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Existence, stability and controllability results of fractional dynamic system on time scales with application to population dynamics

<https://doi.org/10.1515/ijnsns-2019-0199>

Received July 29, 2019; accepted September 25, 2020; published online October 30, 2020

Abstract: In this manuscript, we investigate the existence, uniqueness, Hyer-Ulam stability and controllability analysis for a fractional dynamic system on time scales. Mainly, this manuscript has three segments: In the first segment, we give the existence of solutions. The second segment is devoted to the study of stability analysis while in the last segment, we establish the controllability results. We use the Banach and nonlinear alternative Lery-Schauder-type fixed point theorem to establish these results. Also, we give some numerical examples for different time scales. Moreover, we give two applications to outline the effectiveness of these obtained results.

Keywords: controllability; existence; fractional dynamic system; stability; time scales.

AMS Subject Classification: 34N05; 34A12; 93B05; 34A08.

1 Introduction

The theory of fractional calculus began with a correspondence between Leibniz and L'Hospitalin in 1695. Nowadays, lots of literature is available on numerical as well as theoretical work on this subject. It has many applications in the fields of bioengineering [1], physics [2], viscoelasticity [3], biology [4] and mathematical networks [5, 6], etc. A few years back, many researchers and specialists have appeared incredible interest in fractional theory because of the memory character of fractional derivative, which is the generalization of integer-order derivative and can describe many phenomena of finance, biology and physics, etc. that integer-order derivative cannot clarify. For more recent work, please see [7–12]. In addition, one of the qualitative principles which could be very significant from the numerical factor, and optimization view is committed to stability analysis of the solution to dynamical systems of integer as well as fractional order. During the last few decades, the concept of stability like Mittag-Leffler function, exponential, and Lyapunov stability for different types of dynamical systems have been studied. However, an interesting kind of stability was introduced by Ulam and Hyers is known as Ulam-Hyer's stability. This stability analysis is very useful in many applications e.g., in chemical reactions, fluid flow, semiconductors, population dynamics, heat conduction, and elasticity. There are few researchers who established the Ulam's type of stability results for the ordinary, as well as for the fractional differential and difference equations [13–16] and references therein.

On the other hand, one investigates the continuous and discrete cases separately. The concept of time scale analysis introduced by Hilger [17] is a fairly new idea that combines the continuous and discrete analysis into one theory. Nowadays, the study of dynamic systems on time scales have got a great deal of global

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consideration and various specialists have discovered the applications of time scales in engineering, physics, economics, population dynamics, control system, and as well as in heat transfer system. For more study about time scales, we refer to the books by Bohner and Peterson [18, 19] and the papers [20–25]. In the last few years, few authors have been worked on the existence and Hyer-Ulam (HU)–type stability for the ordinary, as well as fractional order dynamic system on time scales [26–34]. In particular, Ahmadkhanlu and Jahanshahi [29], gave the necessary and sufficient conditions for the existence of unique solution to the following fractional order dynamic system on time scales

$$\begin{aligned} {}^{c,\mathbb{T}}D^\kappa p(t) &= \Phi(t, p(t)), \quad \forall t \in J = [t_0, t_0 + a] \subseteq \mathbb{T}, \quad \kappa \in (0, 1], \\ p(t_0) &= p_0, \end{aligned}$$

where $\Phi: J \times \mathbb{R} \rightarrow \mathbb{R}$ is a right-dense continuous bounded function and ${}^{c,\mathbb{T}}D^\kappa$ denotes the Caputo fractional derivative on time scales.

Eventhough controllability is the principal idea in the mathematical control theory which was introduced by Kalman in 1960 and soon became an active area of research by many mathematicians. Generally speaking, controllability implies the steering the state of a dynamical control system from an arbitrary initial state to the desired final state by using an appropriate control function. Controllability results of linear and nonlinear dynamical systems have been examined by numerous mathematicians, please see [35–39]. In addition, controllability results of the dynamic system on time scales is a relatively newer field, and only few results have been reported [40–44]. Particularly, the controllability result in the finite dimensional space of a dynamic system on time scales is reported in a study by Davis et al. [41]. In [44], the authors considered the dynamic system with impulsive conditions on time scales and established the necessary and sufficient conditions for the state observability and state controllability. There is not a single manuscript which discussed the existence, HU stability and controllability results for a fractional dynamic system on time scales. Inspired by the above works, for the existence and stability results, we consider the fractional dynamic system on time scales of the form

$$\begin{aligned} {}^{c,\mathbb{T}}D^\kappa p(t) &= \Phi(t, p(t), q(t), r(t)), \quad t \in I = [0, T]_{\mathbb{T}}, \kappa \in (0, 1), \\ {}^{c,\mathbb{T}}D^\nu q(t) &= \Psi(t, p(t), q(t), r(t)), \quad t \in I, \nu \in (0, 1), \\ {}^{c,\mathbb{T}}D^\omega r(t) &= \Theta(t, p(t), q(t), r(t)), \quad t \in I, \omega \in (0, 1), \\ p(0) &= p_0, \quad q(0) = q_0, \quad r(0) = r_0 \end{aligned} \quad (1.1)$$

and for the controllability analysis, we consider the fractional dynamic system on time scales of the following form

$$\begin{aligned} {}^{c,\mathbb{T}}D^\kappa p(t) &= \Phi(t, p(t), q(t), r(t)) + \mathcal{B}u(t), \quad t \in I = [0, T]_{\mathbb{T}}, \kappa \in (0, 1), \\ {}^{c,\mathbb{T}}D^\nu q(t) &= \Psi(t, p(t), q(t), r(t)) + \mathcal{C}v(t), \quad t \in I, \nu \in (0, 1), \\ {}^{c,\mathbb{T}}D^\omega r(t) &= \Theta(t, p(t), q(t), r(t)) + \mathcal{D}w(t), \quad t \in I, \omega \in (0, 1), \\ p(0) &= p_0, \quad q(0) = q_0, \quad r(0) = r_0, \end{aligned} \quad (1.2)$$

with $0, T \in \mathbb{T}$. ${}^{c,\mathbb{T}}D^\kappa, {}^{c,\mathbb{T}}D^\nu, {}^{c,\mathbb{T}}D^\omega$ are the Caputo fractional derivatives on time scales. $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are the linear bounded operators from $\mathbb{R}$ to $\mathbb{R}$. $u, v, w \in L^2(I, \mathbb{R})$ are the control inputs. $\Phi, \Psi, \Theta: I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions. The results obtained in this manuscript are completely new even in the case of differential equation (when $\mathbb{T} = \mathbb{R}$) and in the case of difference equation (when $\mathbb{T} = \mathbb{Z}$).

Obviously, we can see that when $\mathbb{T} = \mathbb{R}$, then the system (1.1) includes the well known fractional order three species prey-predator model [45], chaotic fractional order GLV model [46], toy model for three species [47], fractional-order HIV infection of CD4+ T-cells models [48, 49], etc. In the existing literature, there is no manuscript that analysed these mathematical models on time scales. We considered a more general form of the fractional dynamic system on time scales which includes these mathematical models, as well as many more models for the continuous, discrete and combination of these two. In this setting, our results are new and contribute significantly to the existing literature on the topic. The manuscript is divided into 7 sections.

In Section 2, we give some fundamental definitions, useful lemmas and preliminaries which will be used in subsequent sections. Section 3, the existence of solutions for the fractional system (1.1) is obtained. Section 4 is devoted to the investigation of the HU stability for the proposed fractional system. In Section 5, we give the controllability results for the fractional system (1.2). In Section 6, we give an illustrative example for different time scales. In the last Section 7, two applications are given to outline the effectiveness of these obtained results.

2 Preliminaries

In this segment, we define some basic notations, fundamental definitions and introduce useful lemmas. $C(I, \mathbb{R})$ forms a Banach space of all continuous functions endorsed with the norm $\|p\|_C = \sup_{t \in I} |p(t)|$. Therefore, the product space $PS = C(I, \mathbb{R}) \times C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is also Banach space with the norm $\|(p, q, r)\|_C = \|p\|_C + \|q\|_C + \|r\|_C$. Further, $C^1(I, \mathbb{R}) = \{p \in C(I, \mathbb{R}) : p^\Delta \in C(I, \mathbb{R})\}$ is a Banach space with the norm $\|p\|_{C^1} = \max\{\|p\|_C, \|p^\Delta\|_C\}$. Therefore, the product space $PS^1 = C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R})$ is also Banach space with norm $\|(p, q, r)\|_{C^1} = \|p\|_{C^1} + \|q\|_{C^1} + \|r\|_{C^1}$.

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers. A time scale interval is define as $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. In the similar way, we can define $(a, b)_{\mathbb{T}}$, $[a, b)_{\mathbb{T}}$, etc. We define $\mathbb{T}^k = \mathbb{T} \setminus \{\max \mathbb{T}\}$ if $\max \mathbb{T}$ exists, otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 2.1. [18] We define $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ as a forward jump operator by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ along with $\inf \emptyset = \sup \mathbb{T}$. Moreover, a positive function $\mu(t) = \sigma(t) - t, \forall t \in \mathbb{T}$, is known as graininess function.

A function $\phi : \mathbb{T} \rightarrow \mathbb{R}$ is called positive regressive (regressive) if $1 + \mu(t)\phi(t) > 0$ ($\neq 0$), $\forall t \in \mathbb{T}$ and the collection of all positive regressive (regressive) functions are denoted by $\mathcal{R}^+$ ($\mathcal{R}$). A function $\phi : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous on $\mathbb{T}$, if ϕ is continuous at all right-dense points of $\mathbb{T}$ and limit exist (finite) at all left dense point of $\mathbb{T}$. The set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.2. [18] The delta derivative of a function $f : \mathbb{T} \rightarrow \mathbb{R}$ at a point $t \in \mathbb{T}^k$ is a number $f^\Delta(t)$ (provided it exists), if there exists a neighbourhood U of t and $\epsilon > 0$ such that

$$|[f(\sigma(t)) - f(\tau)] - f^\Delta(t)[\sigma(t) - \tau]| \leq \epsilon |\sigma(t) - \tau|, \quad \forall \tau \in U.$$

Definition 2.3. [18] A function $F : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is called antiderivative of a function $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ if $F^\Delta(t) = f(t)$ for each $t \in [a, b]_{\mathbb{T}}$ is holds. Moreover, the Δ -integral is defined as

$$\int_a^b f(\zeta) \Delta \zeta = F(b) - F(a).$$

Theorem 2.4. [18] Let $a, b \in \mathbb{T}$ and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$, then

1. If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t) \Delta t = \int_a^b f(t) dt.$$

2. If $[a, b)$ consists of only isolated points, then

$$\int_a^b f(t) \Delta t = \begin{cases} \sum_{t \in [a, b)} \mu(t) f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{t \in [a, b)} \mu(t) f(t) & \text{if } a > b. \end{cases}$$

3. If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then

$$\int_a^b f(t) \Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} hf(kh) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} hf(kh) & \text{if } a > b. \end{cases}$$

Theorem 2.5. [29] Let $\mathbb{T}$ be an general time scale with $t_1, t_2 \in \mathbb{T}$ and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing, continuous function. Then,

$$\int_{t_1}^{t_2} f(\zeta) \Delta \zeta \leq \int_{t_1}^{t_2} f(\zeta) d\zeta. \quad (2.3)$$

Definition 2.6. [29] The delta fractional integral of an integrable function $\phi: [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ by

$${}^{\Delta}I_{a^+}^{\kappa} \phi(t) = \int_a^t \frac{(t-\zeta)^{\kappa-1}}{\Gamma(\kappa)} \phi(\zeta) \Delta \zeta, \quad (2.4)$$

Definition 2.7. [29] Let $f: \mathbb{T} \rightarrow \mathbb{R}$ be a function. Then, the Caputo fractional derivative on time scales of f is defined as

$${}^{c,\mathbb{T}}D_{a^+}^{\kappa} f(t) = \int_a^t \frac{(t-\zeta)^{n-\kappa-1}}{\Gamma(n-\kappa)} f^{\Delta^n}(\zeta) \Delta \zeta, \quad (2.5)$$

where $n = [\kappa] + 1$ and f^{Δ^n} denotes the n th delta derivative of f .

