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Periodic solutions for stochastic Cohen–Grossberg neural networks with time-varying delays

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Abstract: This paper is concerned with the periodic solutions for a class of stochastic Cohen–Grossberg neural networks with time-varying delays. Since there is a non-linearity in the leakage terms of stochastic Cohen–Grossberg neural networks, some techniques are needed to overcome the difficulty in dealing with the nonlinearity. By applying fixed points principle and Gronwall–Bellman inequality, some sufficient conditions on the existence and exponential stability of periodic solution for the stochastic neural networks are established. Moreover, a numerical example is presented to validate the theoretical results. Our results are also applicable to the existence and exponential stability of periodic solution for the corresponding deterministic systems.

Keywords: exponential stability; periodic solution; stochastic neural networks.

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1 Introduction

Since most of its applications heavily rely on its dynamics, the Cohen–Grossberg neural networks (CGNNs) have been extensively studied by many scholars (see [1–8] and references therein). For example, in Ref. [9], the authors discussed the existence of a unique equilibrium point of a generalized Cohen–Grossberg BAM neural networks of neutral type delays by means of the Homeomorphism theory and inequality technique. They also studied the global asymptotic stability of the equilibrium solution by applying the existence result and constructing a Lyapunov functional. In Ref. [10], by combining Mawhin’s continuation theorem of coincidence degree theory with LMI method and using inequality techniques, some novel sufficient conditions on the existence and asymptotic stability of periodic solutions were established for the complex-valued Cohen–Grossberg neural networks. As is known to us, the stability of a real nervous system could be strengthened or weakened, even disappeared under the influenced of unavoidable external random perturbations [11]. Hence, there is an important role to consider the dynamical behavior of stochastic neural networks. In recent years, great efforts have been made to investigate the dynamics of stochastic neural networks (see [12–22] and reference therein).

However, it is widely recognized that most existing results on stochastic neural networks are mainly pertaining to the stability of considered systems. The direction of the existence of periodic solution or almost periodic solution has not been developed very well. As a matter of fact, the dynamics of neural networks not only involves a discussion of stability, but also involves study on many dynamic behaviors such as periodic or almost periodic oscillation, which are interesting phenomenon in neural networks. Therefore, it is reasonable and meaningful to consider the existence and stability of periodic solutions for stochastic neural networks. Some good results have been obtained on this issue. For example, some authors studied the existence and exponential stability of square-mean almost periodic solution for stochastic Hopfield neural networks with constant delays and constant coefficients in Ref. [23]; some authors considered the square-mean almost periodic solution of one-dimensional stochastic Nicholson’s blowflies model with delays and impulses in Ref. [24].

Unfortunately, since there is a non-linearity in the leakage terms of stochastic Cohen–Grossberg neural networks, we need to overcome much difficulties in
dealing with the nonlinearity. To the best of our knowledge, the results on the existence and stability of periodic solutions for stochastic Cohen–Grossberg neural networks are hard to obtain. To fill this gap, motivated by the aforementioned discussion, this paper is devoted to the existence and stability of periodic solutions for the following stochastic Cohen–Grossberg neural networks with time-varying delays:

$$\begin{align*}
\frac{dx_i(t)}{dt} &= -a_i(x_i(t)) + b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(x_j(t)) \\
&\quad - \sum_{j=1}^{n} d_{ij}(t) g_j(x_j(t - \tau_{ij}(t))) - I_i(t) \\
&\quad + \sum_{j=1}^{n} \sigma_{ij}(x_j(t)) dw_j(t), \quad i = 1, 2, \ldots, n,
\end{align*}$$  \((1)\)

where \(n\) corresponds to the number of units in the neural networks; \(a_i(\cdot)\) represents an amplification function; \(b_i(\cdot)\) is an appropriately behaved function; \(c_{ij}(t)\) and \(d_{ij}(t)\) denote the strengths of connectivity between cell \(i\) and \(j\) at time \(t\), respectively; the activation function \(f_j(\cdot)\) and \(g_j(\cdot)\) show how the \(j\)th neuron reacts to the input; \(\tau_{ij}(t)\) corresponds to the time delay required in processing and transmitting signals at time \(t\); \(I_i(t)\) denotes the \(i\)th component of an external input source introduced from outside the network to cell \(i\) at time \(t\); \(w(t) = (w_1(t), w_2(t), \ldots, w_n(t))^T\) is an \(n\)-dimensional Brownian motions defined on complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), where \(\mathcal{F}\) is the associated \(\sigma\)-algebra generated by \(w(t)\) with the probability measure \(\mathbb{P}\); \(\sigma_{ij}\) is a Borel measurable function, \(\sigma = (\sigma_{ij})_{n \times n}\) is diffusion coefficient matrix, \(i, j = 1, 2, \ldots, n\).

