The growth of entire solutions of some algebraic differential equations

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Abstract. In this paper, we estimate the growth order of entire solutions of some algebraic differential equations and improve the related result of Bergweiler, Barsegian and others. We also give some examples to show that our result is sharp in special cases.

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1 Introduction and the main results

Let $f(z)$ be a function holomorphic in the complex plane. We use the standard notations of Nevanlinna theory and denote the order of $f(z)$ by $\rho(f)$ (see Hayman [7], He [8], Laine [9] and Yang [11]).

Let D be a domain in the complex plane. A family $\mathcal{F}$ of meromorphic functions in D is normal if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges locally uniformly by a spherical distance to a function $g(z)$ meromorphic in D ($g(z)$ is permitted to be identically infinity).

We define the spherical derivative of the meromorphic function $f(z)$ as follows:

$$f^\#(z) := \frac{|f'(z)|}{1 + |f(z)|^2}.$$

An algebraic differential equation for $w(z)$ is of the form

$$P(z, w, w', \dots, w^{(k)}) = 0, \quad (1.1)$$

where P is a polynomial in each of its variables.

It is one of the important and interesting subjects to investigate the growth of a meromorphic solution $w(z)$ of the differential equation (1.1) in the complex plane.

In 1956, Gol'dberg [6] proved that the meromorphic solutions have finite growth order when $k = 1$. Some alternative proofs of this result were given by Bank and Kaufman [1] (see [9, 14]) and by Barsegian [2].

Barsegian [3] and Bergweiler [4] extended Gol'dberg's result to certain algebraic differential equations of higher order. The specific equations will be given below. Recently, the general estimates of growth order of meromorphic solutions $w(z)$ of the same equations, which depend on the degrees at infinity of coefficients of the differential polynomial in z , were given by Yuan et al. [12, 13].

In order to state the result, we first introduce some notation: $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, $t_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for $j = 1, 2, \dots, n$, and $t = (t_1, t_2, \dots, t_n)$. Define $M_t[w]$ by

$$M_t[w](z) := [w'(z)]^{t_1} [w''(z)]^{t_2} \dots [w^{(n)}(z)]^{t_n},$$

with the convention that $M_{\{0\}}[w] = 1$. We call $p(t) := t_1 + 2t_2 + \dots + nt_n$ the weight of $M_t[w]$. A differential polynomial $P[w]$ is an expression of the form

$$P[w](z) := \sum_{t \in I} a_t(z, w(z)) M_t[w](z), \quad (1.2)$$

where the a_t are rational in two variables and I is a finite index set. The total weight $W(P)$ of $P[w]$ is given by $W(P) := \max_{t \in I} p(t)$, and $\deg_{z, \infty} a_t$ denotes the degree at infinity in variable z concerning $a_t(z, w)$,

$$\deg_{z, \infty} a := \max_{t \in I} \max\{\deg_{z, \infty} a_t, 0\}.$$

Theorem A ([13]). *Let $w(z)$ be meromorphic in the complex plane and let $P[w]$ be a differential polynomial. If $w(z)$ satisfies the differential equation*

$$[w'(z)]^n = P[w] \quad \text{where } n \in \mathbb{N} \text{ and } n > W(P),$$

then the growth order $\rho := \rho(w)$ of $w(z)$ satisfies

$$\rho \leq 2 + \frac{2 \deg_{z, \infty} a}{n - W(P)}.$$

Remark. Barsegian [3] and Bergweiler [4] proved $\rho < \infty$ under the same conditions as those of Theorem A.

In this paper, we give a small upper bound of the growth order of entire solutions.

Theorem. *Let $w(z)$ be holomorphic in the complex plane and let $P[w]$ be a differential polynomial. If $w(z)$ satisfies the differential equation*

$$[w'(z)]^n = P[w] \quad \text{where } n \in \mathbb{N} \text{ and } n > W(P),$$

then the growth order $\rho := \rho(w)$ of $w(z)$ satisfies

$$\rho \leq 1 + \frac{\deg_{z,\infty} a}{n - W(P)}.$$

We note that $\deg_{z,\infty} a = 0$ when all $a_t(z, w)$ ($t \in I$) are rational functions with constant coefficients in variable w . Therefore the corollary below follows from the theorem.

