Research Article

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Higher integrability and reverse Hölder inequalities in the limit cases

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Abstract: We study the higher integrability of weights satisfying a *reverse Hölder inequality*

$$\left(\int_I u^\beta\right)^{\frac{1}{\beta}} \leq B \left(\int_I u^\alpha\right)^{\frac{1}{\alpha}}$$

for some $B > 1$ and given $\alpha < \beta$, in the *limit cases* when $\alpha \in \{-\infty, 0\}$ and/or $\beta \in \{0, +\infty\}$. The results apply to the Gehring and Muckenhoupt weights and their corresponding limit classes.

Keywords: Reverse Hölder inequalities, Muckenhoupt weights, higher integrability

MSC 2010: Primary 46E30; secondary 26D15

1 Motivations and main results

A *weight* is a non-negative locally integrable function. Let u be a weight defined on a bounded interval $I_0 \subset \mathbb{R}$. For real and nonzero exponents α and β with $\alpha < \beta$, we say that u verifies a *reverse Hölder inequality* if there exists a constant B such that, for every subinterval $I \subset I_0$,

$$\left(\int_I u^\beta\right)^{\frac{1}{\beta}} \leq B \left(\int_I u^\alpha\right)^{\frac{1}{\alpha}}, \quad (1.1)$$

where the barred integral stands for the integral mean $\frac{1}{|I|} \int_I u$. In this case, we write $u \in \text{RH}^{\alpha, \beta}(B)$. The notation recalls that the reverse of inequality (1.1) is always true for $B = 1$ thanks to the *Hölder's inequality*. Therefore, in (1.1), necessarily, $B \geq 1$, and the equality prevails in the case of constant functions. The quantity

$$\text{RH}^{\alpha, \beta}(u) := \sup_{I \subset I_0} \frac{\left(\int_I u^\beta\right)^{\frac{1}{\beta}}}{\left(\int_I u^\alpha\right)^{\frac{1}{\alpha}}} \quad (1.2)$$

identifies the smallest constant verifying inequality (1.1) and is called the $\text{RH}^{\alpha, \beta}$ *constant* (or *norm* or *characteristic*) of the weight u . By setting

$$\mathcal{M}_\alpha(u)_I = \begin{cases} \text{ess inf}_I u & \text{for } \alpha = -\infty, \\ \left(\int_I u^\alpha\right)^{\frac{1}{\alpha}} & \text{for } \alpha \in \mathbb{R} - \{0\}, \\ \exp\left(\int_I \ln u\right) & \text{for } \alpha = 0, \\ \text{ess sup}_I u & \text{for } \alpha = +\infty, \end{cases}$$

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we can extend the definition of $\text{RH}^{\alpha,\beta}(u)$ replacing (1.2) by

$$\text{RH}^{\alpha,\beta}(u) := \sup_{I \subseteq I_0} \frac{\mathcal{M}_\beta(u)_I}{\mathcal{M}_\alpha(u)_I}, \quad (1.3)$$

which makes the quantity $\text{RH}^{\alpha,\beta}(u)$ well defined for all $\alpha, \beta \in \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with $\alpha < \beta$.

These classes have been introduced by Bojarski in [1], and their interest is based on the fact that they include the *Muckenhoupt* classes and the *Gehring* classes as particular cases, being

$$A_p(w) := \text{RH}^{\frac{1}{1-p},1}(w), \quad A_1(w) := \text{RH}^{-\infty,1}(w), \quad A_\infty(w) := \text{RH}^{0,1}(w), \quad (1.4)$$

$$G_q(v) := \text{RH}^{1,q}(v), \quad G_\infty(v) := \text{RH}^{1,\infty}(v). \quad (1.5)$$

Indeed, it turns out that the class $\text{RH}^{\alpha,\beta}$ seems to be the right environment to study the main properties of A_p and G_q weights (see [13, 14, 19, 20, 24, 28]):

- *self-improving of integrability*: there exist s_p and t_q such that

$$A_p(w) < \infty \implies A_s(w) < \infty \quad \text{for all } s \text{ such that } s_p < s \leq p, \quad (1.6)$$

$$G_q(v) < \infty \implies G_t(v) < \infty \quad \text{for all } t \text{ such that } q \leq t < t_q; \quad (1.7)$$

- *transition*: having $\bigcup_{p>1} A_p = \bigcup_{q>1} G_q$, there exist σ_p and τ_q such that

$$A_p(w) < \infty \implies G_\sigma(w) < \infty \quad \text{for all } \sigma \text{ such that } 1 \leq \sigma < \sigma_p, \quad (1.8)$$

$$G_q(v) < \infty \implies A_\tau(w) < \infty \quad \text{for all } \tau \text{ such that } 1 \leq \tau_q < \tau. \quad (1.9)$$

In fact, (1.6), (1.7), (1.8) and (1.9) can be proved to be special cases of the following higher integrability property enjoyed by weights satisfying condition (1.1):

$$\text{RH}^{\alpha,\beta}(u) < \infty \implies \begin{cases} \text{RH}^{\nu,\beta}(u) < \infty & \text{for all } \nu \text{ such that } \gamma^- < \nu \leq \alpha, \\ \text{RH}^{\alpha,\tau}(u) < \infty & \text{for all } \tau \text{ such that } \beta \leq \tau < \gamma^+. \end{cases} \quad (1.10)$$

We refer to the optimal bounds γ^+ and γ^- as the *sharp higher integrability exponents*, and to $\text{RH}^{\nu,\beta}(u)$ and $\text{RH}^{\alpha,\tau}(u)$ as the *sharp higher integrability constants*.

The problem of finding these sharp bounds has been fully solved in dimension one (see [13, 20, 28]). In particular, in [20] (see Section 2, Theorem 2.1), it has been proved that all those bounds can be obtained by mean of the unique two solutions to the following equation:

$$\omega(\alpha, \beta, x) = \text{RH}^{\alpha,\beta}(u), \quad (1.11)$$

where ω is an auxiliary continuous function defined by

$$\omega(\alpha, \beta, x) := \left(\frac{x - \alpha}{x} \right)^{\frac{1}{\alpha}} \left(\frac{x}{x - \beta} \right)^{\frac{1}{\beta}},$$

which is strictly increasing for $x < \min\{0, \alpha\}$ and strictly decreasing for $x > \max\{0, \beta\}$.

