Research Article

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Floquet theory and stability for a class of first order differential equations with delays

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Abstract: A version of the Floquet theory for first order delay differential equations is proposed. Formula of solutions representation is obtained. On this basis, the stability of first order delay differential equations is studied. An analogue of the classical integral Lyapunov–Zhukovskii test of stability is proved. New, in comparison with all known, tests of the exponential stability are obtained on the basis of the Floquet theory. A possibility to achieve the exponential stability is connected with oscillation of solutions.

Keywords: Floquet theory, delay differential equations, exponential stability, comparison theorems, periodic coefficients and delays

MSC 2020: 34K12, 34K20, 34K38, 34K25

1 Introduction

The Floquet theory is an important part of the qualitative theory of differential equations. For equations with periodic coefficients, this theory allows to get a way to represent solutions and helps essentially in stability analysis. The foundations of the Floquet–Lyapunov theory for a system of ordinary differential equations were obtained in the well-known books by E. A. Coddington and N. Levinson [12], V. A. Yakubovich and V. M. Starjinskii [40], A. Halanay and D. Wexler [25, 26] (augmented English version of the first two chapters of these books can be found in A. Halanay and V. Rasvan [24]). In connection to the Zhukovskii–Lyapunov inequality, we have firstly to refer to M. G. Krein [30], and its discrete time version can be found in [23, 35]. Note that these results deal with discrete time Hamiltonian systems. Other relevant aspects occur in the work of the Brno group of O. Došlá (see, e.g., [20]) and also of W. Kratz [29]. In the last developments of this theory for ordinary differential equations, two directions can be observed. The first one is a generalization to new classes of equations, for example, to nonlinear ones [43], to equations on time scales [1, 6, 13], to discrete and hybrid systems [2, 14]. It should be stressed that the first books on the Floquet–Lyapunov theory for discrete time systems with periodic coefficients are [25, 26]. The second direction was developed around applications. Some results on the Floquet theory, with application to periodic epidemic models were proposed in the paper [38], where a new cholera epidemic model with phage dynamics and seasonality was incorporated for a demonstration.

Applications based on delay feedback control became the main motivation in development of the Floquet theory for equations with memory which goes in various directions. The Floquet theory for integro-differential equations has been developed in [3, 5, 7, 18], where the monodromy matrices were constructed. In [8], the analytical theory of infinite determinants was proposed for integro-differential Floquet theory. For difference equations the Floquet theory was presented in [10, 14, 28] and for continuous and hybrid periodic linear systems in [14].

The foundations of the Floquet theory for delay equations were proposed in [22, 27] and were continued in many publications. Note only a few of them: in [32], the calculation of the Floquet multipliers for functional

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differential equations was developed and in [36] an analytical approach was proposed. It should be stressed that an infinite dimensional fundamental system does not allow obtaining a full analogue of the Floquet theory (for example, the assertions about passing to systems with constant coefficients [27]). In [38], the authors discussed the properties of Floquet exponents for delay linear periodic systems with constant delays. Necessary and sufficient conditions in the case of constant delays and triangular matrices of the coefficients are proposed. Perturbations of multipliers and the stability/instability of vector linear ODEs with periodic matrix coefficients were studied in [9]. Applications to epidemic models are demonstrated. An application of the Floquet theory in its generalized version to systems with memory was proposed in [39]. The conditions of unboundedness of all nontrivial solutions for the second order delay differential equations were obtained on the basis of the Floquet theory in [16].

In this paper, we propose a version of the Floquet theory for the equation

$$x'(t) + \sum_{i=1}^{m} a_i(t)x(t - \tau_i(t)) = f(t), \quad t \in [0, \infty),$$

(1.1)

with periodic coefficients and delays

$$a_i(t + \omega) = a_i(t), \quad \tau_i(t + \omega) = \tau_i(t), \quad t \in [0, \infty), \quad i = 1, \ldots, m,$$

(1.2)

where delays satisfy the condition

$$t - \tau_i(t) \geq 0, \quad t \in [0, \infty), \quad i = 1, \ldots, m,$$

(1.3)

and $f(t)$ is essentially bounded function. We assume that the coefficients $a_i(t)$ are essentially bounded and the delays $\tau_i(t)$ are measurable functions for $i = 1, \ldots, m$. The particular case of (1.1) is the classical one-term delay equation

$$x'(t) + a(t)x(t - \tau(t)) = 0, \quad t \in [0, \infty).$$

(1.4)

Condition (1.3) allows us to remain with the advantages of one-dimensional fundamental system in the study of equation (1.1). Our approach preserves advantages of an explicit representation of solutions and allows then to use them in the study of stability of delay equations with and without assumptions (1.2) and (1.3) through reduction to corresponding integral equations. It could be noted that traditionally the case of constant delays was considered as the main one, but in the feedback control models, described, for example, by (1.4), the delay feedback control is usually constructed in the form $u(t) = x(h(t))$, where the function $h(t) = t - \tau(t)$ is a staircase going up, i.e., $h(t) = h_j$ for $t_j \leq t < t_{j+1}$, $h_j < h_{j+1}$, $j = 1, 2, 3, \ldots$, $t_0 = 0$. If the points $0, \omega, 2\omega, 3\omega, \ldots$ are among them, i.e., $n\omega \in \{h_j\}_{j=0,1,2,3,\ldots}$, $n\omega \in \{t_j\}_{j=0,1,2,3,\ldots}$ for all $n = 0, 1, 2, 3, \ldots$, then the condition $t - \tau(t) \geq 0$ is fulfilled. This is a typical situation in medicine, when the result of a medical examination, obtained at the appropriate point of time, is used as the basis for treatment until the next examination. Indeed, when building control, the results of measuring the state of the process made at certain fixed points in time are used (see Section 6), where Marchuk’s model of infectious diseases is discussed. The case when the state of the process is measured continuously, of course, is also possible, but in many practical problems, both in technology and in medicine, this case does not look the most natural. Despite the fact that the feedback control in the form of staircase going up with delays satisfying (1.3) looks very natural in technology and in medicine, there are practically no results on the specific properties of differential equations based on the form of these delays.

On the basis of our version of the Floquet theory, we propose new results on the exponential stability of the delay equation (1.1). Note one of the classical Lyapunov–Zhukovskii results [42] about stability of the second order ordinary differential equation

$$x''(t) + a(t)x(t) = 0, \quad t \in [0, \infty),$$

(1.5)

with $\omega$-periodic coefficient $a(t)$ that is integrable on $[0, \omega]$.