Corollary. *Let $w(z)$ be holomorphic in the complex plane and let $P[w]$ be a differential polynomial with constant coefficients in variable w . If $w(z)$ satisfies the differential equation*

$$[w'(z)]^n = P[w] \quad \text{where } n \in \mathbb{N} \text{ and } n > W(P),$$

then $\rho(w) \leq 1$.

The following examples show that the result is sharp in special cases.

Example 1 ([10]). For $n > 0$, let $w(z) = \cos z^{\frac{n}{2}}$. Then $\rho(w) = \frac{n}{2}$ and w satisfies the algebraic differential equation

$$(w')^2 + \frac{n^2}{4} z^{n-2} (w^2 - 1) = 0, \quad (1.3)$$

when $n = 1$ or 2 , $\deg_{z,\infty} a = 0$, and the growth order $\rho(w)$ of any entire solution $w(z)$ of equation (1.3) satisfies $\rho(w) \leq 1$; when $n \geq 3$, $\deg_{z,\infty} a = n - 2$, and the growth order $\rho(w)$ of any entire solution $w(z)$ of (1.3) satisfies $\rho(w) \leq \frac{n}{2}$.

Example 2. For $n \geq 2$, the entire function $w(z) = e^{z^n}$ satisfies the algebraic differential equation

$$(w')^n = n^{n-1} (n-1) z^{(n-1)^2-1} w^n - n^{n-1} z^{(n-1)^2-n} w^{n-1} w' + (nz^{n-1} + z^{-1}) w (w')^{n-1}. \quad (1.4)$$

We know that the growth order $\rho(w) = n$, and $\deg_{z,\infty} a = \max\{(n-1)^2 - 1, 1\}$, the growth order $\rho(w)$ of any entire solution $w(z)$ of equation (1.4) satisfies $\rho \leq \max\{(n-1)^2, 2\}$.

Example 3. For $n \geq 3$, the entire function $w(z) = e^{z^n}$ satisfies the algebraic differential equation

$$(w')^n = n^{n-2} z^{(n-1)^2-n+1} w^{n-1} w'' + (1-n) z^{-1} w (w')^{n-1}. \quad (1.5)$$

We know that $\deg_{z,\infty} a = (n-1)^2 - n + 1$, and the growth order $\rho(w)$ of any entire solution $w(z)$ of equation (1.5) satisfies $\rho \leq n^2 - 3n + 3$.

2 Main lemmas

In order to prove our result, we need the following lemmas. Lemma 1 is an extending result of Zalcman [14] concerning a normal family.

Lemma 1 ([15]). *Let $\mathcal{F}$ be a family of meromorphic (or analytic) functions on the unit disc. Then $\mathcal{F}$ is not normal on the unit disc if and only if there exist*

- (a) *a number $0 < r < 1$,*
- (b) *points z_n with $|z_n| < r$,*
- (c) *functions $f_n \in \mathcal{F}$,*
- (d) *positive numbers $\rho_n \rightarrow 0$*

such that $g_n(\zeta) := f_n(z_n + \rho_n \zeta)$ converges locally uniformly to a non-constant meromorphic (or entire) function $g(\zeta)$; the order of $g(\zeta)$ is at most 2. In particular, we may choose w_n and ρ_n such that

$$\rho_n \leq \frac{2}{f_n^\#(w_n)}, \quad f_n^\#(w_n) \geq f_n^\#(0).$$

Lemma 2 ([5]). *Let $f(z)$ be holomorphic in the complex plane, $\sigma > -1$. If*

$$f^\#(z) = O(r^\sigma),$$

then for the Nevanlinna characteristic function $T(r, f)$ we have

$$T(r, f) = O(r^{\sigma+1}).$$

Lemma 3. *Let $f(z)$ be holomorphic in the whole complex plane with growth order $\rho := \rho(f) > 1$. Then for each $0 < \mu < \rho - 1$, there exists a sequence $a_n \rightarrow \infty$ such that*

$$\lim_{n \rightarrow \infty} \frac{f^\#(a_n)}{|a_n|^\mu} = +\infty. \quad (2.1)$$