In a paper of 2009 [14], Korenovskii and Fomichev studied the higher integrability property (1.10) when α, β assume the limit values 0 or $\pm\infty$. They proved, case by case, that the *sharp higher integrability exponents* for functions in the limit classes can be obtained by passing to the limit as appropriate in the equation (1.11).

The aim of this paper is to complete their work by obtaining also the *sharp higher integrability constants* for the same limit classes.

In order to present the result, let us first show how it reads in the special case of Muckenhoupt weights by recalling the relationships existing among the constants $A_1(w)$, $A_p(w)$ and $A_\infty(w)$. Namely, the known theorem proved by Sbordone and Wik [25] states that

$$A_\infty(w) = \lim_{p \rightarrow \infty} A_p(w), \quad (1.12)$$

and more recently, it has been proved that (at least) in dimension one, a corresponding property holds true for the class A_1 (see [21]), i.e.,

$$A_1(w) = \lim_{p \rightarrow 1^+} A_p(w). \quad (1.13)$$

Therefore, it is natural to ask if similar connections hold true for the Gehring classes and in more general classes $\text{RH}^{\alpha,\beta}$. The answer is given by the following theorem.

Theorem 1.1. *Let α, β be real nonzero numbers such that $\alpha < \beta$.*

- *If u belongs to the limit class $\text{RH}^{L,\beta}$ with $L \in \{-\infty, 0\}$, then*

$$\text{RH}^{L,\beta}(u) = \lim_{\alpha \rightarrow L} \text{RH}^{\alpha,\beta}(u).$$

- *If u belongs to the limit class $\text{RH}^{\alpha,L}$ with $L \in \{0, +\infty\}$, then*

$$\text{RH}^{\alpha,L}(u) = \lim_{\beta \rightarrow L} \text{RH}^{\alpha,\beta}(u).$$

It is easy to observe that $A_1(u) = \text{RH}^{-\infty,1}(u)$ and $A_\infty(u) = \text{RH}^{0,1}(u)$ so that (1.12) and (1.13) are immediate corollaries of Theorem 1.1. Notice that this result is another step forward in the unification of the theory of Muckenhoupt and Gehring weights which provides a unique method and general theorems to study the entire set of classes (1.4) and (1.5).

The applications of the higher integrability properties of RHI weights cover several fields of analysis such as, to mention some examples, bounded maximal operator in weighted L^p spaces [18], quasiconformal mappings [8], L^p solvability of the Dirichlet problem [26], quasi-minimizers for one-dimensional Dirichlet integral [17] and many others. Good references on the theory are the books [9, 13].

This paper will proceed as follows. In Section 2, we recall the general theorem on the sharp integrability in RHI classes and show how it is made possible to achieve a unified approach to Muckenhoupt classes. In Section 3, we study the monotonicity and continuity of *sharp integrability exponents* and the *sharp integrability constants*. In Sections 4 to 9, we study each limit class according to the structure of the paper of Korenovskii and Fomichev [14]. For the convenience of the reader, we adopt, as much as possible, their notations. Theorem 1.1 will be proved separately for each limit case, and we will highlight how it works for interesting special cases of Muckenhoupt and Gehring classes.

2 Optimal integrability from reverse Hölder inequalities and the unified theory of Muckenhoupt weights

To state the theorem on the optimal higher integrability of weights satisfying a reverse Hölder inequality, we need to extend the definition of the function ω in order to cover the limit values of α and β . We set

$$\omega(\alpha, \beta, x) = \begin{cases} \left(\frac{x}{x-\beta}\right)^{\frac{1}{\beta}} & \text{for } \alpha = -\infty, \\ e^{-\frac{1}{x}} \left(\frac{x}{x-\beta}\right)^{\frac{1}{\beta}} & \text{for } \alpha = 0, \beta > 0, \\ \left(\frac{x-\alpha}{x}\right)^{\frac{1}{\alpha}} \left(\frac{x}{x-\beta}\right)^{\frac{1}{\beta}} & \text{for } \alpha, \beta \in \mathbb{R} - 0, \\ \left(\frac{x-\alpha}{x}\right)^{\frac{1}{\alpha}} e^{\frac{1}{x}} & \text{for } \alpha < 0, \beta = 0, \\ \left(\frac{x-\alpha}{x}\right)^{\frac{1}{\alpha}} & \text{for } \beta = +\infty. \end{cases} \quad (2.1)$$

With these positions, we can formulate the sharp higher integrability in $\text{RH}^{\alpha,\beta}$ classes.

Theorem 2.1 ([20]). Let $u \in \text{RH}^{\alpha, \beta}$ with $\alpha, \beta \in \mathbb{R} - \{0\}$ and $\alpha < \beta$. If $\gamma^- \leq \alpha$ and $\gamma^+ \geq \beta$ are two unique solutions to the equation $\omega(\alpha, \beta, \gamma^\pm) = \text{RH}^{\alpha, \beta}(u)$, then we have the following improvements of regularity:

(S₊) $u \in \text{RH}^{\alpha, \tau}$ for any τ such that $\beta \leq \tau < \tau_{\alpha, \beta} = \gamma^+$, and $\text{RH}^{\alpha, \tau}(u) = \omega(\alpha, \tau, \gamma^+)$;

(S₋) $u \in \text{RH}^{\nu, \beta}$ for any ν such that $\gamma^- = \nu_{\alpha, \beta} < \nu \leq \alpha$, and $\text{RH}^{\nu, \beta}(u) = \omega(\nu, \beta, \gamma^-)$.

The limit exponents $\nu_{\alpha, \beta}$ and $\tau_{\alpha, \beta}$ as well as the constants $\text{RH}^{\alpha, \tau}(u)$ and $\text{RH}^{\nu, \beta}(u)$ are sharp. Moreover, statement (S₊) also holds true in the limit cases $\alpha = -\infty$ and $\alpha = 0$, while statement (S₋) holds true also in the limit cases $\beta = 0$ and $\beta = +\infty$.

To show how this theorem provides a unification for the theory of Muckenhoupt weights, let us recall the definitions. The class A_p is defined as the set of weights w such that

$$A_p(w) = \sup_{I \subseteq I_0} \int_I w \left(\int_I w^{\frac{1}{1-p}} \right)^{p-1} < \infty.$$

In the limit cases $p = 1$ and $p = \infty$, the Muckenhoupt condition becomes, respectively,

$$A_1(w) = \sup_{I \subseteq I_0} \int_I w (\text{ess inf}_I w)^{-1} < \infty \quad \text{and} \quad A_\infty(w) = \sup_{I \subseteq I_0} \int_I w \exp \left(\int_I \log \frac{1}{w} \right) < \infty$$

(for alternative definitions of A_∞ , see, for example, [10, 23]).

