda\}| &\leq |\{x \in \mathbb{R}^n : C[w]_{p,\alpha}' (M(|f\chi_{E_w}|^p w)(x))^{\frac{1}{p}} > \lambda\}| \\ &\leq \frac{C[w]_{p,\alpha}'}{\lambda^p} \| |f|^p \chi_{E_w} w \|_{L^1} \\ &= \frac{C[w]_{p,\alpha}'}{\lambda^p} \|f\|_{L^p_w(E_w)}^p. \end{aligned}$$

□

Lemma 3.2. Let $1 < p < \infty$ and $0 < \alpha < n/p'$. If $[w]_{p,\alpha}' < \infty$ (see Lemma 3.1), then there is a positive constant $C = C_{p,\alpha}$ such that for all $f \in L^p_w(E_w)$,

$$\|I_\alpha(f\chi_{E_w})\|_{L^{p,\infty}(\mathbb{R}^n)} \leq C_{p,\alpha} \|f\chi_{E_w}\|_{L^p_w(\mathbb{R}^n)}.$$

Proof. It is known (see, e.g., [16, Theorem 1] for a more general setting) that

$$\|I_\alpha g\|_{L^{p,\infty}(\mathbb{R}^n)} \approx \|M_\alpha g\|_{L^{p,\infty}(\mathbb{R}^n)}$$

for a measurable function g . Thus, by Lemma 3.1, the result follows. □

Lemma 3.3. Let $1 < p < \infty$ and $0 < \alpha < n/p'$. If $[w]_{p,\alpha}' < \infty$, then there is a positive constant $C = C_{p,\alpha}$ such that for all $f \in L^{p',1}(E_w)$,

$$\|I_\alpha f\|_{L^{p',1}_{w^{1-p'}}(E_w)} \leq C[w]_{p,\alpha}' \|f\|_{L^{p',1}(E_w)}.$$

Proof. For $f \geq 0$, we have

$$\begin{aligned} \|I_\alpha f\|_{L^{p',1}_{w^{1-p'}}(E_w)} &= \sup_{\|g\|_{L^p_w(E_w)} \leq 1} \left| \int_{\mathbb{R}^n} (I_\alpha f)(x) g(x) w^{-\frac{1}{p}}(x) w^{\frac{1}{p}}(x) \chi_{E_w}(x) dx \right| \\ &\leq \sup_{\|g\|_{L^p_w(E_w)} \leq 1} \left(\int_{\mathbb{R}^n} (I_\alpha f)(x) |g(x)| \chi_{E_w}(x) dx \right) \\ &= \sup_{\|g\|_{L^p_w(E_w)} \leq 1} \int_{E_w} I_\alpha(|g|\chi_{E_w})(y) f(y) dy \\ &\leq \sup_{\|g\|_{L^p_w(E_w)} \leq 1} \|I_\alpha |g|\|_{L^{p,\infty}(E_w)} \|f\|_{L^{p',1}(E_w)} \\ &\leq C_{p,\alpha} [w]_{p,\alpha}' \|f\|_{L^{p',1}(E_w)}. \end{aligned}$$

□

Proof of Theorem 3.1. (iii) $\implies$ (i): Observe that if $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and v is a non-negative function satisfying $[v]_{p,\alpha} < \infty$, then $|\{x \in \mathbb{R}^n : v(x) = \infty\}| = 0$. Hence

$$|\{x \in \mathbb{R}^n : v^{1-p}(x) = 0\}| = 0.$$

Consequently, since

$$[v]_{p,\alpha} = \sup_B \left(\int_B v^{(1-p)(1-p')}(x) dx \right)^{\frac{1}{p}} |B|^{\frac{\alpha}{n} - \frac{1}{p}} < \infty,$$

from Lemma 3.3 we have

$$\|I_\alpha f\|_{L^p_v(\mathbb{R}^n)} \leq C_{p,\alpha} [v]_{p,\alpha} \|f\|_{L^{p,1}(\mathbb{R}^n)}.$$

The pointwise estimate

$$M_\alpha g(x) \leq I_\alpha g(x), \quad g \geq 0,$$

implies the implication (i) $\implies$ (ii).