– When $\mathbb{T} = \cup_{k=0}^{\infty} [2k, 2k+1]$. Then, we have

$$\begin{aligned} {}^{c,\mathbb{T}}D_{a^+}^{\kappa} f(t) &= \int_a^t \frac{(t-\zeta)^{n-\kappa-1}}{\Gamma(n-\kappa)} f^{\Delta^n}(\zeta) \Delta \zeta \\ &= \frac{1}{\Gamma(n-\kappa)} \left[\sum_{j=0}^{k-1} \int_{2j}^{2j+1} (t-\zeta)^{n-\kappa-1} f^{\Delta^n}(\zeta) \Delta \zeta + \int_{2k}^t (t-\zeta)^{n-\kappa-1} f^{\Delta^n}(\zeta) \Delta \zeta \right], \\ &\quad t \in [2k, 2k+1], k = 0, 1, \dots \end{aligned}$$

– When $\mathbb{T} = h\mathbb{Z}, h > 0$, we have

$${}^{c,\mathbb{T}}D_{a^+}^{\kappa} f(t) = \int_a^t \frac{(t-\zeta)^{n-\kappa-1}}{\Gamma(n-\kappa)} f^{\Delta^n}(\zeta) \Delta \zeta = \frac{1}{\Gamma(n-\kappa)} \sum_{k=0}^{\frac{t}{h}-1} h(t-kh)^{n-\kappa-1} f^{\Delta^n}(k), t \in \mathbb{T}.$$

– When $\mathbb{T} = \{q^n : q > 1, n \in \mathbb{Z}\} \cup \mathbb{Z}$, then

$${}^{c,\mathbb{T}}D_{a^+}^{\kappa} f(t) = \int_a^t \frac{(t-\zeta)^{n-\kappa-1}}{\Gamma(n-\kappa)} f^{\Delta^n}(\zeta) \Delta \zeta = \frac{1}{\Gamma(n-\kappa)} \sum_{s \in \mathbb{T}} \mu(s) (t-s)^{n-\kappa-1} f^{\Delta^n}(s).$$

Theorem 2.8. [29] Let $\Phi: I \times \mathbb{R} \rightarrow \mathbb{R}$ be a function. Then, $p(t)$ is a solution of

$$\begin{aligned} {}^{c,\mathbb{T}}D^{\kappa} p(t) &= \Phi(t, p(t)), \quad t \in I, \kappa \in (0, 1), \\ p(0) &= p_0, \end{aligned}$$

iff $p(t)$ is a solution of the following equation

$$p(t) = p_0 + \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta)) \Delta \zeta.$$

Under the following assumptions, we prove the main results of the paper:

(A): $\Phi, \Psi, \Theta : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Also, there exist positive constants $L_{\Phi}, L_{\Psi}, L_{\Theta}, i = 1, 2, 3$, such that

$$\begin{aligned} |\Phi(t, p_1, p_2, p_3) - \Phi(t, q_1, q_2, q_3)| &\leq \sum_{i=1}^3 L_{\Phi_i} |p_i - q_i|, \quad \forall t \in I, p_i, q_i \in \mathbb{R}, i = 1, 2, 3, \\ |\Psi(t, p_1, p_2, p_3) - \Psi(t, q_1, q_2, q_3)| &\leq \sum_{i=1}^3 L_{\Psi_i} |p_i - q_i|, \quad \forall t \in I, p_i, q_i \in \mathbb{R}, i = 1, 2, 3, \\ |\Theta(t, p_1, p_2, p_3) - \Theta(t, q_1, q_2, q_3)| &\leq \sum_{i=1}^3 L_{\Theta_i} |p_i - q_i|, \quad \forall t \in I, p_i, q_i \in \mathbb{R}, i = 1, 2, 3. \end{aligned}$$

(B): $\Phi, \Psi, \Theta : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Also, there exist positive constants $l_i, m_i, n_i, i = 0, 1, 2, 3$ such that

$$\begin{aligned} |\Phi(t, p, q, r)| &\leq l_0 + l_1 |p| + l_2 |q| + l_3 |r|, \quad \forall t \in I, p, q, r \in \mathbb{R}, \\ |\Psi(t, p, q, r)| &\leq m_0 + m_1 |p| + m_2 |q| + m_3 |r|, \quad \forall t \in I, p, q, r \in \mathbb{R}, \\ |\Theta(t, p, q, r)| &\leq n_0 + n_1 |p| + n_2 |q| + n_3 |r|, \quad \forall t \in I, p, q, r \in \mathbb{R}. \end{aligned}$$

(W): The linear operators ${}^{\kappa}\mathcal{W}_0^T, {}^{\nu}\mathcal{W}_0^T, {}^{\omega}\mathcal{W}_0^T : L^2(I, \mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} {}^{\kappa}\mathcal{W}_0^T u &= \frac{1}{\Gamma(\kappa)} \int_0^T (T - \zeta)^{\kappa-1} B u(\zeta) \Delta \zeta, \quad {}^{\nu}\mathcal{W}_0^T v = \frac{1}{\Gamma(\nu)} \int_0^T (T - \zeta)^{\nu-1} C v(\zeta) \Delta \zeta, \\ {}^{\omega}\mathcal{W}_0^T w &= \frac{1}{\Gamma(\omega)} \int_0^T (T - \zeta)^{\omega-1} \mathcal{D} w(\zeta) \Delta \zeta, \end{aligned} \quad (2.6)$$

have the bounded invertible operators $({}^{\kappa}\mathcal{W}_0^T)^{-1}, ({}^{\nu}\mathcal{W}_0^T)^{-1}$ and $({}^{\omega}\mathcal{W}_0^T)^{-1}$ which take values in $L^2(I, \mathbb{R}) \setminus \ker({}^{\kappa}\mathcal{W}_0^T), L^2(I, \mathbb{R}) \setminus \ker({}^{\nu}\mathcal{W}_0^T)$ and $L^2(I, \mathbb{R}) \setminus \ker({}^{\omega}\mathcal{W}_0^T)$, respectively. Also, there exist positive constants $M_{\mathcal{W}}^{\kappa}, M_{\mathcal{W}}^{\nu}$ and $M_{\mathcal{W}}^{\omega}$ such that $\|({}^{\kappa}\mathcal{W}_0^T)^{-1}\| \leq M_{\mathcal{W}}^{\kappa}, \|({}^{\nu}\mathcal{W}_0^T)^{-1}\| \leq M_{\mathcal{W}}^{\nu}$ and $\|({}^{\omega}\mathcal{W}_0^T)^{-1}\| \leq M_{\mathcal{W}}^{\omega}$.

Also, $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are continuous operator from $\mathbb{R}$ to $\mathbb{R}$ and there exist positive constants $M_{\mathcal{B}}, M_{\mathcal{C}}, M_{\mathcal{D}}$ such that $\|\mathcal{B}\| \leq M_{\mathcal{B}}, \|\mathcal{C}\| \leq M_{\mathcal{C}}, \|\mathcal{D}\| \leq M_{\mathcal{D}}$.

The equation (2.6) can be calculate for different time scales by using Theorem 2.4. For example

– When $\mathbb{T} = h\mathbb{Z}, h > 0$, we have:

$${}^{\kappa}\mathcal{W}_0^T u = \frac{1}{\Gamma(\kappa)} \int_0^T (T - \zeta)^{\kappa-1} B u(\zeta) \Delta \zeta = \frac{1}{\Gamma(\kappa)} \sum_{k=0}^{\frac{T}{h}-1} h (T - \zeta h)^{\kappa-1} B u(\zeta h).$$

– When $\mathbb{T} = \cup_{k=0}^{\infty} [2k, 2k+1]$. Let $T \in [4, 5]$. Then, we have

$$\begin{aligned} {}^{\kappa}\mathcal{W}_0^T u &= \frac{1}{\Gamma(\kappa)} \int_0^T (T - \zeta)^{\kappa-1} B u(\zeta) \Delta \zeta \\ &= \frac{1}{\Gamma(\kappa)} \left[\int_0^1 (T - \zeta)^{\kappa-1} B u(\zeta) d\zeta + \int_2^3 (T - \zeta)^{\kappa-1} B u(\zeta) d\zeta + \int_4^T (T - \zeta)^{\kappa-1} B u(\zeta) d\zeta \right]. \end{aligned}$$

– When $\mathbb{T} = \{q^n : q > 1, n \in \mathbb{Z}\} \cup \mathbb{Z}$, then

$${}^{\kappa}\mathcal{W}_0^T u = \frac{1}{\Gamma(\kappa)} \int_0^T (T - \zeta)^{\kappa-1} B u(\zeta) \Delta \zeta = \frac{1}{\Gamma(\kappa)} \sum_{t \in [0, T)} \mu(t) (T - t)^{\kappa-1} B u(t).$$

Similarly, we can find ${}^{\nu}\mathcal{W}_0^T v$ and ${}^{\omega}\mathcal{W}_0^T w$.