**Theorem 1.1** ([42]). If $a(t) \geq 0$ for $t \in [0, \omega]$ and $0 < \int_0^\omega a(t) \, dt \leq \frac{4}{\omega}$, then all solutions of the ordinary second order equation (1.5) are bounded.
The integral appears here as a result of estimating the distance between two adjacent zeros of the solution or two adjacent zeros of its derivative (i.e., between adjacent local maximum and minimum). The same idea can be seen in the proof of Theorem 2.5 below, which gives an assertion on stability based on the estimate of the integral on the period \( \omega \) of the coefficient \( a(t) \) for the delay first order equation (1.4). This assertion can be considered as an analog of the Lyapunov–Zhukovskii theorem.

A. D. Myshkis [34, Theorem 58, p. 229] proved that the inequalities

\[
\sup_{t \leq 0} a(t) \sup_{t \geq 0} \tau(t) < \frac{3}{2}, \quad \inf_{t \leq 0} a(t) > 0 \tag{1.6}
\]

guarantee asymptotic stability of equation (1.4). He also proved that the first inequality could not be improved (see [34, Remark 1, p. 232] demonstrating that in the case of nonstrong first inequality in (1.6), a solution \( x(t) \), which does not tend to zero for \( t \to \infty \), exists). T. Yoneyama obtained asymptotic stability using integral forms of the inequalities.

**Theorem 1.2 ([41]).** Assume that for continuous coefficient \( a(t) \) and delay \( \tau(t) \) the conditions

\[
\lim_{t \to -\infty} \sup_{t \geq q} \int_{t-q}^{t} a(s) \, ds < \frac{3}{2},
\]

\[
\lim_{t \to -\infty} \inf_{t \geq q} \int_{t-q}^{t} a(s) \, ds > 0,
\]

where \( q = \sup_{t \geq 0} \tau(t) \), hold. Then equation (1.4) is asymptotically stable.

In the case of periodic coefficient and delay, using the Floquet theory, we obtain (under the additional condition (1.3) on the delay) the constant \( 2 \) instead of \( \frac{3}{2} \) in (1.7). Note also the recent work of J. I. Stavroulakis and E. Braverman [37], where connection between oscillation and stability is considered.

Consider the equation

\[ x'(t) + ax(t - \tau) = 0, \quad t \in [0, \infty), \]

with positive constant coefficient \( a \) and delay \( \tau \). Concerning nonoscillation and stability conditions, we can note the following known facts [34]:

1. if \( 0 < a \tau \leq \frac{1}{2} \), then this equation is nonoscillatory and exponentially stable,
2. if \( \frac{1}{2} < a \tau < \frac{3}{2} \), then this equation is oscillatory but exponentially stable,
3. if \( a \tau > \frac{3}{2} \), then this equation is oscillatory and amplitudes of oscillating solutions can tend to infinity.

We propose a version of the Floquet theory for delay differential equations with variable periodic delays and coefficients and on this basis obtain results of the following types:

(a) opposite to (1.7), inequality \( \int_{t-\tau(t)}^{t} a(s) \, ds > \frac{3}{2} \) can be fulfilled on corresponding intervals, but equation (1.4) is exponentially stable;

(b) opposite condition to the inequality in (1), i.e., \( \frac{1}{2} + \varepsilon < \int_{t-\tau(t)}^{t} a(s) \, ds \), where \( \varepsilon > 0 \), can be fulfilled on the corresponding intervals, but equation (1.4) stays nonoscillatory.

Improvements of the constants in tests of stability and nonoscillation is one of the directions developed in our paper, but not the main one. The use of the Floquet theory opens new possibilities in the analysis of stability and ways of possible stabilization of processes by delay feedback control. They are based on the following fact obtained in our paper. For equation (1.1) with positive \( \omega \)-periodic coefficients \( a_i(t) \) and nonnegative \( \omega \)-periodic delays \( \tau_i(t) \), satisfying (1.3), there exists infinite number of intervals \( (a_n, \beta_n) \), where \( a_n \to \infty \) as \( n \to \infty \), such that in the case of \( \omega \) situated in one of them, i.e., \( \omega \in \bigcup_{n=1}^{\infty} (a_n, \beta_n) \), equation (1.1) is exponentially stable.

The paper is organized as follows. In Section 2, we propose a version of the Floquet theory for delay differential equations of the first order. A solutions’ representation is obtained. Examples demonstrating advantages of this approach are presented. In Section 3, results on the exponential stability based on the Floquet theory are formulated. Section 4 contains the proofs of the main assertions. In Section 5, the Cauchy function of delay differential equations is constructed and an example is considered. In Section 6, a possibility of applications to model of infectious diseases of G. I. Marchuk is discussed.
2 A version of the Floquet theory for delay differential equation of first order

Consider the equation

\[ x'(t) + \sum_{i=1}^{m} a_i(t)x(t - \tau_i(t)) = 0, \quad t \in [0, \infty), \]

(2.1)

where the coefficients \( a_i(t) \) and delays \( \tau_i(t) \) satisfy the condition which will be called below (H):

(H) The coefficients \( a_i(t) \) are essentially bounded, \( a_i(t) \geq 0, \int_0^\infty \sum_{i=1}^{m} a_i(t) dt = \infty \), delays \( \tau_i(t) \) are measurable, \( t - \tau_i(t) \geq 0 \) for \( t \in [0, \infty) \), \( t - \tau_i(t) \) are nondecreasing and \( t - \tau_i(t) \to \infty \) when \( t \to \infty \).

Consider the nonhomogeneous equation

\[ x'(t) + \sum_{i=1}^{m} a_i(t)x(t - \tau_i(t)) = f(t), \quad t \in [0, \infty), \]

(2.2)

with essentially bounded function \( f(t) \). An absolutely continuous function \( x(t) \) satisfying this equation for almost every \( t \in [0, \infty) \) is called a solution of this equation. It is known that the fundamental system of equation (2.1) is one-dimensional and the general solution of equation (2.2) can be represented in the form

\[ x(t) = \int_0^t C(t, s)f(s) \, ds + C(t, 0)x(0), \quad t \in [0, \infty), \]

(2.3)

where \( C(t, s) \) is the Cauchy (fundamental in another terminology) function of equation (2.1) (see [4, p. 343]). Formula (2.3) emphasizes the finite-dimensional character of our approach which allows to “catch” the specificity of both periodic coefficients and delays of the same period.