For these classes, the transition property (1.8) encodes the well-known fact that Muckenhoupt weights satisfy a reverse Hölder inequality and the Gehring lemma on higher integrability (a classical reference is [3]). There are many papers where sharp bounds and sharp constants for the embeddings of Muckenhoupt classes in Gehring classes are proved case by case (see [2, 4–6, 11, 12, 15, 16, 20, 27]). We point out that these results have been achieved by means of independent theorems, one for each of the classes above. Instead, by using Theorem 2.1, we can reformulate the sharp results for the *transition property* in a new unified shape, which shows the following interesting symmetries (see [20]).

Theorem 2.2. Let $\omega(\alpha, 1, x)$ be defined as in (2.1). The following sharp embeddings hold true:

- $w \in A_1 \Rightarrow w \in G_q$ for all q such that $1 \leq q < \gamma_1$, with γ_1 such that

$$\omega(-\infty, 1, \gamma_1) = A_1(w), \quad G_q(w) = \omega(1, q, \gamma_1);$$

- $w \in A_p \Rightarrow w \in G_q$ for all q such that $1 \leq q < \gamma_p$, with γ_p such that

$$\omega(1/(1-p), 1, \gamma_p) = A_p(w), \quad G_q(w) = \omega(1, q, \gamma_p);$$

- $w \in A_\infty \Rightarrow w \in G_q$ for all q such that $1 \leq q < \gamma_\infty$, with γ_∞ such that

$$\omega(0, 1, \gamma_\infty) = A_\infty(w), \quad G_q(w) = \omega(1, q, \gamma_\infty).$$

A similar theorem holds true for the *self-improving property*, which also uses the same function $\omega(\alpha, \beta, x)$ (see [20]).

3 Monotonicity and continuity of sharp self-improving bounds

We start this section by recalling the monotonicity of the constants $\text{RH}^{\alpha, \beta}(u)$ with $\alpha, \beta \in \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ such that $\alpha < \beta$,

$$\alpha_1 < \alpha_2 < \beta \implies \text{RH}^{\alpha_1, \beta}(u) \geq \text{RH}^{\alpha_2, \beta}(u), \quad (3.1)$$

$$\alpha < \beta_1 < \beta_2 \implies \text{RH}^{\alpha, \beta_1}(u) \leq \text{RH}^{\alpha, \beta_2}(u). \quad (3.2)$$

The immediate consequence of the monotonicity is that, for $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$, we have

$$\text{RH}^{\alpha_1, \beta_2}(B) \subseteq \text{RH}^{\alpha_2, \beta_1}(B).$$

In the limit cases the previous inequalities become

$$\mathrm{RH}^{-\infty, \beta}(u) \geq \mathrm{RH}^{\alpha, \beta}(u) \geq \mathrm{RH}^{0, \beta}(u) \quad \text{for all } \beta > 0 \geq \alpha, \quad (3.3)$$

$$\mathrm{RH}^{\alpha, 0}(u) \leq \mathrm{RH}^{\alpha, \beta}(u) \leq \mathrm{RH}^{\alpha, +\infty}(u) \quad \text{for all } \alpha < 0 \leq \beta. \quad (3.4)$$

Let us now define the *self-improving sharp exponents* as follows:

$$\nu_{\alpha, \beta} = \nu_{\alpha, \beta}(B) = \inf\{\nu \leq \alpha : \mathrm{RH}^{\alpha, \beta}(u) \leq B \Rightarrow \mathrm{RH}^{\nu, \beta}(u) < \infty\}, \quad (3.5)$$

$$\tau_{\alpha, \beta} = \tau_{\alpha, \beta}(B) = \sup\{\tau \geq \beta : \mathrm{RH}^{\alpha, \beta}(u) \leq B \Rightarrow \mathrm{RH}^{\alpha, \tau}(u) < \infty\}. \quad (3.6)$$

We will refer to $\nu_{\alpha, \beta}$ and $\tau_{\alpha, \beta}$, respectively, as the *lower sharp exponent* and the *upper sharp exponent* for the class $\mathrm{RH}^{\alpha, \beta}(B)$.

Lemma 3.1. *Let $B > 1$ be a given real constant. Then we have*

$$\alpha_1 < \alpha_2 < \beta \implies \tau_{\alpha_1, \beta} \geq \tau_{\alpha_2, \beta}, \quad (3.7)$$

$$\alpha < \beta_1 < \beta_2 \implies \nu_{\alpha, \beta_1} \geq \nu_{\alpha, \beta_2}. \quad (3.8)$$

Proof. Let us set

$$U_{\alpha, \beta} = U_{\alpha, \beta}(B) = \{\tau \geq \beta : \mathrm{RH}^{\alpha, \beta}(u) \leq B \implies \mathrm{RH}^{\alpha, \tau}(u) < \infty\}$$

and prove that if $\alpha_1 < \alpha_2$, then $U_{\alpha_2, \beta} \subseteq U_{\alpha_1, \beta}$.

Let $\tau_0 \in U_{\alpha_2, \beta}$ so that

$$\mathrm{RH}^{\alpha_2, \beta}(u) \leq B \implies \mathrm{RH}^{\alpha_2, \tau_0}(u) < \infty,$$

and suppose $\mathrm{RH}^{\alpha_1, \beta}(u) \leq B$. As $\alpha_1 < \alpha_2$ implies $\mathrm{RH}^{\alpha_2, \beta}(u) \leq \mathrm{RH}^{\alpha_1, \beta}(u)$ (see (3.1)), we have

$$\mathrm{RH}^{\alpha_1, \beta}(u) \leq B \implies \mathrm{RH}^{\alpha_2, \beta}(u) \leq B \implies \mathrm{RH}^{\alpha_2, \tau_0}(u) < \infty.$$

Hence $\tau_0 \in U_{\alpha_1, \beta}$ and, since τ_0 is arbitrary, $U_{\alpha_2, \beta} \subseteq U_{\alpha_1, \beta}$. Passing to the supremum, we immediately have

$$\tau_{\alpha_2, \beta} = \sup U_{\alpha_2, \beta} \leq \sup U_{\alpha_1, \beta} = \tau_{\alpha_1, \beta},$$

which is (3.7).