Throughout the manuscript, we set

$$\mathcal{Q}_1 = \frac{T^{\kappa} (L_{\Phi_1} + L_{\Phi_2} + L_{\Phi_3})}{\Gamma(\kappa + 1)}; \quad \mathcal{Q}_2 = \frac{T^{\nu} (L_{\Psi_1} + L_{\Psi_2} + L_{\Psi_3})}{\Gamma(\nu + 1)}; \quad \mathcal{Q}_3 = \frac{T^{\omega} (L_{\Theta_1} + L_{\Theta_2} + L_{\Theta_3})}{\Gamma(\omega + 1)};$$

$$\begin{aligned}\mathcal{N}_1 &= |p_0| + \frac{T^\kappa M_\Phi}{\Gamma(\kappa+1)}; \mathcal{N}_2 = |q_0| + \frac{T^\nu M_\Psi}{\Gamma(\nu+1)}; \mathcal{N}_3 = |r_0| + \frac{T^\omega M_\Theta}{\Gamma(\omega+1)}; \\ \mathcal{N}_4 &= |p_0| + |q_0| + |r_0| + \frac{T^\kappa l_0}{\Gamma(\kappa+1)} + \frac{T^\nu m_0}{\Gamma(\nu+1)} + \frac{T^\omega n_0}{\Gamma(\omega+1)}; \\ \mathcal{N}_5 &= \frac{T^\kappa (l_1 + l_2 + l_3)}{\Gamma(\kappa+1)} + \frac{T^\nu (m_1 + m_2 + m_3)}{\Gamma(\nu+1)} + \frac{T^\omega (n_1 + n_2 + n_3)}{\Gamma(\omega+1)}.\end{aligned}$$

3 Existence of solution

Theorem 3.1. *If the assumption (A) holds, then the system (1.1) has a unique solution provided*

$$\max_{1 \leq i \leq 3} \mathcal{Q}_i < \frac{1}{3}. \quad (3.7)$$

Proof. Consider a subset $\Omega \subseteq PS$ such that

$$\Omega = \{(p, q, r) \in PS : \|(p, q, r)\|_C \leq \delta\},$$

where

$$\delta > \max \left\{ \frac{\mathcal{N}_1}{1-3\mathcal{Q}_1}, \frac{\mathcal{N}_2}{1-3\mathcal{Q}_2}, \frac{\mathcal{N}_3}{1-3\mathcal{Q}_3} \right\}.$$

Now, we define an operator $\Xi : \Omega \rightarrow \Omega$ as

$$\Xi(p, q, r)(t) = \begin{pmatrix} \Xi_\kappa(p, q, r)(t) \\ \Xi_\nu(p, q, r)(t) \\ \Xi_\omega(p, q, r)(t) \end{pmatrix}, t \in I, \quad (3.8)$$

where

$$\begin{aligned}\Xi_\kappa(p, q, r)(t) &= p_0 + \frac{1}{\Gamma(\kappa)} \int_0^T (t-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta, \\ \Xi_\nu(p, q, r)(t) &= q_0 + \frac{1}{\Gamma(\nu)} \int_0^T (t-\zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta, \\ \Xi_\omega(p, q, r)(t) &= r_0 + \frac{1}{\Gamma(\omega)} \int_0^T (t-\zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta.\end{aligned}$$

Also, note that

$$\begin{aligned}|\Phi(t, p, q, r)| &\leq |\Phi(t, p, q, r) - \Phi(t, 0, 0, 0)| + |\Phi(t, 0, 0, 0)| \leq L_{\Phi_1}|p| + L_{\Phi_2}|q| + L_{\Phi_3}|r| + M_\Phi, \\ |\Psi(t, p, q, r)| &\leq L_{\Psi_1}|p| + L_{\Psi_2}|q| + L_{\Psi_3}|r| + M_\Psi, \\ |\Theta(t, p, q, r)| &\leq L_{\Theta_1}|p| + L_{\Theta_2}|q| + L_{\Theta_3}|r| + M_\Theta,\end{aligned}$$

where $M_\Phi = \sup_{t \in I} |\Phi(t, 0, 0, 0)|$, $M_\Psi = \sup_{t \in I} |\Psi(t, 0, 0, 0)|$ and $M_\Theta = \sup_{t \in I} |\Theta(t, 0, 0, 0)|$.

Now, we need to show that Ξ maps Ω into itself, we have

$$\begin{aligned}
|\Xi_{\kappa}(p, q, r)(t)| &\leq |p_0| + \left| \frac{1}{\Gamma(\kappa)} \int_0^T (t-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right| \\
&\leq |p_0| + \frac{1}{\Gamma(\kappa)} \int_0^T (t-\zeta)^{\kappa-1} (L_{\Phi_1}|p(\zeta)| + L_{\Phi_2}|q(\zeta)| + L_{\Phi_3}|r(\zeta)| + M_{\Phi}) \Delta\zeta \\
&\leq |p_0| + \frac{(L_{\Phi_1}\delta + L_{\Phi_2}\delta + L_{\Phi_3}\delta + M_{\Phi})}{\Gamma(\kappa)} \int_0^T (t-\zeta)^{\kappa-1} \Delta\zeta.
\end{aligned}$$

Since $(t-\zeta)^{\kappa-1}$ is an increasing function, by using Theorem 2.5 in the above equation, we get

$$\begin{aligned}
|\Xi_{\kappa}(p, q, r)(t)| &\leq |p_0| + \frac{(L_{\Phi_1}\delta + L_{\Phi_2}\delta + L_{\Phi_3}\delta + M_{\Phi})}{\Gamma(\kappa)} \int_0^T (t-\zeta)^{\kappa-1} d\zeta \\
&\leq |p_0| + \frac{T^{\kappa} (L_{\Phi_1}\delta + L_{\Phi_2}\delta + L_{\Phi_3}\delta + M_{\Phi})}{\Gamma(\kappa+1)} \\
&\leq \mathcal{N}_1 + \delta \mathcal{Q}_1.
\end{aligned}$$

Hence,

$$\|\Xi_{\kappa}(p, q, r)\|_C \leq \mathcal{N}_1 + \delta \mathcal{Q}_1 \leq \frac{\delta}{3}. \quad (3.9)$$

Also,

$$\begin{aligned}
|\Xi_{\nu}(p, q, r)(t)| &\leq |q_0| + \left| \frac{1}{\Gamma(\nu)} \int_0^T (t-\zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right| \\
&\leq |q_0| + \frac{T^{\nu} (L_{\Psi_1}\delta + L_{\Psi_2}\delta + L_{\Psi_3}\delta + M_{\Psi})}{\Gamma(\nu+1)} \\
&\leq \mathcal{N}_2 + \delta \mathcal{Q}_2.
\end{aligned}$$

Hence,

$$\|\Xi_{\nu}(p, q, r)\|_C \leq \mathcal{N}_2 + \delta \mathcal{Q}_2 \leq \frac{\delta}{3}. \quad (3.10)$$

Similarly, one can find

$$\|\Xi_{\omega}(p, q, r)\|_C \leq \mathcal{N}_3 + \delta \mathcal{Q}_3 \leq \frac{\delta}{3}. \quad (3.11)$$

Summarize the above inequalities (3.9), (3.10) and (3.11), we get:

$$\|\Xi(p, q, r)\|_C \leq \delta. \quad (3.12)$$

Therefore, $\Xi : \Omega \rightarrow \Omega$. Also, when $t \in I$, we have

$$\begin{aligned}
|\Xi_{\kappa}(p, q, r)(t) - \Xi_{\kappa}(\bar{p}, \bar{q}, \bar{r})(t)| &\leq \frac{1}{\Gamma(\kappa)} \int_0^T (t-\zeta)^{\kappa-1} |\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) - \Phi(\zeta, \bar{p}(\zeta), \bar{q}(\zeta), \bar{r}(\zeta))| \Delta\zeta \\
&\leq \frac{1}{\Gamma(\kappa)} \int_0^T (t-\zeta)^{\kappa-1} (L_{\Phi_1}|p(\zeta) - \bar{p}(\zeta)| + L_{\Phi_2}|q(\zeta) - \bar{q}(\zeta)| + L_{\Phi_3}|r(\zeta) - \bar{r}(\zeta)|) \Delta\zeta \\
&\leq \frac{T^{\kappa} (L_{\Phi_1}\|p - \bar{p}\|_C + L_{\Phi_2}\|q - \bar{q}\|_C + L_{\Phi_3}\|r - \bar{r}\|_C)}{\Gamma(\kappa+1)}.
\end{aligned}$$

Thus,

$$\|\Xi_{\kappa}(p, q, r) - \Xi_{\kappa}(\bar{p}, \bar{q}, \bar{r})\|_C \leq \mathcal{Q}_1(\|p - \bar{p}\|_C + \|q - \bar{q}\|_C + \|r - \bar{r}\|_C). \quad (3.13)$$

Also,

$$\begin{aligned} |\Xi_{\nu}(p, q, r)(t) - \Xi_{\nu}(\bar{p}, \bar{q}, \bar{r})(t)| &\leq \frac{1}{\Gamma(\nu)} \int_0^T (t - \zeta)^{\nu-1} |\Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) - \Psi(\zeta, \bar{p}(\zeta), \bar{q}(\zeta), \bar{r}(\zeta))| \Delta\zeta \\ &\leq \frac{T^{\nu} (L_{\Psi_1} \|p - \bar{p}\|_C + L_{\Psi_2} \|q - \bar{q}\|_C + L_{\Psi_3} \|r - \bar{r}\|_C)}{\Gamma(\nu + 1)}. \end{aligned}$$

Thus,

$$\|\Xi_{\nu}(p, q, r) - \Xi_{\nu}(\bar{p}, \bar{q}, \bar{r})\|_C \leq \mathcal{Q}_2(\|p - \bar{p}\|_C + \|q - \bar{q}\|_C + \|r - \bar{r}\|_C). \quad (3.14)$$

Similarly, we can easily obtain

$$\|\Xi_{\omega}(p, q, r) - \Xi_{\omega}(\bar{p}, \bar{q}, \bar{r})\|_C \leq \mathcal{Q}_3(\|p - \bar{p}\|_C + \|q - \bar{q}\|_C + \|r - \bar{r}\|_C). \quad (3.15)$$

After summarize the inequalities (3.13), (3.14) and (3.15), we get

$$\|\Xi(p, q, r) - \Xi(\bar{p}, \bar{q}, \bar{r})\|_C \leq (\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3) \|(p, q, r), (\bar{p}, \bar{q}, \bar{r})\|_C.$$

Therefore, from the equation (3.7), we can conclude that Ξ is a strict contraction operator. Hence, by means of Banach fixed point technique, we have a unique fixed point of Ξ which is the solution of the system (1.1). $\square$

Next, by applying the Leray-Schauder alternative-type fixed point technique, existence of at least one solution for the system (1.1) under the weaker condition **(B)** is given.

Theorem 3.2. *If the assumption **(B)** holds, then the system (1.1) has at least one solution on I , provided there exists a positive constant $\mathcal{K}$ such that*

$$\mathcal{N}_4 + \mathcal{N}_5 \mathcal{K} < \mathcal{K}. \quad (3.16)$$

Proof. Firstly, we show that the operator Ξ defined by (3.8) is completely continuous. We note that continuity of functions Φ , Ψ and Θ implies that the operator Ξ is continuous. Also, let $\Omega_1 \subseteq PS$ is bounded. Then, there exist $L_1, L_2, L_3 > 0$, such that $|\Phi(t, p, q, r)| \leq L_1, |\Psi(t, p, q, r)| \leq L_2, |\Theta(t, p, q, r)| \leq L_3$. Then, for any $(p, q, r) \in \Omega_1$, we have

$$|\Xi_{\kappa}(p, q, r)(t)| \leq |p_0| + \left| \frac{1}{\Gamma(\kappa)} \int_0^T (t - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right| \leq |p_0| + \frac{L_1 T^{\kappa}}{\Gamma(\kappa + 1)}.$$

Thus,

$$\|\Xi_{\kappa}(p, q, r)\|_C \leq |p_0| + \frac{L_1 T^{\kappa}}{\Gamma(\kappa + 1)} = C_{\kappa}. \quad (3.17)$$

Also,

$$|\Xi_{\nu}(p, q, r)(t)| \leq |q_0| + \left| \frac{1}{\Gamma(\nu)} \int_0^T (t - \zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right| \leq |q_0| + \frac{L_2 T^{\nu}}{\Gamma(\nu + 1)}.$$

Thus,

$$\|\Xi_{\nu}(p, q, r)\|_C \leq |q_0| + \frac{L_2 T^{\nu}}{\Gamma(\nu + 1)} = C_{\nu}. \quad (3.18)$$

Similarly, we have

$$\|\Xi_{\omega}(p, q, r)\|_C \leq |r_0| + \frac{L_3 T^{\omega}}{\Gamma(\omega + 1)} = C_{\omega}. \quad (3.19)$$

Thus, from the above inequalities (3.17), (3.18) and (3.19), it follows that the operator Ξ is uniformly bounded.