Let \((\mathbb{B}, \|\cdot\|)\) be a Banach space and \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Define \(L^p(\mathbb{P}, \mathbb{B})\) for \(p \geq 1\) as the space of all \(\mathbb{B}\)-value random variable \(X\) such that \(E(\|X\|^p) = \int \|X\|^p d\mathbb{P} < \infty\). Then \(L^p(\mathbb{P}, \mathbb{B})\) is a Banach space with the norm \(\|X\|_p = \left(\int \|X\|^p d\mathbb{P}\right)^{1/p}\), for each \(X \in L^p(\mathbb{P}, \mathbb{B})\).

Let \(C([0,\infty)^p) = \{\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T: \mathbb{R} \to \mathbb{R}^n, \varphi\) is continuous\}. For convenience, for an \(\omega\)-periodic function \(u: \mathbb{R} \to \mathbb{R}\), denote \(\hat{u} = \sup_{t \in \mathbb{R}} |u(t)|, \underline{u} = \inf_{t \in \mathbb{R}} |u(t)|\). The initial condition of Eq. \((1)\) is the following

\[x_i(s) = \varphi_i(s), s \in (-\infty, t_0],\]

where \(\varphi_i(s) \in C((-\infty, t_0], \mathbb{R})\), \(t_0 \in \mathbb{R}\).

Throughout this paper, we assume that

\((H_i)\): \(a_i(\cdot) \in C^1(\mathbb{R}, (0, +\infty))\) has continuous derivative and there exist positive constants \(a_i, \bar{a}_i, L_i^a\) such that

\[a_i \leq a_i(u) \leq \bar{a}_i, \quad |a_i(u) - a_i(v)| \leq L_i^a|u - v|, \quad \text{for all } u, v \in \mathbb{R}, i = 1, 2, \ldots, n;\]

\((H_i)\): \(b_i(\cdot) \in C^1(\mathbb{R}, \mathbb{R})\) and there exist positive constants \(c_i\) such that \(b_i(u) \leq c_i, \quad \text{for all } u \in \mathbb{R}, i = 1, 2, \ldots, n;\)

\((H_i)\): \(f_j(\cdot), g_j(\cdot), \sigma_{ij}(\cdot) \in C(\mathbb{R}, \mathbb{R})\) are all Lipschitz-continuous with positive Lipschitz constants \(L_j^f, L_j^g\) and \(L_{ij}\), respectively. And there exists a positive constant \(L_j^\bar{a}\) such that \(|\sigma_{ij}(u)| \leq L_j^\bar{a}\), \(\text{for all } u \in \mathbb{R}, i = 1, 2, \ldots, n;\)

\((H_i)\): there exists a positive constant \(\delta_i\) such that

\[\frac{\sigma_i(u)\sigma_i(v)}{\sigma_i(u) - \sigma_i(v)} - \frac{\sigma_i(u)\sigma_i(v)}{\sigma_i(v) - \sigma_i(u)} \leq \delta_i|u - v|, \quad \text{for all } u, v \in \mathbb{R}, i = 1, 2, \ldots, n.\]

\[2\] Preliminaries

In this section, we recall some definitions and make some preparations.

**Definition 2.1:** A stochastic process \(x(t)\) is said to be periodic with period \(\omega\) if its finite dimensional distributions are periodic with period \(\omega\), that is, for any positive integer \(m\) and any moments of time \(t_1, t_2, \ldots, t_m\), the joint distribution of the random variables \(x_{t_1+k\omega}(s), x_{t_2+k\omega}(s), \ldots, x_{t_m+k\omega}(s)\) are independent of \(k, k = \pm 1, \pm 2, \ldots\).

**Lemma 2.1:** ([25]) If \(x(t)\) is an \(\omega\)-periodic stochastic process, then its mathematical expectation and variance are \(\omega\)-periodic.