To prove (3.8), we define

$$L_{\alpha, \beta} = L_{\alpha, \beta}(B) = \{\nu \leq \alpha : \mathrm{RH}^{\alpha, \beta}(u) \leq B \implies \mathrm{RH}^{\nu, \beta}(u) < \infty\}$$

and prove that if $\alpha < \beta_1 < \beta_2$, then $L_{\alpha, \beta_1} \subseteq L_{\alpha, \beta_2}$. Let $\nu_0 \in L_{\alpha, \beta_1}$ so that

$$\mathrm{RH}^{\alpha, \beta_1}(u) \leq B \implies \mathrm{RH}^{\nu_0, \beta_1}(u) < \infty,$$

and suppose $\mathrm{RH}^{\alpha, \beta_2}(u) \leq B$. As $\beta_1 < \beta_2$ implies $\mathrm{RH}^{\alpha, \beta_1}(u) \leq \mathrm{RH}^{\alpha, \beta_2}(u)$ (see (3.2)), we have

$$\mathrm{RH}^{\alpha, \beta_2}(u) \leq B \implies \mathrm{RH}^{\alpha, \beta_1}(u) \leq B \implies \mathrm{RH}^{\nu_0, \beta_1}(u) < \infty.$$

Hence $\nu_0 \in L_{\alpha, \beta_2}$ and, since ν_0 is arbitrary, $L_{\alpha, \beta_1} \subseteq L_{\alpha, \beta_2}$. Passing to the supremum, we have

$$\nu_{\alpha, \beta_1} = \inf L_{\alpha, \beta_1} \geq \inf L_{\alpha, \beta_2} = \nu_{\alpha, \beta_2},$$

which is (3.8). □

It is easy to see that definitions (3.5) and (3.6), as well as Lemma 3.1, are also applicable to the limit cases $\alpha \in \{-\infty, 0\}$ and/or $\beta \in \{0, +\infty\}$. It results that the following inequalities for the *lower and upper sharp exponents* correspond to the inequalities for the *sharp constants* (3.3) and (3.4).

Lemma 3.2. *The following inequalities hold:*

$$\tau_{-\infty, \beta} \geq \tau_{\alpha, \beta} \geq \tau_{0, \beta} \quad \text{for all } \beta > 0 \geq \alpha,$$

$$\nu_{\alpha, 0} \geq \nu_{\alpha, \beta} \geq \nu_{\alpha, +\infty} \quad \text{for all } \alpha < 0 \leq \beta.$$

Particular cases of these results have been obtained in [7] by using the duality existing between A_p and G_q weights and their characterization as bi-Sobolev maps (see [22]).

We end this section by showing that $\text{RH}^{\alpha,\beta}(u)$ is the continuous function of α and β .

Lemma 3.3. *Let α, β and L be real nonzero numbers, and assume $\alpha < \beta$.*

- *If u belongs to the class $\text{RH}^{L,\beta}$, then*

$$\lim_{\alpha \rightarrow L} \text{RH}^{\alpha,\beta}(u) = \text{RH}^{L,\beta}(u). \quad (3.9)$$

- *If u belongs to the class $\text{RH}^{\alpha,L}$, then*

$$\lim_{\beta \rightarrow L} \text{RH}^{\alpha,\beta}(u) = \text{RH}^{\alpha,L}(u). \quad (3.10)$$

Remark 3.4. For a weight $w \in A_p$, Lemma 3.3 states that the function

$$r \rightarrow A_r(w) = \sup_{I \subseteq I_0} \int_I w \left(\int_I w^{\frac{1}{1-r}} \right)^{r-1}$$

is a continuous (decreasing) function of r in the interval $]s_p, p]$.

Proof of Lemma 3.3. In order to prove the first limit (3.9), let us start by observing that if u belongs to the class $\text{RH}^{L,\beta}$, it also belongs to $\text{RH}^{\alpha,\beta}$ for all $\alpha > L$ and, thanks to the self-improving property, for all $\alpha < L$ such that $v_{\alpha,\beta} < \alpha < L$. This means that the function $\alpha \rightarrow \text{RH}^{\alpha,\beta}$ is defined in the interval $]L - \epsilon, L + \epsilon[$ for all $\epsilon < v_{\alpha,\beta} - \alpha$. First, assume $\alpha < L$. By the monotonicity of the RH constants, we know that $\alpha < L$ implies $\text{RH}^{\alpha,\beta}(u) \geq \text{RH}^{L,\beta}(u)$, which immediately gives $\lim_{\alpha \rightarrow L^-} \text{RH}^{\alpha,\beta}(u) \geq \text{RH}^{L,\beta}(u)$ so that we have only to prove the opposite inequality

$$\lim_{\alpha \rightarrow L^-} \text{RH}^{\alpha,\beta}(u) \leq \text{RH}^{L,\beta}(u). \quad (3.11)$$

Now we use the self-improving property in both classes $\text{RH}^{\alpha,\beta}$ and $\text{RH}^{L,\beta}$. By Theorem 2.1, we know that

- there exists a unique positive $\gamma_{\alpha,\beta}^+ > \beta$ such that $u \in \text{RH}^{\alpha,\beta} \Rightarrow u \in \text{RH}^{\alpha,\tau}$ for all τ such that $\beta \leq \tau < \tau_{\alpha,\beta}$, where $\tau_{\alpha,\beta} = \gamma_{\alpha,\beta}^+$ and

$$\omega(\alpha, \beta, \gamma_{\alpha,\beta}^+) = \text{RH}^{\alpha,\beta}(u), \quad (3.12)$$

- there exists a unique positive $\gamma_{L,\beta}^+ > \beta$ such that $u \in \text{RH}^{L,\beta} \Rightarrow u \in \text{RH}^{L,\tau}$ for all τ such that $\beta \leq \tau < \tau_{L,\beta}$, where $\tau_{L,\beta} = \gamma_{L,\beta}^+$ and

$$\omega(L, \beta, \gamma_{L,\beta}^+) = \text{RH}^{L,\beta}(u). \quad (3.13)$$

From Lemma 3.1, we know that $\alpha < L$ implies

$$\gamma_{\alpha,\beta}^+ = \tau_{\alpha,\beta} \geq \tau_{L,\beta} = \gamma_{L,\beta}^+.$$

As $\omega(\alpha, \beta, x)$ is strictly decreasing for $x > \beta > \alpha$, from (3.12), we have $\text{RH}^{\alpha,\beta}(u) = \omega(\alpha, \beta, \gamma_{\alpha,\beta}^+) \leq \omega(\alpha, \beta, \gamma_{L,\beta}^+)$. Passing to the limit in the previous inequality and using (3.13), we get

$$\lim_{\alpha \rightarrow L^-} \text{RH}^{\alpha,\beta}(u) \leq \lim_{\alpha \rightarrow L^-} \omega(\alpha, \beta, \gamma_{L,\beta}^+) = \omega(L, \beta, \gamma_{L,\beta}^+) = \text{RH}^{L,\beta}(u),$$

which is (3.11).