**Definition 2.1.** A function \( C(t, s) \) defined as acting \([0, \infty) \times [0, \infty) \to \mathbb{R}\) is called the Cauchy function of equation (2.1) if \( C(t, s) \) as a function of the argument \( t \) for every fixed \( s \geq 0 \) is a solution of the homogeneous equation on \([s, \infty)\) with the zero initial function and 1 in the initial condition, i.e.,

\[
\begin{cases}
  x'(t) + \sum_{i=1}^{m} a_i(t)x(t - \tau_i(t)) = 0, & t \in [s, \infty), \\
  x(\xi) = 0 & \text{for } \xi < s, \\
  x(s) = 1.
\end{cases}
\]

**Definition 2.2.** Equation (2.1) is exponentially stable if there exist two positive constants \( N \) and \( \alpha \) such that

\[ |C(t, s)| \leq Ne^{-\alpha(t-s)} \quad \text{for } 0 \leq s \leq t < \infty. \]

(2.4)

The coefficients and delays in the frame of the Floquet theory are assumed to be periodic with period \( \omega \), i.e., the equalities

\[ a_i(t + \omega) = a_i(t), \quad \tau_i(t + \omega) = \tau_i(t), \quad t \in [0, \infty), \]

(2.5)

hold. We want to find solutions satisfying the equality

\[ x(t + \omega) = \lambda x(t) \quad \text{for } t \in [0, \infty). \]

(2.6)

The following assertion is an analogue of the Floquet theorem about representation of the solution for the delay differential equation (2.1).

**Theorem 2.3.** Let conditions (H) and (2.5) be fulfilled. Then in the case of \( \lambda \neq 0 \), every solution of equation (2.1) can be represented in the form

\[ x(t) = \Phi(t) \exp \left\{ \frac{\ln |\lambda|}{\omega} t \right\}, \]

(2.7)

where the function \( \Phi(t) \) is \( \omega \)-periodic if \( \lambda > 0 \) and \( 2\omega \)-periodic if \( \lambda < 0 \), and in the case of \( \lambda = 0 \) in the form

\[ x(t) = \begin{cases} y(t), & 0 \leq t \leq \omega, \\ 0, & t > \omega, \end{cases} \]

where \( y(t) \) is a solution of equation (2.1) on \([0, \omega]\).
Note that the equality $\lambda = 0$ in (2.6) implies that the solution $y(t)$ satisfies the condition $y(\omega) = 0$.

**Remark 2.4.** It will be demonstrated below in Lemma 4.3 that equation (2.1) is exponentially stable in the case of $|\lambda| < 1$.

**Theorem 2.5.** Let $a(t)$ and $\tau(t)$ be $\omega$-periodic functions, let $a(t)$ be nonnegative and essentially bounded on $[0, \omega]$, let $\tau(t) \geq 0$ be measurable and such that $h(t) \equiv t - \tau(t) \geq 0$ for $t \in [0, \omega]$, let $h(t)$ be nondecreasing function, and let the inequality

$$0 < \int_{0}^{\omega} a(t) \, dt < 2$$

be fulfilled. Then the delay equation (1.4) is exponentially stable.

We demonstrate in Example 2.12 below that the constant 2 in (2.8) is exact. It is clear that in the case of $a(t) \equiv 0$ any constant function is a solution of (1.4) and this demonstrates that the estimate of the integral from below is also exact.

**Remark 2.6.** Let us try to compare this result with the well-known results on stability of the first order delay differential equation (1.4). If

$$\tau(t) = t \text{ for } t \in [0, \omega],$$

and it is continued as $\tau(t + \omega) = \tau(t)$ for $t \in [0, \omega)$, then $h(t) = t - \tau(t) \equiv 0$ for $t \in [0, \omega]$, and the first inequality in (1.7) will be of the form $\int_{0}^{\omega} a(s) \, ds < \frac{3}{2}$, in which the constant $\frac{3}{2}$ is less than 2 in inequality (2.8).

**Remark 2.7.** It may seem that the equation

$$x'(t) + 1.9x(t - 1) = 0$$

that is unstable according to the result of Myshkis (see assertion (3) in Introduction) contradicts Theorem 2.5. But the condition $t - \tau(t) \geq 0$ for $t \in [0, \omega]$ of Theorem 2.5, which looks as $t - 1 \geq 0$, is not fulfilled for $0 \leq t < 1$. This demonstrates that the condition $t - \tau(t) \geq 0$ for all $t \in [0, \omega]$ is essential.

**Theorem 2.8.** Let conditions (H) and (2.5) be fulfilled. Then there exists infinite number of intervals $(\alpha_n, \beta_n)$, where $\alpha_n \to \infty$ as $n \to \infty$, such that equation (2.1) is exponentially stable in the case of $\omega \in \bigcup_{n=1}^{\infty} (\alpha_n, \beta_n)$.

**Definition 2.9.** The interval $[0, \omega]$ is called nonoscillation interval of equation (2.1) if every nontrivial solution $x(t)$ does not have zero on this interval, i.e., $x(t) \neq 0$ for $t \in [0, \omega]$.

This definition can be found in [17], and the existence of nonoscillation intervals for delay differential equations was discussed in [4, p. 342].

The following assertion allows us to estimate $\lambda$ in representation (2.7).

**Theorem 2.10.** Let conditions (H) and (2.5) be fulfilled and let $[0, \omega]$ be a nonoscillation interval of equation (2.1), then $\lambda$ in representation (2.7) satisfies the inequality

$$\exp\left(-\frac{1}{e} \int_{0}^{\omega} a(t) \, dt\right) \leq \lambda \leq \exp\left(-\frac{1}{e} \sum_{i=1}^{m} a_i(t) \, dt\right).$$

**Remark 2.11** ([17]). Each of the following two conditions guarantees that $[0, \omega]$ is a nonoscillation interval of equation (2.1):

$$\int_{0}^{\omega} \sum_{i=1}^{m} a_i(t) \, dt \leq 1$$

and

$$\int_{t-\tau_0(t)}^{t} \sum_{i=1}^{m} a_i(s) \, ds \leq \frac{1}{e},$$

where $\tau_0(t) = \max_{1 \leq i \leq m} \tau_i(t)$ for $t \in [0, \omega]$. 
Example 2.12. Consider the equation

$$x'(t) + x(t - \phi(t)) = 0, \quad t \in [0, \infty),$$  \hspace{1cm} (2.10)

where

$$\phi(t) = \begin{cases} 
  t & \text{for } t \in [0, 2), \\
  \phi(t - 2) & \text{for } t \geq 2.
\end{cases}$$