**Lemma 2.2:** ([26]) (The Itô isometry) If \(\phi(t, \omega)\) is bounded and elementary, then

\[E\left(\int_0^T \phi(t, \omega)dB_t(\omega)^2\right) = E\left[\int_0^T \phi^2(t, \omega)dt\right].\]

**Lemma 2.3:** ([27]) For any \(x \in \mathbb{R}^n\) and \(p > 0\),

\[|x|^p \leq n(\xi^{(p-1)/p})^\nu \sum_{i=1}^n x_i^p, \quad \left(\sum_{i=1}^n x_i\right)^p \leq n^{(p-1)/p} \sum_{i=1}^n x_i^p,\]

where \(\nu\) denotes the Min operator, that is, \(a \vee b = \begin{cases} a, & a \geq b \ \\ b, & a < b. \end{cases}\)

**Definition 2.2:** The periodic solution \(x(t, t_0, \varphi)\) with initial value \(\varphi\) of Eq. \((1)\) is said to be exponential stable, if there are
constants $\lambda > 0$ and $M > 1$ such that for any solution $y(t, t_0, \varphi_i)$ with initial value $\varphi_i$ of Eq. (1) satisfies
\[
E(\|x - y(t)\|^2) \leq M\left(\|\varphi - \varphi_i\|^2\right) e^{-\lambda(t - t_0)}, \quad t \geq t_0.
\]

By (H$_3$), the antiderivative of $\frac{1}{a_i(x_i)}$ exists. Choose the antiderivative $a_i(x_i)$ of $\frac{1}{a_i(x_i)}$ with $a_i(0) = 0$, it follows that $a_i(x_i) = \frac{1}{a_i(x_i)}$. Since $a_i(x_i) > 0$, $a_i(x_i)$ is increasing about $x_i$ and the inverse function $a_i^{-1}(x_i)$ of $a_i(x_i)$ is existent, continuous and derivative. Hence, $(a_i^{-1})'(x_i) = a_i(x_i)$, where $(a_i^{-1})'(x_i)$ is the derivative of $a_i^{-1}(x_i)$ about $x_i$ and composition function $b_i(a_i^{-1}(x_i))$ is differential. Denote $y_i(t) = a_i(x_i(t))$, then $x_i(t) = \alpha_i(y_i(t))$. Substituting these equalities into Eq. (1), combined with Itô formula, we have
\[
\begin{align*}
    dy_i(t) &= \left[ -b_i(y_i(t)) y_i(t) + \sum_{j=1}^{n} c_{ij}(t) f_j(a_i^{-1}(y_i(t))) ight. \\
    & \quad + \sum_{j=1}^{n} d_{ij}(t) g_j(a_i^{-1}(y_i(t) - \tau_j(t))) + \sum_{j=1}^{n} E_j(t) \\
    & \quad \left. + \frac{\alpha_i'(a_i^{-1}(y_i(t)))}{2\alpha_i^2(a_i^{-1}(y_i(t)))} \left( \sum_{j=1}^{n} a_j(a_i^{-1}(y_j(s))) \right)^2 \right] dt \\
    & \quad + \frac{1}{\alpha_i(a_i^{-1}(y_i(t)))} \sum_{j=1}^{n} a_j(a_i^{-1}(y_j(t))) dw_j(t), i = 1, 2, \ldots, n.
\end{align*}
\]

Furthermore, Eq. (3) can be written as follows
\[
\begin{align*}
    dy_i(t) &= \left[ -b_i(y_i(t)) y_i(t) + \sum_{j=1}^{n} c_{ij}(t) f_j(a_i^{-1}(y_i(t))) ight. \\
    & \quad + \sum_{j=1}^{n} d_{ij}(t) g_j(a_i^{-1}(y_i(t) - \tau_j(t))) + \sum_{j=1}^{n} E_j(t) \\
    & \quad \left. + \frac{\alpha_i'(a_i^{-1}(y_i(t)))}{2\alpha_i^2(a_i^{-1}(y_i(t)))} \left( \sum_{j=1}^{n} a_j(a_i^{-1}(y_j(s))) \right)^2 \right] dt \\
    & \quad + \frac{1}{\alpha_i(a_i^{-1}(y_i(t)))} \sum_{j=1}^{n} a_j(a_i^{-1}(y_j(t))) dw_j(t),
\end{align*}
\]

where $b_i(y_i(t)) = [b_i(a_i^{-1}(y_i(t)))]'$, $[b_i(a_i^{-1}(y_i(t)))]'$ denotes the derivative of $b_i(a_i^{-1}(y_i(t))$, $0 \leq \xi_i \leq 1$, $i = 1, 2, \ldots, n$. From (H$_3$) and (H$_4$), we have
\[
\alpha_i|u - v| \leq |\alpha_i^{-1}(u) - \alpha_i^{-1}(v)| \leq \hat{\alpha}_i|u - v|
\]
and $\hat{\beta}_i(y_i(t)) \geq \zeta_i$, $i = 1, 2, \ldots, n$.