For $\alpha > L$, we can proceed exactly in the same way, provided the appropriate inversion of the inequalities. In this case, by the monotonicity of the constants, we know that

$$\lim_{\alpha \rightarrow L^+} \text{RH}^{\alpha,\beta}(u) \leq \text{RH}^{L,\beta}(u) \quad (3.14)$$

and, by the monotonicity of the exponents, we know that $\alpha > L$ implies

$$\gamma_{\alpha,\beta}^+ = \tau_{\alpha,\beta} \leq \tau_{L,\beta} = \gamma_{L,\beta}^+.$$

As in the previous case, from this relation, we obtain

$$\lim_{\alpha \rightarrow L^+} \text{RH}^{\alpha,\beta}(u) \geq \lim_{\alpha \rightarrow L^+} \omega(\alpha, \beta, \gamma_{L,\beta}^+) = \omega(L, \beta, \gamma_{L,\beta}^+) = \text{RH}^{L,\beta}(u),$$

which, together with inequality (3.14), completes the proof of (3.9) in Theorem 3.3.

The proof of (3.10) can be achieved following the same steps. □

4 Limit case 1: the class $\text{RH}^{-\infty, \beta}$

For $\beta \in \mathbb{R} - \{0\}$, a weight u belongs to the class $\text{RH}^{-\infty, \beta}$ when, for every subinterval $I \subset I_0$ and for some $B > 1$ independent of the interval, it satisfies the inequality

$$\left(\int_I u^\beta \right)^{\frac{1}{\beta}} \leq B \operatorname{ess\,inf}_{x \in I} u.$$

The ω -function for this class is

$$\omega(-\infty, \beta, x) = \left(\frac{x}{x - \beta} \right)^{\frac{1}{\beta}},$$

and the characteristic equation $\omega(-\infty, \beta, x) = \text{RH}^{-\infty, \beta}(u)$ becomes

$$\left(\frac{x}{x - \beta} \right)^{\frac{1}{\beta}} = \text{RH}^{-\infty, \beta}(u).$$

In this case, the equation admits a unique positive solution $\gamma_{-\infty, \beta}^+ \geq \beta$ given by

$$\gamma_{-\infty, \beta}^+ = \beta \frac{A^\beta}{A^\beta - 1}$$

with $A = \text{RH}^{-\infty, \beta}(u)$.

Theorem 4.1. *If $u \in \text{RH}^{-\infty, \beta}(I_0)$, then $\text{RH}^{-\infty, \beta}(u) = \lim_{\alpha \rightarrow -\infty} \text{RH}^{\alpha, \beta}(u)$.*

Proof. By recalling (3.3), for all $\alpha < \beta$, we have $\text{RH}^{\alpha, \beta}(u) \leq \text{RH}^{-\infty, \beta}(u)$, which implies

$$\lim_{\alpha \rightarrow -\infty} \text{RH}^{\alpha, \beta}(u) \leq \text{RH}^{-\infty, \beta}(u)$$

so that it will be sufficient to prove the inverse inequality

$$\lim_{\alpha \rightarrow -\infty} \text{RH}^{\alpha, \beta}(u) \geq \text{RH}^{-\infty, \beta}(u). \quad (4.1)$$

As $u \in \text{RH}^{-\infty, \beta}$ implies that $u \in \text{RH}^{\alpha, \beta}$ for all $\alpha < \beta$, let us pick such an α and apply the self-improving property in both classes $\text{RH}^{-\infty, \beta}$ and $\text{RH}^{\alpha, \beta}$. From Theorem 2.1, the *upper sharp exponents* $\tau_{-\infty, \beta}$ and $\tau_{\alpha, \beta}$ for those classes are given, respectively, by the unique positive solutions, $\gamma_{-\infty, \beta}^+$ and $\gamma_{\alpha, \beta}^+$, to the two equations

$$\omega(-\infty, \beta, \gamma_{-\infty, \beta}^+) = \text{RH}^{-\infty, \beta}(u) \quad \text{and} \quad \omega(\alpha, \beta, \gamma_{\alpha, \beta}^+) = \text{RH}^{\alpha, \beta}(u).$$

By Lemma 3.2, the *upper sharp exponents* decrease with α , i.e.,

$$\gamma_{-\infty, \beta}^+ = \tau_{-\infty, \beta} \geq \tau_{\alpha, \beta} = \gamma_{\alpha, \beta}^+.$$

As $\omega(\alpha, \beta, x)$ is strictly decreasing for $x > \beta > \alpha$, from the previous inequality, we get

$$\text{RH}^{\alpha, \beta}(u) = \omega(\alpha, \beta, \gamma_{\alpha, \beta}^+) \geq \omega(\alpha, \beta, \gamma_{-\infty, \beta}^+). \quad (4.2)$$

On the other hand, the function $\omega(\alpha, \beta, x)$ converges pointwise to the function $\omega(-\infty, \beta, x)$, i.e.,

$$\lim_{\alpha \rightarrow -\infty} \omega(\alpha, \beta, x) = \omega(-\infty, \beta, x) \quad \text{for all } x > \beta. \quad (4.3)$$