Define now the function $\tau(t)$ as

$$\tau(t) = \begin{cases} 
  \phi(t) & \text{for } t \in [0, \omega], \\
  \tau(t - \omega) & \text{for } t > \omega \geq 2,
\end{cases}$$  \hspace{1cm} (2.11)

and consider the equation

$$x'(t) + x(t - \tau(t)) = 0, \quad t \in [0, \infty).$$  \hspace{1cm} (2.12)

It is clear that equations (2.10) and (2.12) coincide for $t \in [0, \omega]$. We can see that the deviation of the argument $h(t) = t - \phi(t)$ presents a staircase going up:

$$h(t) = 2(k - 1), \quad 2(k - 1) \leq t < 2k, \quad k = 1, 2, 3, \ldots$$

Any solution of equation (2.10) is proportional to

$$x(t) = \begin{cases} 
  1 - t, & 0 \leq t < 2, \\
  -3 + t, & 2 \leq t < 4,
\end{cases} \quad x(t + 4) = x(t).$$  \hspace{1cm} (2.13)

It is a 4-periodic function, which oscillates with amplitude 1, i.e., $x(2k) = (-1)^k$ and $|x(t)| < 1$ for every $t \neq 2k$, $k = 0, 1, 2, \ldots$. It follows from Theorem 2.3 that for every $\omega \neq 2k$, where $k = 1, 2, 3, \ldots$, equation (2.12) is exponentially stable since $|x(t)| < x(0) = 1$ and $|\lambda| < 1$. For $\omega = 2k$, where $k = 1, 2, 3, \ldots$, the solution of (2.12) is periodic.

This also demonstrates that the constant 2 in inequality (2.8), generally speaking, is sharp. Indeed, in Example 2.12, we have $\int_0^\omega a(t) \, dt = \int_0^\omega 1 \, dt = \omega$. If $\omega = 2$, we obtain the 4-periodic solution (2.13) ($\lambda$ in representation (2.7) is equal to $-1$) and equation (2.12) is not exponentially stable.

Remark 2.13. It can be noted that in several works on stability, the constant “2” is also the exact stability boundary. This is the result of S. N. Shimanov and Yu. F. Dolgii: the equation $x'(t) + a(t)x(t - \omega) = 0$ is stable if $a(t)$ is a 2$\omega$-periodic function, $a(t) \geq 0$ and $0 < \int_0^{2\omega} a(t) \, dt < 2$ (see, e.g., [15, p. 20] and some generalization by K. M. Chudinov [11]). Theorem 2.5 of our paper does not follow from those mentioned results, since in them the integration is carried out over an interval twice as large as the maximum delay. In Theorem 2.5, the integration is carried out over an interval which is equal to the possible maximum delay.

Example 2.14. Consider the equation

$$x'(t) + x(h(t)) = 0$$  \hspace{1cm} (2.14)

with

$$h(t) = \frac{n - 1}{2}, \quad \frac{n - 1}{2} \leq t < \frac{n}{2}, \quad n = 1, 2, 3, \ldots$$

Define now the function $\phi(t) = t - h(t)$. Using MATLAB, we obtain a solution of equation (2.14) (see Figure 1).

It is clear that conditions (a) and (b) of Remark 2.11 and of assertion (I) in Introduction are not fulfilled in this example, but the solution $x(t)$ of equation (2.14) is positive on the interval $[0, 15]$. $\lambda$ in the case of equation (2.14), where $\tau(t)$ is defined by (2.11), can be evaluated by formula (2.9):

$$\frac{1}{e^{2\omega}} \leq \lambda \leq \frac{1}{e\omega}.$$  \hspace{1cm} (2.9)

Let us choose $\omega = 15$ and $\tau(t)$ according to (2.11) and consider (2.12). We get $e^{-15\omega} \leq \lambda \leq e^{-15}$. In order to “touch” these inequalities better, we have made calculations and obtained $1.958 \cdot 10^{-18} \leq \lambda \leq 3.059 \cdot 10^{-7}$, where in the left inequality the rounding was in the direction of increasing, and in the right one in the direction of decreasing. We make the same in all the inequalities obtained in Examples 2.15 and 2.16.
The following two examples demonstrate applications of Theorems 2.3 and 2.8. We see appearance of intervals such that equation (2.12) is exponentially stable in the case of \( \omega \) situating in one of them.

**Example 2.15.** Consider equation (2.14) with
\[
h(t) = \frac{n}{10}, \quad n \leq t < n + 1, \quad n = 0, 1, 2, \ldots \tag{2.15}
\]
Define \( \phi(t) = t - h(t) \) and \( \tau(t) \) according to (2.11). The top graph in Figure 2 contains a solution \( x(t) \). The bottom graph in Figure 2 contains only the intervals where \(-1 < x(t) < 1\). For the exponential stability of equation (2.12) with \( \tau(t) \) defined by (2.11), the period \( \omega \) has to be in one of these intervals.

If we choose \( \omega \) from these in the intervals \( 0 < \omega < 2.125 \), \( 19.5 < \omega < 21.755102 \), we obtain exponential stability.

**Example 2.16.** Consider equation (2.14) with
\[
h(t) = \begin{cases} 
0, & 0 \leq t < 3, \\
[t - 2], & t \geq 3,
\end{cases} \tag{2.16}
\]
where \( [t] \) is the integer part of \( t \). Define \( \phi(t) = t - h(t) \) and \( \tau(t) \) according to (2.11). The top graph in Figure 3 contains the solution \( x(t) \) of equation (2.14) with \( h(t) \) defined by (2.16). The bottom graph in Figure 3 contains only the intervals where \(-1 < x(t) < 1\). To get exponential stability of equation (2.12), with \( \tau(t) \) defined by (2.11), \( \omega \) should...
be in one of them. In this example we can get the intervals for \( \omega \) in which equation (2.12) is exponentially stable: 
\[
0 < \omega < 2, \ 5 < \omega < 6, \ 9.67 < \omega < 10.25, \ 14.28 < \omega < 14.57, \ 18.81 < \omega < 19, \ 23.32 < \omega < 23.4, \ 27.8 < \omega < 27.85, \\
32.28 < \omega < 32.3, \ \text{and so on. If we continue the solution, we can see that the intervals for \( \omega \) appears at approximately every 4.5 on the time axis.}
\]

## 3 Tests of stability based on the Floquet theory

Consider the equation 
\[
\dot{x}(t) + \sum_{i=1}^{m} a_i(t)x(t - \theta_i(t)) = 0, \quad t \in [0, \infty),
\]
where the coefficients \( a_i(t) \) and delays \( \theta_i(t) \) satisfy condition (H), \( \theta_i(t) \) is bounded and \( t - \theta_i(t) \geq 0, i = 1, \ldots, m \).