Next, we need to show that Ξ is completely continuous. We know that any time scales can be categories into three cases namely discrete, continuous and the combination of discrete and continuous. Hence, we are considering the following three cases:

Case 1: When the time scales are discrete. i.e., all the points of the time scales are isolated. In this case, using Theorem 2.4, Ξ_{κ} , Ξ_{ν} and Ξ_{ω} become

$$\begin{aligned} \Xi_{\kappa}(p, q, r)(t) &= p_0 + \frac{1}{\Gamma(\kappa)} \sum_{t \in [0, T)} \mu(t) \Phi(t, p(t), q(t), r(t)), \\ \Xi_{\nu}(p, q, r)(t) &= q_0 + \frac{1}{\Gamma(\nu)} \sum_{t \in [0, T)} \mu(t) \Psi(t, p(t), q(t), r(t)), \\ \Xi_{\omega}(p, q, r)(t) &= r_0 + \frac{1}{\Gamma(\omega)} \sum_{t \in [0, T)} \mu(t) \Theta(t, p(t), q(t), r(t)), \end{aligned}$$

which are the summation operators on a discrete finite set. Since, Φ , Ψ and Θ are continuous. Hence, Ξ_{κ} , Ξ_{ν} and Ξ_{ω} are completely continuous and hence Ξ is also completely continuous.

Case 2: When the time scales are continuous. i.e., all the points of time scales are dense. For this, let $\tau_1, \tau_2 \in I$ such that $\tau_1 < \tau_2$, then

$$\begin{aligned} |\Xi_{\kappa}(p, q, r)(\tau_2) - \Xi_{\kappa}(p, q, r)(\tau_1)| &\leq \left| \frac{1}{\Gamma(\kappa)} \int_0^{\tau_2} (\tau_2 - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right. \\ &\quad \left. - \frac{1}{\Gamma(\kappa)} \int_0^{\tau_1} (\tau_1 - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right| \\ &\leq \left| \frac{1}{\Gamma(\kappa)} \int_0^{\tau_1} ((\tau_2 - \zeta)^{\kappa-1} - (\tau_1 - \zeta)^{\kappa-1}) \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right| \\ &\quad + \left| \frac{1}{\Gamma(\kappa)} \int_{\tau_1}^{\tau_2} (\tau_2 - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right|. \end{aligned} \quad (3.20)$$

Also,

$$\begin{aligned} |\Xi_{\nu}(p, q, r)(\tau_2) - \Xi_{\nu}(p, q, r)(\tau_1)| &\leq \left| \frac{1}{\Gamma(\nu)} \int_0^{\tau_1} ((\tau_2 - \zeta)^{\nu-1} - (\tau_1 - \zeta)^{\nu-1}) \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right| \\ &\quad + \left| \frac{1}{\Gamma(\nu)} \int_{\tau_1}^{\tau_2} (\tau_2 - \zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right|. \end{aligned} \quad (3.21)$$

Similarly, we can easily find

$$\begin{aligned} |\Xi_{\omega}(p, q, r)(\tau_2) - \Xi_{\omega}(p, q, r)(\tau_1)| &\leq \left| \frac{1}{\Gamma(\omega)} \int_0^{\tau_1} ((\tau_2 - \zeta)^{\omega-1} - (\tau_1 - \zeta)^{\omega-1}) \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right| \\ &\quad + \left| \frac{1}{\Gamma(\omega)} \int_{\tau_1}^{\tau_2} (\tau_2 - \zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right|. \end{aligned} \quad (3.22)$$

Clearly, one can see that the right-hand side of the above inequalities (3.20), (3.21) and (3.22) tend to zero as $\tau_2 \rightarrow \tau_1$. Hence, the operators Ξ_{κ} , Ξ_{ν} and Ξ_{ω} are equicontinuous. Therefore, using Arzela-Ascoli theorem, one can find that the operator Ξ is completely continuous.

Case 3: When the time scales are the combination of discrete and continuous points. Then, for the discrete points, using Theorem 2.4, Ξ can be written as the summation which is completely continuous (same as case 1). Also, for the continuous points, we can show that Ξ is completely continuous (same as case 2). Thus, Ξ can be written as a sum of two operators, one for the discrete points and one for the continuous points. Since, the sum of two completely continuous operators is also completely continuous. Hence, we can conclude that Ξ is completely continuous.

Thus, from the above three cases, we can say that Ξ is completely continuous.

Finally, let, for $\beta \in [0,1]$ there exists a $(p(t), q(t), r(t))$ such that $(p(t), q(t), r(t)) = \beta(\Xi(p, q, r))t$. Then for $t \in I$, we have:

$$\begin{aligned} |p(t)| &= |\beta \Xi_{\kappa}(p, q, r)t| = \left| \beta \left(p_0 + \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta \right) \right| \\ &\leq |p_0| + \frac{T^{\kappa}(l_0 + l_1 \|p\|_C + l_2 \|q\|_C + l_3 \|r\|_C)}{\Gamma(\kappa + 1)}. \end{aligned}$$

Thus,

$$\|p\|_C \leq |p_0| + \frac{T^{\kappa}(l_0 + l_1 \|p\|_C + l_2 \|q\|_C + l_3 \|r\|_C)}{\Gamma(\kappa + 1)}. \quad (3.23)$$

Similarly, we get

$$\|q\|_C \leq |q_0| + \frac{T^{\nu}(m_0 + m_1 \|p\|_C + m_2 \|q\|_C + m_3 \|r\|_C)}{\Gamma(\nu + 1)} \quad (3.24)$$

$$\|r\|_C \leq |r_0| + \frac{T^{\omega}(n_0 + n_1 \|p\|_C + n_2 \|q\|_C + n_3 \|r\|_C)}{\Gamma(\omega + 1)}. \quad (3.25)$$

From the above inequalities (3.23), (3.24) and (3.25) we get

$$\begin{aligned} \|p\|_C + \|q\|_C + \|r\|_C &\leq |p_0| + |q_0| + |r_0| + \frac{T^{\kappa}(l_0 + l_1 \|p\|_C + l_2 \|q\|_C + l_3 \|r\|_C)}{\Gamma(\kappa + 1)} \\ &\quad + \frac{T^{\nu}(m_0 + m_1 \|p\|_C + m_2 \|q\|_C + m_3 \|r\|_C)}{\Gamma(\nu + 1)} \\ &\quad + \frac{T^{\omega}(n_0 + n_1 \|p\|_C + n_2 \|q\|_C + n_3 \|r\|_C)}{\Gamma(\omega + 1)} \\ &\leq \mathcal{N}_4 + \mathcal{N}_5 (\|p\|_C + \|q\|_C + \|r\|_C). \end{aligned}$$

Thus,

$$\frac{\|(p, q, r)\|_C}{\mathcal{N}_4 + \mathcal{N}_5 \|(p, q, r)\|_C} \leq 1.$$

Now, the condition (3.16) gives a $\mathcal{K} > 0$ such that $\|(p, q, r)\|_C \neq \mathcal{K}$. Let us suppose that $\mathfrak{N} = \{(p, q, r) \in PS : \|(p, q, r)\|_C < \mathcal{K}\}$. The operator $\Xi : \overline{\mathfrak{N}} \rightarrow PS$ is continuous and completely continuous. Therefore, from the choice of $\mathfrak{N}$, there is no $(p, q, r) \in \partial(\mathfrak{N})$ such that $(p, q, r) = \beta \Xi(p, q, r)$, $\beta \in [0,1]$. Thus, by using the fixed point theorem due to nonlinear alternative of Leray Schauder, Ξ has a fixed point which is the solution of the system (1.1). $\square$

4 Stability analysis

For $\epsilon_{\kappa} > 0$, $\epsilon_{\nu} > 0$ and $\epsilon_{\omega} > 0$, consider the inequality

$$\begin{cases} |{}^{c,\mathbb{T}}D^\kappa \mathbf{p}(t) - \Phi(t, \mathbf{p}(t), \mathbf{q}(t), \mathbf{r}(t))| \leq \epsilon_\kappa, & t \in I, \\ |{}^{c,\mathbb{T}}D^\nu \mathbf{q}(t) - \Psi(t, \mathbf{p}(t), \mathbf{q}(t), \mathbf{r}(t))| \leq \epsilon_\nu, & t \in I, \\ |{}^{c,\mathbb{T}}D^\omega \mathbf{r}(t) - \Theta(t, \mathbf{p}(t), \mathbf{q}(t), \mathbf{r}(t))| \leq \epsilon_\omega, & t \in I. \end{cases} \quad (4.26)$$

Definition 4.1. [13] System (1.1) is called HU stable if there exists a positive number $\mathcal{H}_{(L_{\Phi_i}, L_{\Psi_i}, L_{\Theta_i}, \kappa, \nu, \omega)}$ such that for $\epsilon > 0$ and for each solution $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ of inequality (4.26), there exists a unique solution (p, q, r) of system (1.1), which satisfies the following inequality

$$|(p, q, r)(t) - (\mathbf{p}, \mathbf{q}, \mathbf{r})(t)| \leq \mathcal{H}_{(L_{\Phi_i}, L_{\Psi_i}, L_{\Theta_i}, \kappa, \nu, \omega)} \epsilon, \quad \forall t \in I.$$

Definition 4.2. [13] System (1.1) is said to be generalized HU stable if there exists $\overline{\mathcal{H}}_{(L_{\Phi_i}, L_{\Psi_i}, L_{\Theta_i}, \kappa, \nu, \omega)} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\mathcal{H}_{(L_{\Phi_i}, L_{\Psi_i}, L_{\Theta_i}, \kappa, \nu, \omega)}(0) = 0$ such that for each solution (p, q, r) of inequalities (4.26), there exists a solution $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ of system (1) satisfies the following inequality

$$|(p, q, r)(t) - (\mathbf{p}, \mathbf{q}, \mathbf{r})(t)| \leq \overline{\mathcal{H}}_{(L_{\Phi_i}, L_{\Psi_i}, L_{\Theta_i}, \kappa, \nu, \omega)}(\epsilon), \quad \forall t \in I.$$

Remark 4.3. Definition 4.1 $\Rightarrow$ Definition 4.2.