Passing to the limit in (4.2) and applying (4.3) to $x = \gamma_{-\infty, \beta}^+$, we obtain

$$\lim_{\alpha \rightarrow -\infty} \text{RH}^{\alpha, \beta}(u) \geq \lim_{\alpha \rightarrow -\infty} \omega(\alpha, \beta, \gamma_{-\infty, \beta}^+) = \omega(-\infty, \beta, \gamma_{-\infty, \beta}^+) = \text{RH}^{-\infty, \beta}(u),$$

which is (4.1). □

4.1 Special case: the Muckenhoupt class A_1

In the special case $\alpha = -\infty$ and $\beta = 1$, with the change of variable $\alpha = \frac{1}{1-p}$, the class $\text{RH}^{-\infty,1}$ coincides with the well-known *Muckenhoupt class* A_1 which is the class of weights w such that, for $A > 1$ and for every subinterval $I \subset I_0$, it is

$$\int_I w \leq A \operatorname{ess\,inf}_{x \in I} w.$$

The constant $\text{RH}^{-\infty,1}(w)$ defined by (1.3) is usually denoted by

$$A_1(w) = \sup_{I \subseteq I_0} \int_I w (\operatorname{ess\,inf}_I w)^{-1}.$$

The ω -function for this class becomes

$$\omega(-\infty, 1, x) = \frac{x}{x-1},$$

and the *characteristic equation* $\omega(-\infty, 1, x) = A_1(w)$ is

$$\frac{x}{x-1} = A_1(w).$$

In this case, the unique solution is given by (see [2])

$$\gamma_{-\infty} = \frac{A_1(w)}{A_1(w) - 1}.$$

Clearly, we have $\alpha \rightarrow -\infty \iff p \rightarrow 1^+$, so in this case, Theorem 4.1 reads as follows.

Theorem 4.2 ([21]). *Let $w \in A_1$; then $A_1(w) = \lim_{p \rightarrow 1^+} A_p(w)$.*

5 Limit case 2: the class $\text{RH}^{\alpha,+\infty}$

For $\alpha \in \mathbb{R} - \{0\}$, the case $\text{RH}^{\alpha,+\infty}$ corresponds to the class of weights u such that, given a constant $B > 1$, for every subinterval $I \subset I_0$, it is

$$\operatorname{ess\,sup}_{x \in I} u \leq B \left(\int_I u^\alpha \right)^{\frac{1}{\alpha}}.$$

The ω -function for this class is

$$\omega(\alpha, +\infty, x) = \left(\frac{x-\alpha}{x} \right)^{\frac{1}{\alpha}},$$

and the *characteristic equation* $\omega(\alpha, +\infty, x) = \text{RH}^{\alpha,+\infty}(u)$ becomes

$$\left(\frac{x-\alpha}{x} \right)^{\frac{1}{\alpha}} = \text{RH}^{\alpha,+\infty}(u).$$

This equation admits a unique positive solution $\gamma_{\alpha,+\infty}^- \leq \alpha$ given by

$$\gamma_{\alpha,+\infty}^- = \frac{\alpha}{1 - G^\alpha}$$

with $G = \text{RH}^{\alpha,+\infty}(u)$.

Theorem 5.1. *If $u \in \text{RH}^{\alpha,+\infty}(I_0)$, then $\text{RH}^{\alpha,+\infty}(u) = \lim_{\beta \rightarrow +\infty} \text{RH}^{\alpha,\beta}(u)$.*

Proof. By recalling that $\text{RH}^{\alpha,\beta}(u) \leq \text{RH}^{\alpha,+\infty}(u)$ for any $\alpha \leq \beta$, we already know that

$$\lim_{\beta \rightarrow +\infty} \text{RH}^{\alpha,\beta}(u) \leq \text{RH}^{\alpha,+\infty}(u)$$

so that it will be sufficient to prove the inverse inequality

$$\lim_{\beta \rightarrow +\infty} \text{RH}^{\alpha, \beta}(u) \geq \text{RH}^{\alpha, +\infty}(u). \quad (5.1)$$

As $u \in \text{RH}^{\alpha, +\infty}(u)$, we know that $u \in \text{RH}^{\alpha, \beta}$ for all $\beta > \alpha$, so let us pick such a β . From Theorem 2.1, the *lower sharp exponents* of the self-improving property in the classes $\text{RH}^{\alpha, +\infty}$ and $\text{RH}^{\alpha, \beta}$, respectively, $v_{\alpha, +\infty}^-$ and $v_{\alpha, \beta}^-$, coincide with the two negative solutions $\gamma_{\alpha, +\infty}^-$ and $\gamma_{\alpha, \beta}^-$ to the equations

$$\omega(\alpha, +\infty, \gamma_{\alpha, +\infty}^-) = \text{RH}^{\alpha, +\infty}(u) \quad \text{and} \quad \omega(\alpha, \beta, \gamma_{\alpha, \beta}^-) = \text{RH}^{\alpha, \beta}(u).$$

By Lemma 3.2, for the *lower sharp exponents*, we have

$$\gamma_{\alpha, \beta}^- = v_{\alpha, \beta} \geq v_{\alpha, +\infty} = \gamma_{\alpha, +\infty}^-.$$

As $\omega(\alpha, \beta, x)$ is strictly increasing for $x < \alpha$, from the previous inequality, we get

$$\text{RH}^{\alpha, \beta}(u) = \omega(\alpha, \beta, \gamma_{\alpha, \beta}^-) \geq \omega(\alpha, \beta, \gamma_{\alpha, +\infty}^-). \quad (5.2)$$

On the other hand, the function $\omega(\alpha, \beta, x)$ converges pointwise to the function $\omega(\alpha, +\infty, x)$, i.e.,

$$\lim_{\beta \rightarrow +\infty} \omega(\alpha, \beta, x) = \omega(\alpha, +\infty, x) \quad \text{for all } x < \alpha. \quad (5.3)$$

Passing to the limit in (5.2) and applying (5.3) for $x = \gamma_{\alpha, +\infty}^-$, we finally obtain

$$\lim_{\beta \rightarrow +\infty} \text{RH}^{\alpha, \beta}(u) \geq \lim_{\beta \rightarrow +\infty} \omega(\alpha, \beta, \gamma_{\alpha, +\infty}^-) = \omega(\alpha, +\infty, \gamma_{\alpha, +\infty}^-) = \text{RH}^{\alpha, +\infty}(u),$$

which is (5.1). □

5.1 Special case: the Gehring class G_∞

For $\alpha = 1$ and $\beta = +\infty$, the class $\text{RH}^{1, +\infty}$ coincides with the class G_∞ which is the class of weights v such that, for $G > 1$ and for every subinterval $I \subset I_0$, it is

$$\text{ess sup}_{x \in I} v \leq G \int_I v$$

with the G_∞ -constant given by

$$G_\infty(v) = \sup_{I \subseteq I_0} \left(\int_I v \right)^{-1} \text{ess sup}_{x \in I} v.$$

The ω -function for this class is

$$\omega(1, +\infty, x) = \frac{x-1}{x},$$

and the *characteristic equation* $\omega(1, +\infty, \gamma) = G_\infty(v)$ becomes

$$\frac{x-1}{x} = G_\infty(v),$$

which admits a unique positive solution given by

$$\gamma_{+\infty} = \frac{1}{1 - G_\infty(v)}.$$

In this class, Theorem 5.1 reads as follows.