Denote the functions of deviations as \( g_i(t) = t - \tau_i(t) \), and analogously for equation (2.1) we set \( h_i(t) = t - \theta_i(t), i = 1, \ldots, m \).

Introduce the operator \( K : C \to C \) in the space of continuous functions \( x : [0, \infty) \to \mathbb{R} \) equipped with the standard norm \( \|x\| = \sup_{t \geq 0} |x(t)| \) by the equality 
\[
(Kx)(t) = - \int_{0}^{t} C_r(t, s) \left\{ \sum_{i=1}^{m} a_i(s) \int_{g_i(s)}^{h_i(s)} \sum_{j=1}^{m} a_j(\xi) x(g_j(\xi)) \, d\xi \right\} ds, \quad t \in [0, \infty),
\]
where \( C_r(t, s) \) is the Cauchy function of equation (2.1).

**Theorem 3.1.** If equation (2.1) is exponentially stable and the norm of the operator \( K : C \to C \) is less than one, then equation (3.1) is also exponentially stable.

Denote 
\[
A_* = \text{ess inf}_{t \geq 0} \sum_{i=1}^{m} a_i(t), \quad A^* = \text{ess sup}_{t \geq 0} \sum_{i=1}^{m} a_i(t),
\]
and 
\[
(h - g)^* = \max_{1 \leq i \leq m} \text{ess sup}_{t \geq 0} |h_i(t) - g_i(t)|.
\]

**Corollary 3.2.** If equation (2.1) is exponentially stable and
\[
A^* \left( h - g \right)^* \sup_{t \geq 0} \int_{0}^{t} |C_r(t, s)| \, ds < 1,
\]
then equation (3.1) is exponentially stable.
Remark 4.1. It is clear that estimating of the integral \( \int_0^t |C_t(t, s)| \, ds \) is a way to study stability of equation (3.1). We propose two methods to obtain its estimates. The first one is based on the positivity of \( C_t(t, s) \). We develop a corresponding technique in Theorems 3.4 and 3.7. In the general case, we estimate this integral directly, constructing the Cauchy function \( C_t(t, s) \). We deal with this in Section 5.

**Theorem 3.4.** Let \( [0, \omega] \) be a nonoscillation interval of equation (2.1) with the coefficients \( a_i(t) \) and delays \( \tau_i(t) \) satisfying conditions (H), (1.2) and \( A_* > 0 \). Then the nontrivial solution of (2.1) is positive for \( t \in [0, \infty) \), \( C_t(t, s) > 0 \) for \( 0 \leq s \leq t < \infty \) and

\[
\sup_{t \geq 0} \int_0^t C_t(t, s) \, ds \leq \frac{1}{A_*}.
\]

Remark 3.5. Using the transform from the paper [31], we can reduce the analysis of stability of equation (3.1), where \( m = 1 \), to one for equation (2.12). This is a reason that we write Corollaries 3.6 and 3.8 for equation (2.12).

**Corollary 3.6.** Let \( [0, \omega] \) be a nonoscillation interval of the homogeneous equation (2.12) with \( \omega \)-periodic delay \( \tau(t) \) such that \( \tau(t) \leq t \) for every \( t \in [0, \omega] \). Then \( C_t(t, s) > 0 \) for \( 0 \leq s \leq t < \infty \) and

\[
\sup_{t \geq 0} \int_0^t C_t(t, s) \, ds \leq 1.
\]

**Theorem 3.7.** Let \( [0, \omega] \) be a nonoscillation interval of equation (2.1) with the coefficients \( a_i(t) \) and delays \( \tau_i(t) \) satisfying conditions (H), (1.2), and

\[
\text{ess sup}_{s \geq 0} \sum_{i=1}^m a_i(s) \sum_{j=1}^m \left| \int_{g_i(s)} h_i(s) \, d\xi \right| < A_*.
\]

Then equation (3.1) is exponentially stable.

Consider the equation

\[
x'(t) + x(g(t)) = 0, \quad t \in [0, \infty),
\]

where \( g(t) \leq t \) is a measurable function. Denote in equation (1.4): \( h(t) = t - \tau(t) \).

**Corollary 3.8.** If \( [0, \omega] \) is a nonoscillation interval of the homogeneous equation (1.4) with the \( \omega \)-periodic delay \( \tau(t) \) such that \( \tau(t) \leq t \) for every \( t \in [0, \omega] \), and \( |h(t) - g(t)| < 1 \) for \( t \geq 0 \), then equation (3.5) is exponentially stable.

4 Proofs

**Proof of Theorem 2.3.** The space of solutions to equation (2.1) under condition (1.3) is one-dimensional according to the general theory of functional differential equations (see [17, p. 773]). Periodicity of the delays \( \tau_i(t + \omega) = \tau_i(t) \) and the inequalities \( \tau_i(t) \leq t \) for \( i = 1, \ldots, m \) and \( t \in [0, \infty) \) imply that \( k\omega \leq t - \tau_i(t) \leq (k + 1)\omega \) for \( t \in [k\omega, (k + 1)\omega] \) and for every \( i = 1, \ldots, m, k = 0, 1, 2, \ldots \). This and \( \omega \)-periodicity of the coefficients \( a_i(t + \omega) = a_i(t) \) imply that the function \( x_k(t) = x_0(t - k\omega) \), where \( x_0 \) is a solution of equation (2.1) on \( [0, \omega] \), satisfies this equation on the interval \([k\omega, (k + 1)\omega]\) for every \( k = 0, 1, 2, \ldots \).

Remark 4.1. The reference to the original work by Floquet [21] can complete the proof, but we prefer to obtain formula (2.7) directly. This will allow us to get the form of the periodic function \( \Phi(t) \) in (2.7).