**Remark 2.1:** Eq. (1) has a periodic solution which is exponentially stable, if and only if, Eq. (4) has a periodic solution which is exponentially stable.

Together with Eq. (4), the solution of the linear system
\[
y_i'(t) = -\hat{b}_i(y_i(t)) y_i(t),
\]
is in the form
\[
y_i(t, t_0, y_i(t_0)) = W_i(t, t_0) y_i(t_0), t_0 \in \mathbb{R}, i = 1, 2, \ldots, n,
\]
where
\[
W_i(t, s) = \exp \left\{ -\int_{s}^{t} \hat{b}_i(y_i(\sigma)) d\sigma \right\}.
\]

From (H$_3$), it can be obtained that the Cauchy matrix $W_i(t, s)$ of system Eq. (5) satisfies
\[
W_i(t, s) \leq e^{-\xi_i(t-s)}, \quad t \geq s, s, t \in \mathbb{R}, i = 1, 2, \ldots, n.
\]

### 3 Existence of periodic solution

In this section, we will state and prove the existence of periodic solutions for Eqs. (1) and (7).

**Theorem 3.1:** Let (H$_3$)–(H$_5$) hold. Suppose further that
\[
p = \max_{t \in \mathbb{R}} \frac{1}{\lambda(t)} < 1,
\]
where
\[
\Theta_i = \left( \sum_{j=1}^{n} \hat{\alpha}_j L_j + \frac{\hat{\alpha}_j L_j}{2} \right)^2 + \left( \frac{1}{\alpha_i} \sum_{j=1}^{n} L_j \right)^2,
\]
\[
i = 1, 2, \ldots, n.
\]

Then Eq. (1) has a unique periodic solution.

**Proof:** Set $\mathbb{X} = \{ \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T \in C(\mathbb{R}, \mathbb{R}^n) | \varphi_i(t + \omega) = \varphi_i(t), i = 1, 2, \ldots, n \}$, with the norm $\|\varphi\|_\mathbb{X} = E[\varphi]^2$, where $\|\varphi\|_1 = \max_{1 \leq i \leq n, 1 \leq j \leq n} \{ \sup_{t \in \mathbb{R}} |\varphi_i(t)| \}$. Then $\mathbb{X}$ is a Banach space. Define a map $\Phi$ on $\mathbb{X}$ by
\[
(\Phi \varphi)_i(t) = \left( (\Phi \varphi_1)_i(t), (\Phi \varphi_2)_i(t), \ldots, (\Phi \varphi_n)_i(t) \right)^T, \quad t \in \mathbb{R},
\]
where
\[
(\Phi \varphi)_i(t) = \int_{-\infty}^{t} W_i(t, s) \left[ \sum_{j=1}^{n} c_{ij}(s) f_j(a_i^{-1}(\varphi_j(s))) \right.
\]
\[
+ \sum_{j=1}^{n} d_{ij}(s) g_j(a_i^{-1}(\varphi_j(s) - \tau_{ij}(s))) + E_j(s) \\
\left. - \frac{\alpha_i'(a_i^{-1}(\varphi_i(s)))}{2\alpha_i^2(a_i^{-1}(\varphi_i(s)))} \left( \sum_{j=1}^{n} a_j(a_i^{-1}(\varphi_j(s))) \right)^2 \right] ds \\
+ \int_{-\infty}^{t} W_i(t, s) \sum_{j=1}^{n} a_j(a_i^{-1}(\varphi_j(s))) dw_j(s),
\]
i = 1, 2, \ldots, n.