Proof. If the assertion of Lemma 3 does not hold, then there exists a positive number $M > 0$, for large enough $|z|$ such that

$$f^\#(z) \leq M|z|^\mu. \quad (2.2)$$

By Lemma 2 and (2.2), we see that

$$T(r, f) = O(r^{\mu+1}).$$

Thus the growth order $\rho(f)$ of $f(z)$ satisfies $\rho \leq \mu + 1$. This contradicts the choice of μ . The proof of Lemma 3 is complete. $\square$

3 Proof of the theorem

Proof. Suppose that the conclusion of the theorem is not true, then there exists an entire solution $w(z)$ satisfying the differential equation $[w'(z)]^n = P[w]$ such that

$$\rho > 1 + \frac{\deg a}{n - W(P)}. \quad (3.1)$$

By Lemma 3 we know that for each $0 < \mu < \rho - 1$, there exists a sequence $a_m \rightarrow \infty$ ($m \rightarrow \infty$) such that (2.1) holds. Therefore $w_m^\#(0)$ are unbounded. Marty's criterion [11, 15] implies that the family $\{w(a_m + z)\}_{m \in \mathbb{N}}$ is not normal at $z = 0$. By Lemma 1, there exist sequences $\{b_m\}$ and $\{\rho_m\}$ such that

$$|a_m - b_m| < 1, \quad \rho_m \rightarrow 0, \quad (3.2)$$

and $g_m(\zeta) := w_m(b_m - a_m + \rho_m \zeta) = w(b_m + \rho_m \zeta)$ converges locally uniformly to a non-constant entire function $g(\zeta)$ (since $w(z)$ is entire, $g(z)$ is entire); the order of $g(\zeta)$ is at most 2. In particular, we may choose b_m and ρ_m such that

$$\rho_m \leq \frac{2}{w^\#(b_m)}, \quad w^\#(b_m) \geq w^\#(a_m). \quad (3.3)$$

According to (2.1), (3.1), (3.2) and (3.3), we can make the following conclusion:

For any fixed constant $0 \leq \mu < \rho - 1$, we have

$$\lim_{m \rightarrow \infty} b_m^\mu \rho_m = 0. \quad (3.4)$$

In the differential equation $[w'(z)]^n = P[w]$ we now replace z by $b_m + \rho_m \zeta$. Assuming that $P[w]$ has the form (1.2), we obtain

$$\rho_m^{-n} [g'_m(\zeta)]^n = \sum_{t \in I} a_t (b_m + \rho_m \zeta, g_m(\zeta)) \rho_m^{-p(t)} M_t[g_m](\zeta),$$

namely

$$[g'_m(\zeta)]^n = \sum_{t \in I} \frac{a_t (b_m + \rho_m \zeta, g_m(\zeta))}{b_m^{\deg_{z, \infty} a_t}} \left\{ b_m^{\frac{\deg_{z, \infty} a_t}{n - p(t)}} \rho_m \right\}^{n - p(t)} M_t[g_m](\zeta). \quad (3.5)$$

Because

$$0 \leq \mu = \frac{\deg_{z, \infty} a_t}{n - p(t)} \leq \frac{\deg_{z, \infty} a}{n - W(P)} < \rho - 1, \quad p(t) < n,$$

by (3.4) and (3.5) we can get $[g'(\zeta)]^n = 0$. Thus we get that $g(\zeta)$ is a constant. This contradicts the conclusion which we have made. The proof of the theorem is complete. $\square$

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