Remark 4.4. A function $(p, q, r) \in PS^1$ is a solution of inequality (4.26) iff there is $f, g, h \in PS$ such that

$$(a) |f(t)| \leq \epsilon_\kappa, |g(t)| \leq \epsilon_\nu, |h(t)| \leq \epsilon_\omega, \forall t \in I.$$

$$\begin{aligned} (b) \quad & D_{c,\mathbb{T}}^\kappa \mathbf{p}(t) = \Phi(t, \mathbf{p}(t), \mathbf{q}(t), \mathbf{r}(t)) + f(t), t \in I, \\ & D_{c,\mathbb{T}}^\nu \mathbf{q}(t) = \Psi(t, \mathbf{p}(t), \mathbf{q}(t), \mathbf{r}(t)) + g(t), t \in I, \\ & D_{c,\mathbb{T}}^\omega \mathbf{r}(t) = \Theta(t, \mathbf{p}(t), \mathbf{q}(t), \mathbf{r}(t)) + h(t), t \in I, \end{aligned}$$

Lemma 4.5. Let $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ be the solution of the inequality (4.26), then the following inequalities will be satisfied:

$$\left| \mathbf{p}(t) - p_0 - \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} \Phi(\zeta, \mathbf{p}(\zeta), \mathbf{q}(\zeta), \mathbf{r}(\zeta)) \Delta\zeta \right| \leq \frac{T^\kappa}{\Gamma(\kappa+1)} \epsilon_\kappa, \quad (4.27)$$

$$\left| \mathbf{q}(t) - q_0 - \frac{1}{\Gamma(\nu)} \int_0^t (t-\zeta)^{\nu-1} \Psi(\zeta, \mathbf{p}(\zeta), \mathbf{q}(\zeta), \mathbf{r}(\zeta)) \Delta\zeta \right| \leq \frac{T^\nu}{\Gamma(\nu+1)} \epsilon_\nu, \quad (4.28)$$

$$\left| \mathbf{r}(t) - r_0 - \frac{1}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} \Theta(\zeta, \mathbf{p}(\zeta), \mathbf{q}(\zeta), \mathbf{r}(\zeta)) \Delta\zeta \right| \leq \frac{T^\omega}{\Gamma(\omega+1)} \epsilon_\omega. \quad (4.29)$$

Proof. From the Remark 4.4, we have:

$$\begin{aligned} D_{c,\mathbb{T}}^\kappa \mathbf{p}(t) &= \Phi(t, \mathbf{p}(t), \mathbf{q}(t), \mathbf{r}(t)) + f(t), t \in I, \\ D_{c,\mathbb{T}}^\nu \mathbf{q}(t) &= \Psi(t, \mathbf{p}(t), \mathbf{q}(t), \mathbf{r}(t)) + g(t), t \in I, \\ D_{c,\mathbb{T}}^\omega \mathbf{r}(t) &= \Theta(t, \mathbf{p}(t), \mathbf{q}(t), \mathbf{r}(t)) + h(t), t \in I. \end{aligned}$$

From Theorem 3.1, the solution of the above coupled system is given by

$$\mathbf{p}(t) = p_0 + \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} (\Phi(\zeta, \mathbf{p}(\zeta), \mathbf{q}(\zeta), \mathbf{r}(\zeta)) + f(\zeta)) \Delta\zeta, \quad (4.30)$$

$$\mathbf{q}(t) = q_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-\zeta)^{\nu-1} (\Psi(\zeta, \mathbf{p}(\zeta), \mathbf{q}(\zeta), \mathbf{r}(\zeta)) + g(\zeta)) \Delta\zeta, \quad (4.31)$$

$$\mathbf{r}(t) = r_0 + \frac{1}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} (\Theta(\zeta, \mathbf{p}(\zeta), \mathbf{q}(\zeta), \mathbf{r}(\zeta)) + h(\zeta)) \Delta\zeta. \quad (4.32)$$

Therefore, one can easily prove the inequalities (4.27), (4.28) and (4.29).

Our next theorem gives sufficient condition under which the system (1.1) is HU stable. $\square$

Theorem 4.6. *If the assumption (A) and the inequality (3.7) are fulfilled, then the system (1.1) is HU stable.*

Proof. Suppose that (p, q, r) be a unique solution of the system (1.1) and $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ be the solution of inequality (4.26). Therefore, from Theorem 3.1, we have

$$\begin{aligned} p(t) &= p_0 + \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta, \\ q(t) &= q_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-\zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta, \\ r(t) &= r_0 + \frac{1}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta. \end{aligned}$$

For $t \in I$, we find

$$\begin{aligned} |\mathbf{p}(t) - p(t)| &\leq \left| \mathbf{p}(t) - p_0 - \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right| \\ &\leq \frac{T^\kappa}{\Gamma(\kappa+1)} \epsilon_\kappa + \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} |\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) - \Phi(\zeta, \mathbf{p}(\zeta), \mathbf{q}(\zeta), \mathbf{r}(\zeta))| \Delta\zeta. \end{aligned}$$

Subsequently, we get

$$\begin{aligned} \|\mathbf{p} - p\|_C &\leq \frac{T^\kappa}{\Gamma(\kappa+1)} \epsilon_\kappa + \frac{T^\kappa (L_{\Phi_1} \|\mathbf{p} - p\|_C + L_{\Phi_2} \|\mathbf{q} - q\|_C + L_{\Phi_3} \|\mathbf{r} - r\|_C)}{\Gamma(\kappa+1)} \\ &\leq \frac{T^\kappa}{\Gamma(\kappa+1)} \epsilon_\kappa + \mathcal{Q}_1 (\|\mathbf{p} - p\|_C + \|\mathbf{q} - q\|_C + \|\mathbf{r} - r\|_C). \end{aligned} \quad (4.33)$$

Similarly, we can get

$$\|\mathbf{q} - q\|_C \leq \frac{T^\nu}{\Gamma(\nu+1)} \epsilon_\nu + \mathcal{Q}_2 (\|\mathbf{p} - p\|_C + \|\mathbf{q} - q\|_C + \|\mathbf{r} - r\|_C). \quad (4.34)$$

$$\|\mathbf{r} - r\|_C \leq \frac{T^\omega}{\Gamma(\omega+1)} \epsilon_\omega + \mathcal{Q}_3 (\|\mathbf{p} - p\|_C + \|\mathbf{q} - q\|_C + \|\mathbf{r} - r\|_C). \quad (4.35)$$

From the inequalities (4.33), (4.34) and (4.35), we get

$$\begin{aligned} \|\mathbf{p} - p\|_C + \|\mathbf{q} - q\|_C + \|\mathbf{r} - r\|_C &\leq \frac{T^\kappa}{\Gamma(\kappa+1)} \epsilon_\kappa + \frac{T^\nu}{\Gamma(\nu+1)} \epsilon_\nu + \frac{T^\omega}{\Gamma(\omega+1)} \epsilon_\omega \\ &\quad + (\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3) (\|\mathbf{p} - p\|_C + \|\mathbf{q} - q\|_C + \|\mathbf{r} - r\|_C). \end{aligned}$$

Hence,

$$\begin{aligned} \|(p, q, r) - (\mathbf{p}, \mathbf{q}, \mathbf{r})\|_C &\leq \left(\frac{T^\kappa}{\Gamma(\kappa+1)} + \frac{T^\nu}{\Gamma(\nu+1)} + \frac{T^\omega}{\Gamma(\omega+1)} \right) \frac{\epsilon}{1 - (\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3)} \\ &\leq \mathcal{H}_{(L_{\Phi_1}, L_{\Psi_1}, L_{\Theta_1}, \kappa, \nu, \omega)} \epsilon, \end{aligned}$$

where $\mathcal{H}_{(L_{\Phi_1}, L_{\Psi_1}, L_{\Theta_1}, \kappa, \nu, \omega)} = \left(\frac{T^\kappa}{\Gamma(\kappa+1)} + \frac{T^\nu}{\Gamma(\nu+1)} + \frac{T^\omega}{\Gamma(\omega+1)} \right) \frac{\epsilon}{1 - (\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3)}$, $\epsilon = \max\{\epsilon_\kappa, \epsilon_\nu, \epsilon_\omega\}$. Hence, the fractional system (1.1) is HU stable. Also, if we set $\overline{\mathcal{H}}_{(L_{\Phi_1}, L_{\Psi_1}, L_{\Theta_1}, \kappa, \nu, \omega)}(\epsilon) = \mathcal{H}_{(L_{\Phi_1}, L_{\Psi_1}, L_{\Theta_1}, \kappa, \nu, \omega)} \epsilon$, $\mathcal{H}_{(L_{\Phi_1}, L_{\Psi_1}, L_{\Theta_1}, \kappa, \nu, \omega)}(0) = 0$, then system (1.1) is generalized HU stable. $\square$

5 Controllability results

Definition 5.1. The functions $p, q, r \in C(I, \mathbb{R})$ are said to be the solution of the system (1.2), if $p(t), q(t), r(t)$ satisfies $p(0) = p_0, q(0) = q_0, r(0) = r_0$ and p, q, r is the solution of the following integral equations

$$p(t) = p_0 + \frac{1}{\Gamma(\kappa)} \int_0^t (t - \zeta)^{\kappa-1} (\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) + \mathcal{B}u(\zeta)) \Delta\zeta, \quad \forall t \in I, \quad (5.36)$$

$$q(t) = q_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t - \zeta)^{\nu-1} (\Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) + \mathcal{C}v(\zeta)) \Delta\zeta, \quad \forall t \in I, \quad (5.37)$$

$$r(t) = r_0 + \frac{1}{\Gamma(\omega)} \int_0^t (t - \zeta)^{\omega-1} (\Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) + \mathcal{D}w(\zeta)) \Delta\zeta, \quad \forall t \in I. \quad (5.38)$$

Definition 5.2. System (1.2) is called controllable on I , if for every $p_0, q_0, r_0, p_T, q_T, r_T \in \mathbb{R}$, there exists rd-continuous functions $u, v, w \in L^2(I, \mathbb{R})$ such that the corresponding solution of system (1.2) satisfies $p(0) = p_0, q(0) = q_0, r(0) = r_0, p(T) = p_T, q(T) = q_T$ and $r(T) = r_T$.

We set:

$$\mathcal{Q}_4 = \frac{T^\kappa (L_{\Phi_1} + L_{\Phi_2} + L_{\Phi_3})}{\Gamma(\kappa + 1)} \left(1 + \frac{M_B M_{\mathcal{W}}^\kappa T^\kappa}{\Gamma(\kappa + 1)} \right), \quad \mathcal{Q}_5 = \frac{T^\nu (L_{\Psi_1} + L_{\Psi_2} + L_{\Psi_3})}{\Gamma(\nu + 1)} \left(1 + \frac{M_C M_{\mathcal{W}}^\nu T^\nu}{\Gamma(\nu + 1)} \right),$$

$$\mathcal{Q}_6 = \frac{T^\omega (L_{\Theta_1} + L_{\Theta_2} + L_{\Theta_3})}{\Gamma(\omega + 1)} \left(1 + \frac{M_D M_{\mathcal{W}}^\omega T^\omega}{\Gamma(\omega + 1)} \right),$$

$$\mathcal{N}_6 = \left(1 + \frac{M_B M_{\mathcal{W}}^\kappa T^\kappa}{\Gamma(\kappa + 1)} \right) \left(|p_0| + \frac{T^\kappa M_\Phi}{\Gamma(\kappa + 1)} \right) + \frac{M_B M_{\mathcal{W}}^\kappa T^\kappa |p_T|}{\Gamma(\kappa + 1)},$$

$$\mathcal{N}_7 = \left(1 + \frac{M_C M_{\mathcal{W}}^\nu T^\nu}{\Gamma(\nu + 1)} \right) \left(|q_0| + \frac{T^\nu M_\Psi}{\Gamma(\nu + 1)} \right) + \frac{M_C M_{\mathcal{W}}^\nu T^\nu |q_T|}{\Gamma(\nu + 1)},$$

$$\mathcal{N}_8 = \left(1 + \frac{M_D M_{\mathcal{W}}^\omega T^\omega}{\Gamma(\omega + 1)} \right) \left(|r_0| + \frac{T^\omega M_\Theta}{\Gamma(\omega + 1)} \right) + \frac{M_D M_{\mathcal{W}}^\omega T^\omega |r_T|}{\Gamma(\omega + 1)}.$$