For $i = 1, 2, \cdots, n$, it is obvious that $(\Phi \varphi)_i(t + \omega) = (\Phi \varphi)_i(t)$, which shows that $\Phi$ is a self-mapping from $\mathbb{X}$ to $\mathbb{X}$. In the following, we need to prove that $\Phi$ is a contraction mapping. For any $\varphi, \psi \in \mathbb{X}$, we have

$$
| (\Phi \varphi)_i(t) - (\Phi \psi)_i(t) | \\
\leq \int_{t-\omega}^{t} W_i(s, t) \left[ \sum_{j=1}^{n} c_j(s) (f_j(\alpha_j^{-1}(\varphi_j(s))) - f_j(\alpha_j^{-1}(\psi_j(s)))) + \sum_{j=1}^{n} d_j(s) (g_j(\alpha_j^{-1}(\varphi_j(s) - \tau_j(s))) - g_j(\alpha_j^{-1}(\psi_j(s) - \tau_j(s)))) \right] ds \\
+ \sum_{j=1}^{n} \frac{\alpha_j'(\alpha_j^{-1}(\varphi_j(s)))}{2\alpha_j'(\alpha_j^{-1}(\psi_j(s)))} \sum_{i=1}^{n} \sigma_i'(\alpha_j^{-1}(\psi_j(s))) ds
$$

We evaluate the first term of the right-hand side of Eq. (6). Taking expectations, by Lemma 2.3, we have

$$
E|H_1|^2 = E \left| \int_{t-\omega}^{t} W_i(s, t) \left[ \frac{1}{\alpha_j'(\alpha_j^{-1}(\varphi_j(s)))} \sum_{i=1}^{n} \sigma_i'(\alpha_j^{-1}(\varphi_j(s))) \right] ds \right|^2
$$

As to the second term of the right-hand side of Eq. (6), for $i = 1, 2, \cdots, n$, we also have

$$
E|H_2|^2 = E \left| \int_{t-\omega}^{t} W_i(s, t) \left( \frac{1}{\alpha_j'(\alpha_j^{-1}(\varphi_j(s)))} \right) \right|^2
$$

As to the last term of the right-hand side of Eq. (6), for $i = 1, 2, \cdots, n$, we have

$$
E|H_3|^2 = E \left| \int_{t-\omega}^{t} W_i(s, t) \left( \frac{1}{\alpha_j'(\alpha_j^{-1}(\varphi_j(s)))} \right) \right|^2
$$

Therefore, by Eq. (6), we have

$$
E|\Phi \varphi - \Phi \psi|^2 \leq \rho E|\varphi - \psi|^2.
$$

It follows that $||\Phi \varphi - \Phi \psi||_{\mathbb{X}} \leq \rho||\varphi - \psi||_{\mathbb{X}}$. In view of $\rho < 1$, Eq. (4) has a unique periodic solution. That is, Eq. (1) has a unique periodic solution. This completes the proof of Theorem 3.1.
\begin{align}
    x_i(t) &= -a_i(x_i(t)) \left[ b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}(t)f_j(x_j(t)) - \sum_{j=1}^{n} d_{ij}(t)g_j(x_j(t - \tau_{ij}(t))) - I_i(t) \right],
    \tag{7}
\end{align}

where \( i = 1, 2, \ldots, n \). According to the above proof method, we can get the following theorem about the deterministic neural network system Eq. (7).

**Remark 3.1:** The corresponding deterministic systems of Eq. (1) is the following neural networks:

**Theorem 3.2:** Let \((H_1)-(H_3)\) hold and the following \((H_4)\) hold.

\((H_4)^1 f_j(\cdot), g_j(\cdot) \in C(\mathbb{R}, \mathbb{R})\) are all Lipschitz-continuous with positive Lipchitz constants \( L_j^f, L_j^g \), respectively, \( i, j = 1, 2, \ldots, n \);

Suppose further that \( \rho_1 = \max_{1 \leq i \leq n} \left\{ \frac{a_{\theta}^{(i)}}{a_i} \right\} < 1 \), where

\[
    \Theta^{(i)} = \sum_{j=1}^{n} a_i(c_j L_j^f + d_j L_j^g), \quad i = 1, 2, \ldots, n.
\]

Then (3.4) has a unique periodic solution.

Then (3.4) has a unique periodic solution.

**4 Exponential stability of periodic solution**

In this section, we will state and prove the exponential stability of periodic solutions for Eqs. (1) and (7).

**Theorem 4.1:** Let \((H_1)-(H_3)\) and \( \rho < 1 \) hold. Suppose further that \( \gamma = \min_{1 \leq i \leq n} \left\{ 2\zeta_i - \Theta_i \right\} > 0 \). Then, the periodic solution of Eq. (1) is exponentially stable.