Taking the test functions

$$f_{B_0} = \chi_{B_0}$$

in (3.2), where B_0 is a ball in $\mathbb{R}^n$, using the estimate

$$M_\alpha f_{B_0}(x) \geq C_{n,\alpha} |B_0|^{\frac{\alpha}{n}}, \quad x \in B_0,$$

and observing that

$$\|f_{B_0}\|_{L^{p,1}} = c |B_0|^{\frac{1}{p}},$$

we can easily see that (ii) implies

$$v(B_0)^{1/p} \leq C |B_0|^{1/p - \alpha/n},$$

with a positive constant C independent of B_0 . The theorem has been proved. $\square$

4 The case of spaces of homogeneous type

Let (X, d, μ) be a quasi-metric measure space with a quasi-metric d and measure μ . A quasi-metric d is a function $d: X \times X \rightarrow [0, \infty)$ which satisfies the following conditions:

- (i) $d(x, y) = 0$ if and only if $x = y$.
- (ii) For all $x, y \in X$, one has $d(x, y) = d(y, x)$.
- (iii) There is a positive constant κ such that $d(x, y) \leq \kappa(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

In the sequel, we will assume that the balls $B(x, r) := \{y \in X : d(x, y) < r\}$ are measurable with positive measure μ for all $x \in X$ and $r > 0$.

If μ satisfies the doubling condition

$$\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)),$$

with a positive constant C_μ independent of x and r , then we say that (X, d, μ) is a space of homogeneous type (SHT). Throughout the paper, we assume that (X, d, μ) is an SHT.

For example, rectifiable curves in $\mathbb{C}$ with Euclidean distance and arc-length measure satisfying the Carleson (regularity) condition, nilpotent Lie groups with Haar measure, and domains in $\mathbb{R}^n$ with the so-called $\mathcal{A}$ condition are examples of an SHT. For the definition, examples and some properties of an SHT, see, e.g., the paper [11] and the monographs [3, 19].

For a given quasi-metric measure space (X, d, μ) and q satisfying $1 \leq q \leq \infty$, as usual, we will denote by $L^q = L^q(X, \mu)$ the Lebesgue space equipped with the standard norm. Let $L^{p,s}(X, \mu)$ be the Lorentz space defined on an SHT (X, d, μ) .

Let us denote by $K_\alpha f$ the Riesz potential of a μ -measurable function f defined by the formula

$$K_\alpha f(x) = \int_X \mu(B_{xy})^{\alpha-1} f(y) d\mu(y), \quad x \in X,$$

where $0 < \alpha < 1$ and $B_{xy} := B(x, d(x, y))$.

An appropriate fractional maximal function has the form

$$\mathcal{M}_\alpha f(x) = \sup_{B \ni x} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)| d\mu(y), \quad x \in X.$$

The following trace inequality for an SHT was proved by Gabidzashvili (see [6]).

Theorem B. Let $1 < p < q < \infty$ and $0 < \alpha < \frac{1}{p}$. Suppose that (X, d, μ) is an SHT and ν is another measure on X . Then the inequality

$$\|K_\alpha f\|_{L^q(X, \nu)} \leq C \|f\|_{L^p(X, \mu)}$$

holds if and only if

$$\sup_B \nu(B)^{1/q} \mu(B)^{\alpha-1/p} < \infty,$$

where the supremum is taken over all balls $B \subset X$.

Using the proof of Theorem 3.1, we can formulate the same result for an SHT. In particular, the following theorem holds true.

Theorem 4.1. Let $1 < p < \infty$ and $0 < \alpha < \frac{1}{p}$. Suppose that (X, d, μ) is an SHT. Assume that ν is a non-negative μ -locally integrable a.e. finite function on X . Then the following statements are equivalent:

(i) There is a positive constant C such that for all $f \in L^{p,1}(X, \mu)$,

$$\|K_\alpha f\|_{L^p_v(X, \mu)} \leq C \|f\|_{L^{p,1}(X, \mu)}. \quad (4.1)$$

(ii) There is a positive constant c such that for all $f \in L^{p,1}(X, \mu)$,

$$\|\mathcal{M}_\alpha f\|_{L^p_v(X, \mu)} \leq c \|f\|_{L^{p,1}(X, \mu)}. \quad (4.2)$$

(iii)

$$[v]_{p, \alpha, X, \mu} = \sup_B \left(\int_B v(x) d\mu(x) \right)^{1/p} \mu(B)^{\alpha-1/p} < \infty.$$

Moreover, if C and c are best constants in (4.1) and (4.2), respectively, then

$$C \approx c \approx [v]_{p, \alpha, X, \mu}.$$

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