Lemma 5.3. Let the assumptions (A) and (W) be satisfied and $p_T, q_T, r_T \in \mathbb{R}$ are arbitrary points. Then, the solution $p(t), q(t), r(t)$ of the system (1.2) on I defined by (5.36), (5.37) and (5.38)) respectively with the control functions

$$u(t) = \left({}^\kappa \mathcal{W}_0^T \right)^{-1} \left[p_T - p_0 - \frac{1}{\Gamma(\kappa)} \int_0^T (T - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), \quad t \in I, \quad (5.39)$$

$$v(t) = \left({}^\nu \mathcal{W}_0^T \right)^{-1} \left[q_T - q_0 - \frac{1}{\Gamma(\nu)} \int_0^T (T - \zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), \quad t \in I, \quad (5.40)$$

$$w(t) = \left({}^\omega \mathcal{W}_0^T \right)^{-1} \left[r_T - r_0 - \frac{1}{\Gamma(\omega)} \int_0^T (T - \zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), \quad t \in I, \quad (5.41)$$

satisfies $p(T) = p_T, q(T) = q_T, r(T) = r_T$. Also, the control functions $u(t), v(t), w(t)$ has the estimate $\|u(t)\| \leq M_u, \|v(t)\| \leq M_v, \|w(t)\| \leq M_w$ respectively, where

$$\begin{aligned}
M_u &= M_{\mathcal{W}}^{\kappa} \left[|p_T| + |p_0| + \frac{T^{\kappa}}{\Gamma(\kappa+1)} \left(M_{\Phi} + L_{\Phi_1} \sup_{t \in I} |p(t)| + L_{\Phi_2} \sup_{t \in I} |q(t)| + L_{\Phi_3} \sup_{t \in I} |r(t)| \right) \right], \\
M_v &= M_{\mathcal{W}}^{\nu} \left[|q_T| + |q_0| + \frac{T^{\nu}}{\Gamma(\nu+1)} \left(M_{\Psi} + L_{\Psi_1} \sup_{t \in I} |p(t)| + L_{\Psi_2} \sup_{t \in I} |q(t)| + L_{\Psi_3} \sup_{t \in I} |r(t)| \right) \right], \\
M_w &= M_{\mathcal{W}}^{\omega} \left[|r_T| + |r_0| + \frac{T^{\omega}}{\Gamma(\omega+1)} \left(M_{\Theta} + L_{\Theta_1} \sup_{t \in I} |p(t)| + L_{\Theta_2} \sup_{t \in I} |q(t)| + L_{\Theta_3} \sup_{t \in I} |r(t)| \right) \right].
\end{aligned}$$

Proof. Consider the solution $p(t)$ of the system (1.2) on I defined by (5.36). For $t = T$ we get:

$$\begin{aligned}
p(T) &= p_0 + \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} [\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) + \mathcal{B}u(\zeta)] \Delta\zeta \\
&= p_0 + \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \\
&\quad + \int_0^T (T-\tau)^{\kappa-1} \mathcal{B}(\mathcal{W}_0^T \kappa)^{-1} \left[p_T - p_0 - \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] \Delta\tau \\
&= p_0 + \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \\
&\quad + (\mathcal{W}_0^T \kappa)(\mathcal{W}_0^T \kappa)^{-1} \left[p_T - p_0 - \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] \\
&= p_T.
\end{aligned}$$

Similarly, by putting $t = T$ in the solution (5.37) and (5.38) of the system (1.2), we get

$$q(T) = q_T \text{ and } r(T) = r_T.$$

Also,

$$\begin{aligned}
|u(t)| &\leq M_{\mathcal{W}}^{\kappa} \left[|p_T| + |p_0| + \frac{1}{\Gamma(\kappa)} \int_0^T (T-r)^{\kappa-1} |\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta))| \Delta\zeta \right] \\
&\leq M_{\mathcal{W}}^{\kappa} \left[|p_T| + |p_0| + \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} [M_{\Phi} + L_{\Phi_1} |p(\zeta)| + L_{\Phi_2} |q(\zeta)| + L_{\Phi_3} |r(\zeta)|] \Delta\zeta \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
|u(t)| &\leq M_{\mathcal{W}}^{\kappa} \left[|p_T| + |p_0| + \frac{T^{\kappa}}{\Gamma(\kappa+1)} (M_{\Phi} + L_{\Phi_1} \sup_{t \in I} |p(t)| + L_{\Phi_2} \sup_{t \in I} |q(t)| + L_{\Phi_3} \sup_{t \in I} |r(t)|) \right] \\
&= M_u.
\end{aligned}$$

Similarly, we can show that $|v(t)| \leq M_v$ and $|w(t)| \leq M_w$. □

Theorem 5.4. If the assumptions (A) and (W) are satisfied, then the system (1.2) is controllable on I provide

$$\max_{4 \leq i \leq 6} \{Q_i\} < \frac{1}{3}. \quad (5.42)$$

Proof. Let us consider a subset $\Omega'' \subseteq PS$ such that

$$\Omega'' = \{(p, q, r) \in PS : \|(p, q, r)\|_C \leq \delta''\},$$

where

$$\delta'' > \max \left\{ \frac{\mathcal{N}_6}{1-3\mathcal{Q}_4}, \frac{\mathcal{N}_7}{1-3\mathcal{Q}_5}, \frac{\mathcal{N}_8}{1-3\mathcal{Q}_6} \right\}.$$

Now, we define an operator $\Xi'' : \Omega'' \rightarrow \Omega''$ as

$$\Xi''(p, q, r)(t) = \begin{pmatrix} \Xi''_{\kappa}(p, q, r)(t) \\ \Xi''_{\nu}(p, q, r)(t) \\ \Xi''_{\omega}(p, q, r)(t) \end{pmatrix}, \quad t \in I, \quad (5.43)$$

where

$$\Xi''_{\kappa}(p, q, r)(t) = p_0 + \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} (\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) + \mathcal{B}u(\zeta)) \Delta\zeta,$$

$$\Xi''_{\nu}(p, q, r)(t) = q_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-\zeta)^{\nu-1} (\Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) + \mathcal{C}v(\zeta)) \Delta\zeta,$$

$$\Xi''_{\omega}(p, q, r)(t) = r_0 + \frac{1}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} (\Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) + \mathcal{D}w(\zeta)) \Delta\zeta,$$

$u(t)$, $v(t)$, $w(t)$ are given by the (equations 5.39), (5.40) and (5.41) in the interval I .

Now, we need to prove that Ξ'' maps Ω'' into itself, we have

$$\begin{aligned} |\Xi''_{\kappa}(p, q, r)(t)| &\leq |p_0| + \left| \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right| + \left| \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} \mathcal{B}u(\zeta) \Delta\zeta \right| \\ &\leq |p_0| + \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} (L_{\Phi_1}|p(\zeta)| + L_{\Phi_2}|q(\zeta)| + L_{\Phi_3}|r(\zeta)| + M_{\Phi}) \Delta\zeta + \frac{M_{\mathcal{B}}M_uT^{\kappa}}{\Gamma(\kappa+1)} \\ &\leq |p_0| + \frac{(L_{\Phi_1}\delta'' + L_{\Phi_2}\delta'' + L_{\Phi_3}\delta'' + M_{\Phi})}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} d\zeta + \frac{M_{\mathcal{B}}M_{\nu}T^{\kappa}}{\Gamma(\kappa+1)} [|p_T| + |p_0| \\ &\quad + \frac{T^{\kappa}}{\Gamma(\kappa+1)} (L_{\Phi_1}\delta'' + L_{\Phi_2}\delta'' + L_{\Phi_3}\delta'' + M_{\Phi})] \\ &\leq \mathcal{N}_6 + \delta'' \mathcal{Q}_4. \end{aligned}$$

Hence,

$$\|\Xi''_{\kappa}(p, q, r)\|_C \leq \mathcal{N}_6 + \delta'' \mathcal{Q}_4. \quad (5.44)$$

Similarly, one can find

$$\|\Xi''_{\nu}(p, q, r)\|_C \leq \mathcal{N}_7 + \delta'' \mathcal{Q}_5. \quad (5.45)$$

$$\|\Xi''_{\omega}(p, q, r)\|_C \leq \mathcal{N}_8 + \delta'' \mathcal{Q}_6. \quad (5.46)$$

Summarize the above inequalities (5.44), (5.45) and (5.46), we get:

$$\|\Xi''(p, q, r)\|_C \leq \delta''. \quad (5.47)$$

Therefore, $\Xi'' : \Omega'' \rightarrow \Omega''$. Also, for $t \in I$,

$$\begin{aligned}
|\Xi''_{\kappa}(p, q, r)(t) - \Xi''_{\kappa}(\bar{p}, \bar{q}, \bar{r})(t)| &\leq \frac{1}{\Gamma(\kappa)} \int_0^t (t-\zeta)^{\kappa-1} |\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) - \Phi(\zeta, \bar{p}(\zeta), \bar{q}(\zeta), \bar{r}(\zeta))| \Delta\zeta \\
&\quad + \frac{1}{\Gamma(\kappa)} \int_0^t (t-\tau)^{\kappa-1} |\mathcal{B}| |(\mathcal{W}_0^T \kappa)^{-1}| \times \left[\frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} |\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) - \Phi(\zeta, \bar{p}(\zeta), \bar{q}(\zeta), \bar{r}(\zeta))| \Delta\zeta \right] \Delta\tau \\
&\leq \frac{T^{\kappa} (L_{\Phi_1} \|p - \bar{p}\|_C + L_{\Phi_2} \|q - \bar{q}\|_C + L_{\Phi_3} \|r - \bar{r}\|_C)}{\Gamma(\kappa+1)} + \frac{T^{\kappa} M_B M_{\mathcal{W}}^{\kappa}}{\Gamma(\kappa+1)} \left[\frac{T^{\kappa} (L_{\Phi_1} \|p - \bar{p}\|_C + L_{\Phi_2} \|q - \bar{q}\|_C + L_{\Phi_3} \|r - \bar{r}\|_C)}{\Gamma(\kappa+1)} \right] \\
&\leq \left(1 + \frac{T^{\kappa} M_B M_{\mathcal{W}}^{\kappa}}{\Gamma(\kappa+1)} \right) \frac{T^{\kappa} (L_{\Phi_1} \|p - \bar{p}\|_C + L_{\Phi_2} \|q - \bar{q}\|_C + L_{\Phi_3} \|r - \bar{r}\|_C)}{\Gamma(\kappa+1)}.
\end{aligned}$$

Thus,

$$\|\Xi''_{\kappa}(p, q, r) - \Xi''_{\kappa}(\bar{p}, \bar{q}, \bar{r})\|_C \leq \mathcal{Q}_4 (\|p - \bar{p}\|_C + \|q - \bar{q}\|_C + \|r - \bar{r}\|_C). \quad (5.48)$$