Theorem 5.2. *Let $v \in G_\infty$; then $G_\infty(v) = \lim_{q \rightarrow \infty} G_q(w)$.*

6 Limit case 3: the class $\text{RH}^{0,\beta}$

For $\alpha = 0$ and $\beta > 0$, we obtain the class of weights u such that, given a constant $B > 1$, for every subinterval $I \subset I_0$, it is

$$\left(\int_I u^\beta \right)^{\frac{1}{\beta}} \leq B \exp \left(\int_I \ln u \right).$$

The ω -function for this class is

$$\omega(0, \beta, x) = e^{-\frac{1}{x}} \cdot \left(\frac{x}{x-\beta} \right)^{\frac{1}{\beta}},$$

and the characteristic equation $\omega(0, \beta, x) = \text{RH}^{0,\beta}(u)$ becomes

$$e^{-\frac{1}{x}} \cdot \left(\frac{x}{x-\beta} \right)^{\frac{1}{\beta}} = \text{RH}^{0,\beta}(u).$$

Theorem 6.1. *If $u \in \text{RH}^{0,\beta}(I_0)$, then $\text{RH}^{0,\beta}(u) = \lim_{\alpha \rightarrow 0^-} \text{RH}^{\alpha,\beta}(u)$.*

Proof. From $\text{RH}^{0,\beta}(u) \leq \text{RH}^{\alpha,\beta}(u)$, we already know that $\text{RH}^{0,\beta}(u) \leq \lim_{\alpha \rightarrow 0^-} \text{RH}^{\alpha,\beta}(u)$ so that it remains to prove the inverse inequality

$$\lim_{\alpha \rightarrow 0^-} \text{RH}^{\alpha,\beta}(u) \leq \text{RH}^{0,\beta}(u). \quad (6.1)$$

By assumption, $u \in \text{RH}^{0,\beta}$, so $u \in \text{RH}^{\alpha,\beta}$ for some $\alpha < 0$. From Theorem 2.1, we know that the *upper sharp exponents* in the two classes coincide with the positive solutions to the equations

$$\omega(0, \beta, \gamma_{0,\beta}^+) = \text{RH}^{0,\beta}(u) \quad \text{and} \quad \omega(\alpha, \beta, \gamma_{\alpha,\beta}^+) = \text{RH}^{\alpha,\beta}(u).$$

Again by Lemma 3.2, we get

$$\gamma_{0,\beta}^+ = \tau_{0,\beta} \leq \tau_{\alpha,\beta} = \gamma_{\alpha,\beta}^+.$$

As $\omega(\alpha, \beta, x)$ is strictly decreasing for $x > \beta$, this implies that

$$\text{RH}^{\alpha,\beta}(u) = \omega(\alpha, \beta, \gamma_{\alpha,\beta}^+) \leq \omega(\alpha, \beta, \gamma_{0,\beta}^+). \quad (6.2)$$

On the other hand, the function $\omega(\alpha, \beta, x)$ converges pointwise to the function $\omega(0, \beta, x)$, i.e.,

$$\lim_{\alpha \rightarrow 0^-} \omega(\alpha, \beta, x) = \omega(0, \beta, x) \quad \text{for all } x > \beta. \quad (6.3)$$

Thus we pass to the limit in (6.2) for $\alpha \rightarrow 0$ and apply (6.3) for $x = \gamma_{0,\beta}^+$, so we have

$$\lim_{\alpha \rightarrow 0^-} \text{RH}^{\alpha,\beta}(u) \leq \lim_{\alpha \rightarrow 0^-} \omega(\alpha, \beta, \gamma_{0,\beta}^+) = \omega(0, \beta, \gamma_{0,\beta}^+) = \text{RH}^{0,\beta}(u),$$

which is (6.1). □

6.1 Special case: the Muckenhoupt class A_∞

For $\alpha = 0$ and $\beta = 1$, adopting again the change of variable $\alpha = \frac{1}{1-p}$, the class $\text{RH}^{0,1}$ coincides with the class A_∞ , the class of weights w such that, for every subinterval $I \subset I_0$, for some constant $A > 1$, it is

$$\int_I w \leq A \exp \left(\int_I \ln w \right)$$

with the A_∞ -constant defined by

$$A_\infty(w) = \sup_{I \subset I_0} \int_I w \exp \left(\int_I \log \frac{1}{w} \right).$$

The ω -function for this class is

$$\omega(0, 1, x) = e^{-\frac{1}{x}} \cdot \frac{x}{x-1},$$

and the characteristic equation $\omega(0, 1, \gamma) = A_\infty(w)$ becomes

$$e^{-\frac{1}{\gamma}} \cdot \frac{\gamma}{\gamma-1} = A_\infty(w).$$

In this class, having again $\alpha \rightarrow -\infty \Leftrightarrow p \rightarrow +\infty$, Theorem 6.1 gives back the following well-known theorem.

Theorem 6.2 ([25]). *Let $w \in A_\infty$; then $A_\infty(w) = \lim_{p \rightarrow \infty} A_p(w)$.*

7 Limit case 4: the class $\text{RH}^{\alpha,0}$

For $\alpha < 0$ and $\beta = 0$, this case corresponds to the class of weights u such that, given a constant $B > 1$, for every subinterval $I \subset I_0$, it is

$$\exp\left(\int_I \ln u\right) \leq B \left(\int_I u^\alpha\right)^{\frac{1}{\alpha}}.$$

The ω -function for this class is

$$\omega(\alpha, 0, x) := \lim_{\beta \rightarrow 0^+} \omega(\alpha, \beta, x) = \left(\frac{x-\alpha}{x}\right)^{\frac{1}{\alpha}} \cdot e^{\frac{1}{x}};$$

hence the characteristic equation $\omega(\alpha, 0, x) = \text{RH}^{\alpha,0}(u)$ becomes

$$\left(\frac{x-\alpha}{x}\right)^{\frac{1}{\alpha}} \cdot e^{\frac{1}{x}} = \text{RH}^{\alpha,0}(u).$$