**Proof:** By Theorem 3.1, Eq. (4) has a periodic solution \( y(t) = [y_i(t)]_{i=1}^{n} \) with initial condition \( \varphi(t) = \{\varphi_i(t)\}_{i=1}^{n} \). Suppose that \( \gamma^* (t) = [y_i^*(t)]_{i=1}^{n} \) is an arbitrary solution of Eq. (4) with the initial condition \( \varphi^*(t) = [\varphi_i(t)]_{i=1}^{n} \). One can easily to see that any solution of Eq. (4) can be expressed as follows

\[
y_i(t) = W_i(t, t_0)y_i(t_0) + \int_{t_0}^{t} W_i(s, t) \left[ \sum_{j=1}^{n} c_{ij}(s)f_j(y_j(s)) \right. \\
+ I_i(s) + \sum_{j=1}^{n} d_{ij}(s)g_j(y_j(s - \tau_{ij}(s))) \\
- \frac{a_i(\alpha_i^{-1}(\varphi_i(s)))}{2a_i^2(\alpha_i^{-1}(\varphi_i(s)))} \left( \sum_{j=1}^{n} \sigma_j^2(\alpha_j^{-1}(\varphi_j(s))) \right)^2 \\
\left. + \int_{t_0}^{t} W_i(s, t) \sum_{j=1}^{n} d_{ij}(s)g_j(y_j(s)) \right] \sigma_i(y_i(s))\,dw_i(s),
\tag{8}
\]

\[ t \geq t_0, i = 1, 2, \ldots, n. \]

Denote \( z(t) = \{z_i(t)\}_{i=1}^{n} \), where \( z_i(t) = y_i(t) - y_i^*(t), i = 1, 2, \ldots, n \). Then from Eq. (8), for \( i = 1, 2, \ldots, n \) and \( t \geq t_0 \), we have

\[
z_i(t) = z_i(t_0) + \int_{t_0}^{t} W_i(s, t) \left[ \sum_{j=1}^{n} c_{ij}(s)f_j(\alpha_j^{-1}(y_j(s))) \\
- f_j(\alpha_j^{-1}(y_j^*(s))) \\
+ \sum_{j=1}^{n} d_{ij}(s)(g_j(\alpha_j^{-1}(y_j(s - \tau_{ij}(s)))) - g_j(\alpha_j^{-1}(y_j^*(s - \tau_{ij}(s))))) \\
- \frac{a_i(\alpha_i^{-1}(y_i(s)))}{2a_i^2(\alpha_i^{-1}(y_i(s)))} \left( \sum_{j=1}^{n} \sigma_j^2(\alpha_j^{-1}(y_j(s))) \right)^2 \\
+ \int_{t_0}^{t} W_i(s, t) \sum_{j=1}^{n} d_{ij}(s)g_j(y_j(s)) \right] \sigma_i(y_i(s))\,dw_i(s),
\]

\[ t \geq t_0, i = 1, \ldots, n. \]

Taking expectations, by Lemma 2.3, we have

\[
E|z_i(t)|^2 \leq 4E \left( |G_1|^2 + |G_2|^2 + |G_3|^2 + |G_4|^2 \right), i = 1, 2, \ldots, n.
\tag{9}
\]

Proceeding as the proof of Theorem 3.1, for \( i = 1, 2, \ldots, n \), we have

\[
E|G_i|^2 \leq e^{-2\gamma(t-t_0)}E|z_i(t_0)|^2.
\]
For the second term of (9), we have
\[
E[G_2] = E\int_{t_0}^{t_1} W_i(t, s) \left[ \sum_{j=1}^{n} c_i(s) (f_j(a_i^{-1}(y_j(s))) - f_j(a_i^{-1}(y_i(s)))) \right] ds
\]
\[
+ \sum_{j=1}^{n} d_i(s) (g_j(a_i^{-1}(y_j(s))) - g_j(a_i^{-1}(y_i(s))))
\]
\[
+ \frac{\sigma_i(a_i^{-1}(y_i(s)))}{2\Theta_i} \left( \sum_{j=1}^{n} \sigma^2_j(a_i^{-1}(y_j(s))) \right)^2
\]
\[
+ \frac{\sigma_i(a_i^{-1}(y_i(s)))}{2\Theta_i} \left( \sum_{j=1}^{n} \sigma^2_j(a_i^{-1}(y_j(s))) \right)^2 |ds|
\]
\[
\leq \left( \sum_{j=1}^{n} \sigma_j \right)^2 \left( \sum_{j=1}^{n} \frac{L_j^2 + \delta_j}{2} \right) \int_{t_0}^{t_1} e^{-2\gamma(t-s)} E|\bar{z}_i|^2 ds, \quad i = 1, 2, \ldots, n.
\]