Let us continue the proof. The equality \( x(t + \omega) = \lambda x(t) \) implies that \( |x(k\omega)| = |\lambda x((k - 1)\omega)| = \lambda^2 |x((k - 2)\omega)| = \cdots = |\lambda|^k |x(0)| \) and, consequently, for \( t \in [k\omega, (k + 1)\omega] \) we obtain

\[
|x(t)| = |\lambda|^k |x_0(t - k\omega)| = \exp(|\ln(|\lambda|^k)| |x_0(t - k\omega)| = \exp \left( \frac{|\ln(|\lambda|^k)|}{k\omega} \cdot k\omega \right) |x_0(t - k\omega)|
\]

\[
= \exp \left( \frac{k|\ln(|\lambda|)|}{k\omega} \cdot k\omega \right) |x_0(t - k\omega)| = \exp \left( \frac{|\ln(|\lambda|)|}{\omega} \cdot k\omega \right) |x_0(t - k\omega)|.
\]
Note the following simple fact: any solution $x(t)$ of equation (2.1) can be written by formula (2.7), but we do not know either function $\Phi(t)$ is periodic or not. Thus, to complete the proof of Theorem 2.3, we have to prove that $\Phi(t + \omega) = \Phi(t)$ for every $t \geq 0$ in the case of $\lambda > 0$ and $\Phi(t + 2\omega) = \Phi(t)$ in the case of $\lambda < 0$.

We have two representations of the solution $x(t)$: $|x(t)| = \exp[\ln|\lambda|/\omega]x_0(t - k\omega)$ and (2.7). Comparing these representations, we obtain

$$\Phi(t) = \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - t)\right\}|x_0(t - k\omega)| \text{ for } t \in [k\omega, (k + 1)\omega].$$

Let us write

$$|\Phi(t + \omega)| = \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - (t + \omega))\right\}|x_0(t + \omega - k\omega)|$$

$$= \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - (t + \omega))\right\}|\lambda||x_0(t - k\omega)|$$

$$= \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - (t + \omega))\right\}\exp\left\{\frac{\ln|\lambda|}{\omega} \cdot \omega\right\}|x_0(t - k\omega)|$$

$$= \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - t)\right\}|x_0(t - k\omega)| = |\Phi(t)|.$$

If $\lambda > 0$, then it follows from the equality $x(t + \omega) = \lambda x(t)$ that $\Phi(t + \omega) = \Phi(t)$, if $\lambda < 0$, then $\Phi(t + 2\omega) = \Phi(t)$. This completes the proof of Theorem 2.3.

The proofs of Theorems 2.5 and 2.8 are based on two lemmas. The following assertion concerns equation (1.4), which is a particular case of (2.1) for $m = 1$. Thus presentation (2.7) is true for (1.4) with $a(t) = a_1(t)$ and $\tau(t) = \tau_1(t)$.

**Lemma 4.2.** Let all conditions of Theorem 2.5 be fulfilled. Then the number $\lambda$ in formula (2.7) satisfies the inequality $|\lambda| < 1$.

**Proof.** Without loss of generality, we suppose that $x(0) = 1$.

In the case of positivity of the solution $x(t)$ in $t \in [0, \infty)$, we have $x'(t) = -a(t)x(t - \tau(t)) \leq 0$ and the solution $x(t)$ decreases. It is clear that $0 < x(\omega) < 1$ for every $\omega > 0$ and $0 < \lambda < 1$.

In the case of the solution changing sign at the points $t_1, t_2, \ldots, t_n$ of the interval $[0, \omega]$, we can see that the amplitudes of its oscillations $|x(t_1)|, |x(t_2)|, \ldots, |x(t_n)|$, which are achieved at the points satisfying the inequalities $t_1 < t_1^* < t_2 < \cdots < t_n < t_n^* \leq \omega$, decrease. Indeed,

$$x(t) = x(0) - \int_0^t a(s)x(s - \tau(s)) \, ds, \quad t \in [0, t_1^*),$$

and then

$$x(t_1^*) = 1 - \int_0^{t_1^*} a(s)x(s - \tau(s)) \, ds > 1 - \int_0^{t_1^*} a(s) \, ds.$$ 

Note that $t - \tau(t) \in [0, t_1]$ if $t \in [0, t_1^*)$. This and inequality (2.8) imply that $-1 < x(t_1^*) < 0$. Estimate now $x(t_2^*)$:

$$x(t) = x(t_1^*) - \int_{t_1^*}^t a(s)x(s - \tau(s)) \, ds, \quad t \in [t_1^*, t_2^*),$$

then

$$x(t_2^*) = x(t_1^*) - \int_{t_1^*}^{t_2^*} a(s)x(s - \tau(s)) \, ds < x(t_1^*) - \int_{t_1^*}^{t_2^*} a(s)x(t_1^*) \, ds = x(t_1^*)(1 - \int_{t_1^*}^{t_2^*} a(s) \, ds).$$

Note that $t - \tau(t) \in [t_1, t_2]$ if $t \in [t_1^*, t_2^*)$. This and inequality (2.8) imply that $0 < x(t_2^*) < 1$. Similarly, we obtain that $|x(t_j^*)| < 1, j = 1, \ldots, n$. It is clear that $|x(\omega)| < 1$ and $|\lambda| < 1$. 

\qed
**Lemma 4.3.** Let all conditions of Theorem 2.8 be fulfilled and let \( \lambda \) in formula (2.7) satisfy the inequality \(|\lambda| < 1\), then equation (2.1) is exponentially stable.

**Proof.** Conditions (1.3) and (2.5) allow us to conclude that the inequality \(|C(t, 0)| \leq Ne^{-\alpha t}\), \(0 \leq t < \infty\), implies the inequality \(|C(t, \omega)| \leq Ne^{\alpha(t-\omega)}\), \(n\omega \leq t < \infty\) for every \( n = 1, 2, 3, \ldots \). Conditions (1.3) and (2.5) imply the equality \( C(t, s) = C(t, n\omega)C(n\omega, s)\) for \((n - 1)\omega < s < n\omega\) and \(t > n\omega\). It follows from the assumption that \(a_i(t)\) are essentially bounded (see condition (II)) that the equation \( \sup_{(n-1)\omega \leq s \leq n\omega} |C(n\omega, s)| = m + \infty\). Thus,

\[
|C(t, s)| = |C(t, n\omega)C(n\omega, s)| \leq mNe^{-\alpha(t-n\omega)} \leq mNe^{-\alpha(t-s)}.
\]

This completes the proof of Lemma 4.3.

**Proof of Theorem 2.5.** The proof follows from Lemmas 4.2 and 4.3.

**Proof of Theorem 2.8.** For the exponential stability of equation (2.1) we have only to obtain that \( \lambda \) in presentation (2.7) satisfies the inequality \(|x(\omega)| < |x(0)| \) implies \(|\lambda| < 1\), since (2.6). There exist only two possibilities:

(a) a solution \( x(t) \) is eventually positive (or negative), in this case there exists \( t_0 \) such that \( x(t) > 0 \) (\( x(t) < 0 \)) for \( t > t_0 \geq 0 \),

(b) a solution \( x(t) \) is oscillating, i.e., has infinite number of zeros \( x(\eta_i) = 0, \eta_i \to \infty \) as \( i \to \infty \).