Similarly, we can easily obtain

$$\|\Xi''_{\nu}(p, q, r) - \Xi''_{\nu}(\bar{p}, \bar{q}, \bar{r})\|_C \leq \mathcal{Q}_5 (\|p - \bar{p}\|_C + \|q - \bar{q}\|_C + \|r - \bar{r}\|_C). \quad (5.49)$$

$$\|\Xi''_{\omega}(p, q, r) - \Xi''_{\omega}(\bar{p}, \bar{q}, \bar{r})\|_C \leq \mathcal{Q}_6 (\|p - \bar{p}\|_C + \|q - \bar{q}\|_C + \|r - \bar{r}\|_C). \quad (5.50)$$

After summarize the inequalities (5.48), (5.49) and (5.50), we get

$$\|\Xi''(p, q, r) - \Xi''(\bar{p}, \bar{q}, \bar{r})\|_C \leq (\mathcal{Q}_4 + \mathcal{Q}_5 + \mathcal{Q}_6) \| (p, q, r), (\bar{p}, \bar{q}, \bar{r}) \|_C.$$

Therefore, Ξ'' is a strict contraction operator. Thus, by means of Banach fixed point technique, we have a unique fixed point of Ξ'' which is the solution of the system (1.2). Also, from the Lemma 5.3, we can see that $p(t)$, $q(t)$, $r(t)$ satisfies $p(T) = p_T$, $q(T) = q_T$ and $r(T) = r_T$. Thus, we conclude that the system (1.2) is controllable on I . $\square$

6 An illustrative example

Example 6.1. Let us consider the fractional system

$$\begin{aligned}
{}^{c, \mathbb{T}} D^{\frac{1}{2}} p(t) &= \frac{|p(t)|}{(t+9)^2(1+|p(t)|)} + \frac{1}{65(1+q^2(t))} + \frac{\sin(r(t))}{e^{t+4}} + \frac{1}{20e^t}, \quad t \in I = [0, 10]_{\mathbb{T}}, \\
{}^{c, \mathbb{T}} D^{\frac{1}{2}} q(t) &= \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2e^{t+4}(1+|p(t)| + |q(t)| + |r(t)|)} + \frac{1}{2(t^2+4)}, \quad t \in I, \\
{}^{c, \mathbb{T}} D^{\frac{1}{2}} r(t) &= \frac{\sin(p(t))}{e^{t+4}} + \frac{|q(t)|}{(t+9)^2(1+|q(t)|)} + \frac{1}{65(1+r^2(t))} + \frac{1}{20e^t}, \quad t \in I, \\
p(0) &= 0, \quad q(0) = 0, \quad r(0) = 0.
\end{aligned} \quad (6.51)$$

For $t \in I$ and $p, q, r \in \mathbb{R}$, we set

$$\begin{aligned}
\Phi(t, p(t), q(t), r(t)) &= \frac{|p(t)|}{(t+9)^2(1+|p(t)|)} + \frac{1}{65(1+q^2(t))} + \frac{\sin(r(t))}{e^{t+4}} + \frac{1}{20e^t}, \\
\Psi(t, p(t), q(t), r(t)) &= \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2e^{t+4}(1+|p(t)| + |q(t)| + |r(t)|)} + \frac{1}{2(t^2+4)}, \\
\Theta(t, p(t), q(t), r(t)) &= \frac{\sin(p(t))}{e^{t+4}} + \frac{|q(t)|}{(t+9)^2(1+|q(t)|)} + \frac{1}{65(1+r^2(t))} + \frac{1}{20e^t}.
\end{aligned}$$

Then, $\forall t \in I, p, q, r, \bar{p}, \bar{q}, \bar{r} \in \mathbb{R}$, we have:

$$|\Phi(t, p, q, r) - \Phi(t, \bar{p}, \bar{q}, \bar{r})| \leq \frac{1}{81} |p - \bar{p}| + \frac{1}{65} |q - \bar{q}| + \frac{1}{e^4} |r - \bar{r}|,$$

$$|\Psi(t, p, q, r) - \Psi(t, \bar{p}, \bar{q}, \bar{r})| \leq \frac{1}{e^4} [|p - \bar{p}| + |q - \bar{q}| + |r - \bar{r}|],$$

$$|\Theta(t, p, q, r) - \Theta(t, \bar{p}, \bar{q}, \bar{r})| \leq \frac{1}{e^4} |p - \bar{p}| + \frac{1}{81} |q - \bar{q}| + \frac{1}{65} |r - \bar{r}|.$$

Thus, the assumption (A) is holds. Also, for $\kappa = 1/3$, $\nu = 1/3$, $\omega = 1/4$ and $T = 10$,

$$\mathcal{Q}_1 = \frac{T^\kappa (L_{\Phi_1} + L_{\Phi_2} + L_{\Phi_3})}{\Gamma(\kappa + 1)} = 0.1111,$$

$$\mathcal{Q}_2 = \frac{T^\nu (L_{\Psi_1} + L_{\Psi_2} + L_{\Psi_3})}{\Gamma(\nu + 1)} = 0.0663,$$

$$\mathcal{Q}_3 = \frac{T^\omega (L_{\Theta_1} + L_{\Theta_2} + L_{\Theta_3})}{\Gamma(\omega + 1)} = 0.0903.$$

Therefore, all the conditions of the Theorem 3.1 are satisfied. Hence, fractional system (6.51) has a unique solution.

Now, we consider the following cases for different time scales:

Case A. When $\mathbb{T} = \mathbb{R}$. We choose the final points as $p(T) = 2$, $q(T) = 1$, $r(T) = 2$. Clearly, we can see that the trajectory of the system (6.51) shown in Figure 1 does not passes throw the desired points $p(T) = 2$, $q(T) = 1$ and $r(T) = 2$. But if we add the control functions $u(t)$, $v(t)$ and $w(t)$ given by

$$u(t) = ({}^\kappa \mathcal{W}_0^T)^{-1} \left[2 - \frac{1}{\Gamma(\kappa)} \int_0^T (T - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta \right] (t), t \in I,$$

$$v(t) = ({}^\nu \mathcal{W}_0^T)^{-1} \left[1 - \frac{1}{\Gamma(\nu)} \int_0^T (T - \zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta \right] (t), t \in I,$$

$$w(t) = ({}^\omega \mathcal{W}_0^T)^{-1} \left[2 - \frac{1}{\Gamma(\omega)} \int_0^T (T - \zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta \right] (t), t \in I,$$

where

Figure 1: Trajectory of the system (6.51), when $\mathbb{T} = \mathbb{R}$.

$${}^{\kappa}\mathcal{W}_0^T = \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Delta\zeta, {}^{\nu}\mathcal{W}_0^T = \frac{1}{\Gamma(\nu)} \int_0^T (T-\zeta)^{\nu-1} \Delta\zeta \text{ and } {}^{\omega}\mathcal{W}_0^T = \frac{1}{\Gamma(\omega)} \int_0^T (T-\zeta)^{\omega-1} \Delta\zeta,$$

in the system (6.51), it becomes

$$\begin{aligned} {}^{c,\mathbb{T}}D^{\frac{1}{2}}p(t) &= \frac{|p(t)|}{(t+9)^2(1+|p(t)|)} + \frac{1}{65(1+q^2(t))} + \frac{\sin(r(t))}{e^{t+4}} + \frac{1}{20e^t} + u(t), \quad t \in I = [0, 10]_{\mathbb{R}}, \\ {}^{c,\mathbb{T}}D^{\frac{1}{2}}q(t) &= \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2e^{t+4}(1+|p(t)| + |q(t)| + |r(t)|)} + \frac{1}{2(t^2+4)} + v(t), \quad t \in I, \\ {}^{c,\mathbb{T}}D^{\frac{1}{2}}r(t) &= \frac{\sin(p(t))}{e^{t+4}} + \frac{|q(t)|}{(t+9)^2(1+|q(t)|)} + \frac{1}{65(1+r^2(t))} + \frac{1}{20e^t} + w(t), \quad t \in I, \\ p(0) &= 0, \quad q(0) = 0, \quad r(0) = 0. \end{aligned} \quad (6.52)$$

Now, we can find that

$$\begin{aligned} {}^{\kappa}\mathcal{W}_0^T &= 2.413, \quad {}^{\nu}\mathcal{W}_0^T = 2.413, \quad {}^{\omega}\mathcal{W}_0^T = 1.9619, \\ \mathcal{Q}_4 &= \frac{T^{\kappa}(L_{\Phi_1} + L_{\Phi_2} + L_{\Phi_3})}{\Gamma(\kappa+1)} \left(1 + \frac{M_B M_{\mathcal{W}}^{\kappa} T^{\kappa}}{\Gamma(\kappa+1)} \right) = 0.22218, \\ \mathcal{Q}_5 &= \frac{T^{\nu}(L_{\Psi_1} + L_{\Psi_2} + L_{\Psi_3})}{\Gamma(\nu+1)} \left(1 + \frac{M_C M_{\mathcal{W}}^{\nu} T^{\nu}}{\Gamma(\nu+1)} \right) = 0.13257, \\ \mathcal{Q}_6 &= \frac{T^{\omega}(L_{\Theta_1} + L_{\Theta_2} + L_{\Theta_3})}{\Gamma(\omega+1)} \left(1 + \frac{M_D M_{\mathcal{W}}^{\omega} T^{\omega}}{\Gamma(\omega+1)} \right) = 0.18068. \end{aligned}$$

Therefore, all the assumptions of Theorem 5.4 are hold. Hence, system (6.52) is controllable and the controlled trajectory is shown in Figure 2.

Case 2. When $\mathbb{T} = \mathbb{Z}$. We choose the final points as $p(T) = 1$, $q(T) = 2$, $r(T) = 1.4$. Clearly, we can see that the trajectory of the system (6.51) shown in Figure 3 does not passes throw the desired points $p(T) = 1$, $q(T) = 2$ and $r(T) = 1.4$. But if we add the control functions $u(t)$, $v(t)$ and $w(t)$ given by

$$u(t) = ({}^{\kappa}\mathcal{W}_0^T)^{-1} \left[1 - \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), t \in I,$$

Figure 2: Trajectory of the controlled system (6.52), when $\mathbb{T} = \mathbb{R}$, $p(T) = 2$, $q(T) = 1$, $r(T) = 2$.

$$v(t) = \left({}^v\mathcal{W}_0^T\right)^{-1} \left[2 - \frac{1}{\Gamma(v)} \int_0^T (T-\zeta)^{v-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), t \in I,$$

$$w(t) = \left({}^\omega\mathcal{W}_0^T\right)^{-1} \left[1.4 - \frac{1}{\Gamma(\omega)} \int_0^T (T-\zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), t \in I,$$

where

$${}^\kappa\mathcal{W}_0^T = \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Delta\zeta, {}^v\mathcal{W}_0^T = \frac{1}{\Gamma(v)} \int_0^T (T-\zeta)^{v-1} \Delta\zeta \text{ and } {}^\omega\mathcal{W}_0^T = \frac{1}{\Gamma(\omega)} \int_0^T (T-\zeta)^{\omega-1} \Delta\zeta,$$

in the system (6.51), it becomes

$$\begin{aligned} {}^{c,\mathbb{T}}D^{\frac{1}{3}}p(t) &= \frac{|p(t)|}{(t+9)^2(1+|p(t)|)} + \frac{1}{65(1+q^2(t))} + \frac{\sin(r(t))}{e^{t+4}} + \frac{1}{20e^t} + u(t), \quad t \in I = [0, 10]_{\mathbb{Z}}, \\ {}^{c,\mathbb{T}}D^{\frac{1}{3}}q(t) &= \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2e^{t+4}(1+|p(t)| + |q(t)| + |r(t)|)} + \frac{1}{2(t^2+4)} + v(t), \quad t \in I, \\ {}^{c,\mathbb{T}}D^{\frac{1}{3}}r(t) &= \frac{\sin(p(t))}{e^{t+4}} + \frac{|q(t)|}{(t+9)^2(1+|q(t)|)} + \frac{1}{65(1+r^2(t))} + \frac{1}{20e^t} + w(t), \quad t \in I, \\ p(0) &= 0, \quad q(0) = 0, \quad r(0) = 0. \end{aligned} \quad (6.53)$$