As to the third term of Eq. (9), for \( i = 1, 2, \ldots, n \), we also have
\[
E[G_3] = E\int_{t_0}^{t_1} W_i(t, s) \left[ \sum_{j=1}^{n} c_i(s) (f_j(a_i^{-1}(y_j(s))) - f_j(a_i^{-1}(y_i(s)))) \right] ds
\]
\[
- \sigma_i(a_i^{-1}(y_i(s))) dw_i(s)
\]
\[
= E\int_{t_0}^{t_1} W_i(t, s) \left[ \sum_{j=1}^{n} \sigma_j \left( a_i^{-1}(y_i(s)) \right) \right] ds
\]
\[
\leq \left( \sum_{j=1}^{n} \frac{L_j^2 + \delta_j}{2} \right) \int_{t_0}^{t_1} e^{-2\gamma(t-s)} E|\bar{z}_i|^2 ds, \quad i = 1, 2, \ldots, n.
\]

For the fourth term of Eq. (9), we have
\[
E[G_4] = E\int_{t_0}^{t_1} W_i(t, s) \left[ \frac{1}{a_i(a_i^{-1}(y_i(s)))} - \frac{1}{a_i^{-1}(y_i(s))} \right] ds
\]
\[
= E\int_{t_0}^{t_1} W_i(t, s) \left[ \sum_{j=1}^{n} \sigma_j \left( a_i^{-1}(y_i(s)) \right) \right] ds
\]
\[
\leq \left( \sum_{j=1}^{n} \frac{L_j^2 + \delta_j}{2} \right) \int_{t_0}^{t_1} e^{-2\gamma(t-s)} E|\bar{z}_i|^2 ds, \quad i = 1, 2, \ldots, n.
\]

It follows from Eq. (9) that,
\[
E|\bar{z}_i(t)|^2 \leq 4e^{-2\gamma(t-t_0)} E|\bar{z}_i(t_0)|^2 + 4 \left[ \sum_{j=1}^{n} \frac{c_j I_j + d_j L_j^2 + \delta_j}{2} \right]^2
\]
\[
+ \left( \sum_{j=1}^{n} l_j \right)^2 \int_{t_0}^{t_1} e^{-2\gamma(t-s)} E|\bar{z}_i|^2 ds,
\]
where \( i = 1, 2, \ldots, n \). Set \( V_i(t) = e^{2\gamma(t)} E|\bar{z}_i(t)|^2 \). By Gronwall–Bellman inequality, we have
\[
E|\bar{z}_i(t)|^2 \leq 4E|\bar{z}_i(t_0)|^2 (e^{2\gamma(t-t_0)}).
\]

Hence, the periodic solution of Eq. (4) is exponentially stable. That is, the periodic solution of Eq. (1) is exponentially stable. This completes the proof of Theorem 4.1.

**Theorem 4.2.** Let \((H_1)-(H_3), (H'_4)\) hold. Suppose further that
\[
y_1 = \min_{1 \leq i \leq n} \left| c_i - \Theta_i \right| > 0,
\]
then the periodic solution of Eq. (7) is exponentially stable.

**Remark 4.1.** Since there is a nonlinearity in the leakage terms, we need to overcome much more difficulties in proving the existence and stability. By using some techniques of calculus and Itô formula, we can transform the nonlinearity to a linearity. Combining the fixed points principle and Gronwall–Bellman inequality, We discussed both the existence and the stability of periodic solutions for the stochastic Cohen–Grossberg neural networks and its corresponding deterministic systems. This is different from the previous results which only consider the stability of the system. Hence, our results can be applied to the system considered in [23], but not vice versa.