Let us start with the proof of case (a). Assume, without losing generality, that the solution of equation (1.1) satisfies the inequality \( x(t) > 0 \) for \( t \in [t_0, \infty) \), where \( t_0 \geq 0 \). Then there exists \( t_1 \geq t_0 \) such that \( t - \tau_i(t) > t_1 \) for \( t \geq t_1 \) and \( i = 1, \ldots, m \). This solution \( x(t) \) is nonincreasing, since \( x'(t) = -\sum_{i=1}^{m} a_i(t)x(t-\tau_i(t)) \leq 0 \) for \( t \geq t_1 \), and it follows from the conditions \( a_i(t) \geq 0 \) and \( \int_{0}^{\infty} \sum_{i=1}^{m} a_i(t) dt > 0 \) for \( t \geq t_1 \) that there exists \( \omega_0 \) such that \( x(\omega_0) < |x(0)| \). Let us choose now \( \omega > \omega_0 \), then \( x(\omega) < x(\omega_0) < x(0) \). We obtain \(|\lambda| < 1\). The same conclusion we obtain for every eventually negative solution.

Let us prove now case (b). We can choose the intervals \( (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots \) around these points \( \eta_i \) such that \( |x(t)| < |x(0)| \) in each of them. Taking \( \omega \) in any of these intervals, we have \(|x(\omega)| < |x(0)| \) and \(|\lambda| < 1\).

Reference to Lemma 4.3 completes the proof of Theorem 2.8.

**Proof of Theorem 3.10.** The proof follows from [17, Theorem 3.3].

**Proof of Theorem 3.1.** Let us consider the equation

\[
x'(t) + \sum_{i=1}^{m} a_i(t)x(g_i(t)) = f(t), \quad t \in [0, \infty).
\]

Denote \( h_i(t) = t - \tau_i(t) \) and rewrite (4.1) in the forms:

\[
x'(t) + \sum_{i=1}^{m} a_i(t)x(h_i(t)) - \sum_{i=1}^{m} a_i(t)x(h_i(t)) + \sum_{i=1}^{m} a_i(t)x(g_i(t)) = f(t),
\]

\[
x'(t) + \sum_{i=1}^{m} a_i(t)x(h_i(t)) = \sum_{i=1}^{m} a_i(t)|x(h_i(t)) - x(g_i(t))| + f(t),
\]

\[
x'(t) + \sum_{i=1}^{m} a_i(t)x(h_i(t)) = \sum_{i=1}^{m} a_i(t) \int_{g_i(t)}^{h_i(t)} x' (\xi) \, d\xi + f(t),
\]

\[
x'(t) + \sum_{i=1}^{m} a_i(t)x(h_i(t)) = -\sum_{i=1}^{m} a_i(t) \int_{g_i(t)}^{h_i(t)} \left\{ \sum_{j=1}^{m} a_i(\xi) [x(g_j(\xi)) - f(\xi)] \right\} d\xi + f(t)
\]

for \( t \in [0, \infty) \). We see that (4.5) is of the form (2.2) with the notation \( h_i(t) = t - \tau_i(t) \).

Let us make a regularization of equation (4.5) using the formula of the general representation (2.3) of solutions

\[
x(t) = \int_{a}^{t} C_r(t, s)f(s) \, ds + C_r(t, 0)x(0), \quad t \in [0, \infty),
\]

where \( C_r(t, s) \) is defined by (2.3).

**Proof of Theorem 3.11.** The proof follows from [17, Theorem 3.4].

**Proof of Theorem 3.12.** The proof follows from [17, Theorem 3.5].
where $C_r(t, s)$ is the Cauchy function of equation (2.1). We obtain the equality

$$x(t) = -\int_0^t C_r(t, s) \left\{ \sum_{i=1}^m a_i(s) \int_{g_i(s)} h_i(\xi) x(g_i(\xi)) \, d\xi \right\} \, ds + \int_0^t C_r(t, s) \left\{ \sum_{i=1}^m a_i(s) \int_{g_i(s)} f(\xi) \, d\xi \right\} \, ds + \int_0^t C_r(t, s) f(s) \, ds + C_r(t, 0) x(0),$$

(4.6)

which can be rewritten as

$$x(t) = (Kx)(t) + \psi(t), \quad t \in [0, \infty),$$

(4.7)

where the operator $K : C \rightarrow C$ is defined by (3.2) and

$$\psi(t) = \int_0^t C_r(t, s) \left\{ \sum_{i=1}^m a_i(s) \int_{g_i(s)} h_i(\xi) \, d\xi \right\} \, ds + \int_0^t C_r(t, s) f(s) \, ds + C_r(t, 0) x(0), \quad t \in [0, \infty).$$

The Cauchy function $C_r(t, s)$ satisfies estimate (2.4). It is clear that $\psi(t)$ is bounded on the semi-axis $t \in [0, \infty)$ for every bounded $f(t), \ t \in [0, \infty)$. We obtain that the solution

$$x(t) = ((I - K)^{-1} \psi)(t) = ((I + K + K^2 + K^3 + \cdots) \psi)(t)$$

of equation (4.7) is bounded on $[0, \infty)$ for every bounded on the semi-axis right-hand side $f$. Now, according to the Bohr–Perron theorem (see [4, Theorem B.21, p. 500]), equation (3.1) is exponentially stable.

This completes the proof of Theorem 3.1. □

**Proof of Corollary 3.2.** The inequality in the formulation of this corollary implies that the norm of the operator $K : C \rightarrow C$ is less than one. Reference to Theorem 3.1 completes the proof. □

**Proof of Theorem 3.4.** If the solution $x(t)$ is positive for $t \in [0, \omega]$, then according to the formula of the Frobenius representation (2.7) we obtain that $x(t) > 0$ for $t \in [0, \infty)$. Then [17, Theorem 4.1], where equivalence of positivity of a nontrivial solution $x(t)$ of the homogeneous equation (2.1) and positivity of the Cauchy function $C_r(t, s)$ was obtained, implies that $C_r(t, s) > 0$ for $0 \leq s \leq t < \infty$.