Theorem 7.1. *If $u \in \text{RH}^{\alpha,0}(I_0)$, then $\text{RH}^{\alpha,0}(u) = \lim_{\beta \rightarrow 0^+} \text{RH}^{\alpha,\beta}(u)$.*

Proof. As $\text{RH}^{\alpha,0}(u) \leq \text{RH}^{\alpha,\beta}(u)$ for all $\beta > 0$, we already have that $\text{RH}^{\alpha,0}(u) \leq \lim_{\beta \rightarrow 0^+} \text{RH}^{\alpha,\beta}(u)$, so we need to prove that

$$\lim_{\beta \rightarrow 0^+} \text{RH}^{\alpha,\beta}(u) \leq \text{RH}^{\alpha,0}(u). \quad (7.1)$$

As $u \in \text{RH}^{\alpha,0}(u)$, it also belongs to $\text{RH}^{\alpha,\beta}$ for some $\beta > 0$. Let us pick such a β . From Theorem 2.1, there exist $\gamma_{\alpha,0}^- < 0$ and $\gamma_{\alpha,\beta}^- > \beta$ such that $\omega(\alpha, 0, \gamma_{\alpha,0}^-) = \text{RH}^{\alpha,0}(u)$ with $v_{\alpha,0} = \gamma_{\alpha,0}^- < 0$, and $\omega(\alpha, \beta, \gamma_{\alpha,\beta}^-) = \text{RH}^{\alpha,\beta}(u)$ with $v_{\alpha,\beta} = \gamma_{\alpha,\beta}^- > \beta$. Now Lemma 3.2 gives

$$\gamma_{\alpha,\beta}^- = v_{\alpha,\beta} \leq v_{\alpha,0} = \gamma_{\alpha,0}^-.$$

As $\omega(\alpha, \beta, x)$ is strictly increasing for $x < \alpha$, this implies that

$$\text{RH}^{\alpha,\beta}(u) = \omega(\alpha, \beta, \gamma_{\alpha,\beta}^-) \leq \omega(\alpha, \beta, \gamma_{\alpha,0}^-). \quad (7.2)$$

On the other hand, the function $\omega(\alpha, \beta, x)$ converges pointwise to the function $\omega(\alpha, 0, x)$, i.e.,

$$\lim_{\beta \rightarrow 0^+} \omega(\alpha, \beta, x) = \omega(\alpha, 0, x) \quad \text{for all } x < \alpha. \quad (7.3)$$

If we pass to the limit in (7.2) for $\beta \rightarrow 0^+$ and apply (7.3) to $x = \gamma_{\alpha,0}^-$, we finally obtain

$$\lim_{\beta \rightarrow 0^+} \text{RH}^{\alpha,\beta}(u) \leq \lim_{\beta \rightarrow 0^+} \omega(\alpha, \beta, \gamma_{\alpha,0}^-) = \omega(\alpha, 0, \gamma_{\alpha,0}^-) = \text{RH}^{\alpha,0}(u),$$

which is (7.1). □

8 Limit case 5: the class $\text{RH}^{-\infty,0}$

When $\alpha = -\infty$ and $\beta = 0$, we have the class of weights u such that, given a constant $B > 1$, for every subinterval $I \subset I_0$, it is

$$\exp\left(\int_I \ln u\right) \leq B \operatorname{ess\,inf}_I u.$$

The ω -function for this class assumes the form

$$\omega(-\infty, 0, x) = e^{\frac{1}{x}},$$

and the characteristic equation $\omega(-\infty, 0, x) = \text{RH}^{-\infty,0}(u)$ becomes

$$e^{\frac{1}{x}} = \text{RH}^{-\infty,0}(u),$$

which admits a unique solution

$$\gamma_*^+ = \frac{1}{\ln(\text{RH}^{-\infty,0}(u))}.$$

Theorem 8.1. *If $u \in \text{RH}^{-\infty,0}(I_0)$, then $\text{RH}^{-\infty,0}(u) = \lim_{\alpha \rightarrow -\infty} \text{RH}^{\alpha,0}(u)$.*

Proof. By recalling (3.3) for $\beta = 0$, we have $\text{RH}^{\alpha,0}(u) \leq \text{RH}^{-\infty,0}(u)$, which implies

$$\lim_{\alpha \rightarrow -\infty} \text{RH}^{\alpha,0}(u) \leq \text{RH}^{-\infty,0}(u).$$

The inverse inequality $\lim_{\alpha \rightarrow -\infty} \text{RH}^{\alpha,0}(u) \geq \text{RH}^{-\infty,0}(u)$ follows exactly by the same steps of Theorem 4.1. $\square$

9 Limit case 6: the class $\text{RH}^{0,+\infty}$

The last case, when $\alpha = 0$ and $\beta = +\infty$, corresponds to the class of weights u such that, given $B > 1$, for every subinterval $I \subset I_0$, it is

$$\operatorname{ess\,sup}_I u \leq B \exp\left(\int_I \ln u\right).$$

The ω -function for this class assumes the form

$$\omega(0, +\infty, x) = e^{-\frac{1}{x}},$$

and the characteristic equation $\omega(0, +\infty, x) = \text{RH}^{0,+\infty}(u)$ becomes

$$e^{-\frac{1}{x}} = \text{RH}^{0,+\infty}(u),$$

which admits a unique solution

$$\gamma_*^- = -\frac{1}{\ln(\text{RH}^{0,+\infty}(u))}.$$

Theorem 9.1. *If $u \in \text{RH}^{0,+\infty}(I_0)$, then $\text{RH}^{0,+\infty}(u) = \lim_{\beta \rightarrow +\infty} \text{RH}^{0,\beta}(u)$.*

Proof. By applying (3.4) for $\alpha = 0$, we have $\text{RH}^{0,\beta}(u) \leq \text{RH}^{0,+\infty}(u)$, which implies

$$\lim_{\beta \rightarrow +\infty} \text{RH}^{0,\beta}(u) \leq \text{RH}^{0,+\infty}(u).$$

Finally, the inverse inequality $\lim_{\beta \rightarrow +\infty} \text{RH}^{0,\beta}(u) \geq \text{RH}^{0,+\infty}(u)$ can be obtained proceeding exactly as in Theorem 5.1. $\square$

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