### 5 An example

In this section, we will give an example to illustrate the feasibility of our results.

\[
dx_i(t) = -a_i(x_i(t)) \left[ b_i(x_i(t)) - \sum_{j=1}^{n} c_i (f_j(x_j(t)) \right]
\]
\[
- \sum_{j=1}^{n} d_j(t) g_j(x_j(t) - \tau_j(t)) - I_i(t) \right] dt
\]
\[
+ \sum_{j=1}^{n} \sigma_j(x_j(t)) dw_j(t), \quad i = 1, 2, \ldots
\]
and the corresponding deterministic systems

\[
x^*_i(t) = -a_i(x_i(t)) \left[ b_i(x_i(t)) - \sum_{j=1}^{n} c_i (f_j(x_j(t)) \right]
\]
\[
- \sum_{j=1}^{n} d_j(t) g_j(x_j(t) - \tau_j(t)) - I_i(t) \right], \quad i = 1, 2, \ldots
\]
where
The states of Figure 2: Besides, all states of Example 5.1: Let $n = 2$. Consider the following stochastic Cohen–Grossberg neural networks

$$a_i(u) = 0.3 + 0.1 \sin u, \quad a_j(u) = 0.16 + 0.04 \cos u, \quad b_i(u) = 0.8u, \quad b_j(u) = 0.6u, \quad f_i(u) = 0.025(|u + 1| + |u|), \quad f_j(u) = 0.005(|u + 1| + |u|), \quad g_i(u) = 0.02|u|, \quad g_j(u) = 0.01|u|,$$

$$h_i(t) = 0.3 \cos 2t, \quad h_j(t) = 0.2 \sin 2t,$$

$$(c_{ij}(t))_{2 \times 2} = \begin{pmatrix} 0.2 + 0.1 \cos 2t & 0.1 + 0.05 \sin 2t \\ 0.3 + 0.1 \sin 2t & 0.2 + 0.01 \cos 2t \end{pmatrix},$$

$$(d_{ij}(t))_{2 \times 2} = \begin{pmatrix} 0.15 + 0.05 \cos 2t & 0.2 - 0.1 \sin 2t \\ 0.25 + 0.05 \sin 2t & 0.1 + 0.05 \cos 2t \end{pmatrix},$$

$$(\sigma_{ij}(u))_{2 \times 2} = \begin{pmatrix} 0.1 + 0.04 \sin u & 0.2 + 0.06 \sin u \\ 0.1 + 0.01 \cos u & 0.06 + 0.03 \cos u \end{pmatrix},$$

$$(\tau_{ij}(t))_{2 \times 2} = \begin{pmatrix} 0.02 + 0.01 \cos 2t & 0.25 + 0.05 \sin 2t \\ 0.15 + 0.05 \sin 2t & 0.1 + 0.05 \cos 2t \end{pmatrix}.$$

Remark 5.1: It can be seen that under certain conditions, the random term has no effect on the periodicity of the system Eq. (11). That is, both the stochastic Cohen–Grossberg neural networks and its corresponding deterministic systems have the similar periodicity. However, in the previous literature, only deterministic systems or the stability of stochastic systems were studied. To some extent, our results complement the previous results.

6 Conclusion and discussion

By utilizing Itô formula, we transform the nonlinear link terms in stochastic Cohen–Grossberg neural networks to the linear terms. Combining the fixed points principle and Gronwall–Bellman inequality, we establish some sufficient conditions on the existence and exponential stability of periodic solutions for the stochastic Cohen–Grossberg neural networks and its corresponding deterministic systems. In the end, an example and its numerical simulations is presented to support our theoretical results. Our results show that under certain conditions, the random term has no effect on the periodicity of the corresponding deterministic systems. Our method can be applied to other stochastic Cohen–Grossberg neural networks, such as stochastic Cohen–Grossberg BAM neural networks and so on.

Figure 1: The states of $x_1$ of Eq. (10) with different initial values.

Figure 2: The states of $x_2$ of Eq. (10) with different initial values.

Figure 3: The states of $x_1$ of Eq. (10) with different initial values.

Figure 4: The states of $x_2$ of Eq. (10) with different initial values.
Moreover, the periodicity of other stochastic neural networks with linear terms, could be studied by the fixed points principle and Gronwall–Bellman inequality, such as stochastic Hopf neural networks, stochastic BAM neural networks and so on. Most existing results on stochastic neural networks only studies the stability of considered models. We discussed not only the stability of solutions, but also the existence of solutions. Hence, our results complement and improve the previous results in a sense. Furthermore, under what conditions that the stability of deterministic neural networks be destroyed is our further work.

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