Consider the equation

$$x'(t) + \sum_{i=1}^m a_i(t)x(h_i(t)) = 1, \quad t \in [0, \infty).$$

(4.8)

The constant function $v(t) = \frac{1}{A_*}$ satisfies the inequality

$$v'(t) + \sum_{i=1}^m a_i(t)v(h_i(t)) = \sum_{i=1}^m a_i(t) \leq 1, \quad t \in [0, \infty).$$

This according to the fact of positivity of $C_r(t, s) > 0$ for $0 \leq s \leq t < \infty$ implies that $v(t) \geq x(t)$ for $t \in [0, \infty)$, where $x(t)$ is any solution of (4.8) with the initial condition satisfying the inequality $0 \leq x(0) \leq \frac{1}{A_*}$. Formula (2.3) of representation of the general solution implies that

$$x(t) = \int_0^t C_r(t, s) \, ds + C_r(t, 0)x(0) \leq v(t) = \frac{1}{A_*}, \quad t \in [0, \infty),$$

and, if we take into account the positivity of $C_r(t, s)$, we obtain

$$\int_0^t C_r(t, s) \, ds \leq \frac{1}{A_*}, \quad t \in [0, \infty).$$

This completes the proof of Theorem 3.4. □

**Proof of Corollary 3.6.** The assertion follows from Theorem 3.4 in the case of $m = 1, a_1 = 1$. □
Proof of Theorem 3.7. It follows from Theorem 3.4 that $C_{\tau}(t, s) > 0$ for $0 \leq s \leq t < \infty$ and inequality (3.3) is satisfied. Rewriting equation (3.1) in the forms (4.1)–(4.5) (as in the proof of Theorem 3.1) and then in (4.6), we come to equation (4.7) in which the operator $K : C \rightarrow C$ is defined by formula (3.2). Taking out essential supremum of the assertion in the brackets and using estimate (3.3) in (3.2), we obtain that the norm of the operator $K : C \rightarrow C$ is less than one. Reference to Theorem 3.1 completes the proof.

Proof of Corollary 3.8. If $m = 1$ and $a_1 = 1$, then inequality (3.4), where $h_1(t) = h(t) = t - \tau(t)$, becomes

$$|h(t) - g(t)| < 1.$$  

Reference to Theorem 3.7 completes the proof.

5 Constructing Cauchy functions and example

The use of the Floquet theory allows us to obtain tests of stability that look different from the known ones.

Example 5.1. Consider the equation

$$x'(t) + x(h(t)) = 0, \quad t \in [0, \infty),$$  

(5.1)

with the deviation defined by the formula

$$h(t) = [t], \quad \text{where } [t] \text{ is the integer part of } t.$$  

We can now write the delay $\tau(t) = t - h(t) = \{t\}$, where $\{t\}$ is the fractional part of $t$, which is a 1-periodic function. The corresponding Cauchy function is

$$C_{\tau}(t, s) = \begin{cases} 1, & s \leq t < [s] + 1, \\ [s] - t + 2, & [s] + 1 \leq t < [s] + 2, \\ 0, & [s] + 2 \leq t. \end{cases}$$

It is clear that $C_{\tau}(t, s) \geq 0$ for $0 \leq s \leq t < \infty$, $\int_0^t C_{\tau}(t, s) \, ds \leq 1$ and (5.1) is exponentially stable.

The deviation $g(t) \equiv t - \theta(t)$ in equation (3.5) can be in the pink zone (see Figure 4) in order to guarantee the exponential stability of the equation $x'(t) + x(t - \theta(t)) = 0$. This follows now from Corollary 3.2.

It is clear that in the intervals $n + \frac{1}{3} < t < n + 1$, with natural $n$, we may have $t - \theta(t) > \frac{2}{3}$. This demonstrates that our criterion may yield exponential stability in cases when conditions (1.6) and (1.7) are not applicable.

Figure 4: Zones of stability of the equation $x'(t) + x(g(t)) = 0$.  

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6 Around Marchuk’s model of infection diseases

The well-known Marchuk model of infectious diseases is described by a system of delay differential equations containing dynamics of antigen, antibody, plasma cell concentration rates and relative features of the body (see [33, p. 69]), and discussions on this model and its possible development see in the recent paper [19]. The equation

$$V'(t) = \beta V(t) - \gamma F(t) V(t)$$  \hspace{1cm} (6.1)

in the system describes the change in the antigen concentration rate. Here, $V(t)$ is the antigen concentration rate and $F(t)$ represents the antibody concentration rate. The concentration of antigens in the body, on the other hand, increases in proportion to their number, on the other hand, it decreases as a result of interaction with antibodies.

Let us assume that a “natural” change of antibody concentration $F(t)$ is small enough. This situation is very typical for elderly and patients with a weak immune system. This assumption allows us to simplify the model and consider equation (6.1), which is most interesting for us, separately. Let us linearize now equation (6.1) in the neighborhood of the stationary solution $V = 0, F = F^*$, where $F^*$ is the antibody concentration of the healthy body. We get

$$V'(t) = \beta V(t) - \gamma F^* V(t).$$

To strengthen the resistance to viruses, there is a way to decrease the concentration of the viruses, for example, in the support of the immune system. Mathematically, this can be expressed by adding a delay feedback control $u(t)$ which is proportional to the concentration of the antigen at the time $h(t)$, i.e.,

$$u(t) = -kV(h(t)).$$

It is clear that $h(t) \leq t$. Concerning the form of the function $h(t)$, we can note the following. The function $h(t)$ usually represents a staircase going up: $h(t) = h_n, (n-1)\omega \leq t < n\omega$. Indeed, the patient is tested for the presence of the virus and a dose of medication is prescribed, depending on the result of the examinations. In each period of time until the next examination, which can be several days or even weeks, the results of the previous one are used. In simple models, we can suppose that this period is $\omega$ and $h(t) = \omega(n - 1)$ for $(n-1)\omega \leq t < n\omega$. But more complicated models, in which ambulatory examinations with results obtained several days after and examinations in hospital, where results and actions on their basis, follow just after, can be combined. Mathematically it means that the values $h_k$, that are different from $n\omega$, can also exist. The values at $(k-1)\omega < t < k\omega$ are $V(h_k)$ and they describe the results of such ambulatory examinations. Thus we come to the analysis of the delay equation

$$V'(t) = (\beta - \gamma F^*) V(t) - kV(h(t)),$$

where the deviation $h(t)$ is in the form described above. Stability with respect to the right-hand side, which is equivalent to exponential stability, is one of the most important problems here. In all “standard” methods for stability analysis, a smallness of the delay $t - h(t)$ is assumed and this does not allow to use them. Special methods for representation of solutions and stability analysis of equations with these deviations of arguments are needed. Our paper is a step in this direction.

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