Now, we can find that

$$\begin{aligned} {}^\kappa\mathcal{W}_0^T &= 1.847, \quad {}^v\mathcal{W}_0^T = 1.847, \quad {}^\omega\mathcal{W}_0^T = 1.365, \\ \mathcal{Q}_4 &= \frac{T^\kappa(L_{\Phi_1} + L_{\Phi_2} + L_{\Phi_3})}{\Gamma(\kappa+1)} \left(1 + \frac{M_B M_{\mathcal{W}}^\kappa T^\kappa}{\Gamma(\kappa+1)} \right) = 0.2562, \\ \mathcal{Q}_5 &= \frac{T^v(L_{\Psi_1} + L_{\Psi_2} + L_{\Psi_3})}{\Gamma(v+1)} \left(1 + \frac{M_C M_{\mathcal{W}}^v T^v}{\Gamma(v+1)} \right) = 0.1529, \\ \mathcal{Q}_6 &= \frac{T^\omega(L_{\Theta_1} + L_{\Theta_2} + L_{\Theta_3})}{\Gamma(\omega+1)} \left(1 + \frac{M_D M_{\mathcal{W}}^\omega T^\omega}{\Gamma(\omega+1)} \right) = 0.2202. \end{aligned}$$

Therefore, all the assumptions of Theorem 5.4 are hold. Hence, system (6.53) is controllable and the controlled trajectory is shown in Figure 4.

Figure 3: Trajectory of the system (6.51), when $\mathbb{T} = \mathbb{Z}$.

Case 3. When $\mathbb{T} = \frac{1}{4}\mathbb{Z}$. We choose the final points as $p(T) = 3$, $q(T) = 1.5$, $r(T) = 2$. Clearly, we can see that the trajectory of the system (6.51) shown in Figure 5 does not pass through the desired points $p(T) = 3$, $q(T) = 1.5$ and $r(T) = 2$.

But if we add the control functions $u(t)$, $v(t)$ and $w(t)$ given by

$$\begin{aligned} u(t) &= \left({}^{\kappa}\mathcal{W}_0^T\right)^{-1} \left[3 - \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), t \in I, \\ v(t) &= \left({}^{\nu}\mathcal{W}_0^T\right)^{-1} \left[1.5 - \frac{1}{\Gamma(\nu)} \int_0^T (T-\zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), t \in I, \\ w(t) &= \left({}^{\omega}\mathcal{W}_0^T\right)^{-1} \left[2 - \frac{1}{\Gamma(\omega)} \int_0^T (T-\zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), t \in I, \end{aligned}$$

where

$${}^{\kappa}\mathcal{W}_0^T = \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Delta\zeta, {}^{\nu}\mathcal{W}_0^T = \frac{1}{\Gamma(\nu)} \int_0^T (T-\zeta)^{\nu-1} \Delta\zeta \text{ and } {}^{\omega}\mathcal{W}_0^T = \frac{1}{\Gamma(\omega)} \int_0^T (T-\zeta)^{\omega-1} \Delta\zeta,$$

in the system (6.51), it becomes

$$\begin{aligned} {}^{c,\mathbb{T}}D^{\frac{1}{2}}p(t) &= \frac{|p(t)|}{(t+9)^2(1+|p(t)|)} + \frac{1}{65(1+q^2(t))} + \frac{\sin(r(t))}{e^{t+4}} + \frac{1}{20e^t} + u(t), \quad t \in I = [0, 10]_{\mathbb{Z}}, \\ {}^{c,\mathbb{T}}D^{\frac{1}{2}}q(t) &= \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2e^{t+4}(1+|p(t)| + |q(t)| + |r(t)|)} + \frac{1}{2(t^2+4)} + v(t), \quad t \in I, \\ {}^{c,\mathbb{T}}D^{\frac{1}{2}}r(t) &= \frac{\sin(p(t))}{e^{t+4}} + \frac{|q(t)|}{(t+9)^2(1+|q(t)|)} + \frac{1}{65(1+r^2(t))} + \frac{1}{20e^t} + w(t), \quad t \in I, \\ p(0) &= 0, \quad q(0) = 0, \quad r(0) = 0. \end{aligned} \tag{6.54}$$

Now, we can find that

Figure 4: Trajectory of the controlled system (6.53), when $\mathbb{T} = \mathbb{Z}$, $p(T) = 1$, $q(T) = 2$, $r(T) = 1.4$.

Figure 5: Trajectory of the system (6.51), when $\mathbb{T} = \frac{1}{4}\mathbb{Z}$.

$${}^{\kappa}\mathcal{W}_0^T = 1.539, {}^{\nu}\mathcal{W}_0^T = 1.539, {}^{\omega}\mathcal{W}_0^T = 1.037,$$

$$\mathcal{Q}_4 = \frac{T^{\kappa}(L_{\Phi_1} + L_{\Phi_2} + L_{\Phi_3})}{\Gamma(\kappa + 1)} \left(1 + \frac{M_B M_{\mathcal{W}}^{\kappa} T^{\kappa}}{\Gamma(\kappa + 1)} \right) = 0.2853,$$

$$\mathcal{Q}_5 = \frac{T^{\nu}(L_{\Psi_1} + L_{\Psi_2} + L_{\Psi_3})}{\Gamma(\nu + 1)} \left(1 + \frac{M_C M_{\mathcal{W}}^{\nu} T^{\nu}}{\Gamma(\nu + 1)} \right) = 0.1702,$$

$$\mathcal{Q}_6 = \frac{T^{\omega}(L_{\Theta_1} + L_{\Theta_2} + L_{\Theta_3})}{\Gamma(\omega + 1)} \left(1 + \frac{M_D M_{\mathcal{W}}^{\omega} T^{\omega}}{\Gamma(\omega + 1)} \right) = 0.2613.$$

Therefore, all the assumptions of Theorem 5.4 are hold. Hence, system (6.54) is controllable and the controlled trajectory is shown in Figure 6.

Case 4. When $[0, 6]_{\frac{1}{2}\mathbb{Z}} \cup [7, 10]_{\mathbb{R}}$ (say D). We choose the final points as $p(T) = 3$, $q(T) = 1$, $r(T) = 2$. Clearly, we can see that the trajectory of the system (6.51) shown in Figure 7 does not passes throw the desired points $p(T) = 3$, $q(T) = 1$ and $r(T) = 2$.

But if we add the control functions $u(t)$, $v(t)$ and $w(t)$ given by

Figure 6: Trajectory of the controlled system (6.54), when $\mathbb{T} = \frac{1}{4}\mathbb{Z}$, $p(T) = 3$, $q(T) = 1.5$, $r(T) = 2$.

Figure 7: Trajectory of the system (6.51), when $\mathbb{T} = D$.

$$\begin{aligned}
 u(t) &= \left({}^{\kappa}\mathcal{W}_0^T\right)^{-1} \left[3 - \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), t \in I, \\
 v(t) &= \left({}^{\nu}\mathcal{W}_0^T\right)^{-1} \left[1 - \frac{1}{\Gamma(\nu)} \int_0^T (T-\zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), t \in I, \\
 w(t) &= \left({}^{\omega}\mathcal{W}_0^T\right)^{-1} \left[2 - \frac{1}{\Gamma(\omega)} \int_0^T (T-\zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta \right] (t), t \in I,
 \end{aligned}$$

where

$${}^{\kappa}\mathcal{W}_0^T = \frac{1}{\Gamma(\kappa)} \int_0^T (T-\zeta)^{\kappa-1} \Delta\zeta, \quad {}^{\nu}\mathcal{W}_0^T = \frac{1}{\Gamma(\nu)} \int_0^T (T-\zeta)^{\nu-1} \Delta\zeta \quad \text{and} \quad {}^{\omega}\mathcal{W}_0^T = \frac{1}{\Gamma(\omega)} \int_0^T (T-\zeta)^{\omega-1} \Delta\zeta,$$

in the system (6.51), it becomes

$$\begin{aligned}
 {}^{c,\mathbb{T}}D^{\frac{1}{2}}p(t) &= \frac{|p(t)|}{(t+9)^2(1+|p(t)|)} + \frac{1}{65(1+q^2(t))} + \frac{\sin(r(t))}{e^{t+4}} + \frac{1}{20e^t} + u(t), \quad t \in I = [0, 10]_D, \\
 {}^{c,\mathbb{T}}D^{\frac{1}{3}}q(t) &= \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2e^{t+4}(1+|p(t)| + |q(t)| + |r(t)|)} + \frac{1}{2(t^2+4)} + v(t), \quad t \in I, \\
 {}^{c,\mathbb{T}}D^{\frac{1}{3}}r(t) &= \frac{\sin(p(t))}{e^{t+4}} + \frac{|q(t)|}{(t+9)^2(1+|q(t)|)} + \frac{1}{65(1+r^2(t))} + \frac{1}{20e^t} + w(t), \quad t \in I, \\
 p(0) &= 0, \quad q(0) = 0, \quad r(0) = 0.
 \end{aligned} \tag{6.55}$$

Now, we can find that

$$\begin{aligned}
 {}^{\kappa}\mathcal{W}_0^T &= 0.5414, \quad {}^{\nu}\mathcal{W}_0^T = 0.5414, \quad {}^{\omega}\mathcal{W}_0^T = 0.7711, \\
 \mathcal{Q}_4 &= \frac{T^{\kappa}(L_{\Phi_1} + L_{\Phi_2} + L_{\Phi_3})}{\Gamma(\kappa+1)} \left(1 + \frac{M_B M_{\mathcal{W}}^{\kappa} T^{\kappa}}{\Gamma(\kappa+1)} \right) = 0.2562, \\
 \mathcal{Q}_5 &= \frac{T^{\nu}(L_{\Psi_1} + L_{\Psi_2} + L_{\Psi_3})}{\Gamma(\nu+1)} \left(1 + \frac{M_C M_{\mathcal{W}}^{\nu} T^{\nu}}{\Gamma(\nu+1)} \right) = 0.15292, \\
 \mathcal{Q}_6 &= \frac{T^{\omega}(L_{\Theta_1} + L_{\Theta_2} + L_{\Theta_3})}{\Gamma(\omega+1)} \left(1 + \frac{M_D M_{\mathcal{W}}^{\omega} T^{\omega}}{\Gamma(\omega+1)} \right) = 0.2270.
 \end{aligned}$$

Therefore, all the assumptions of Theorem 5.4 are hold. Hence, system (6.55) is controllable and the controlled trajectory is shown in Figure 8.

7 Applications

7.1 Let us consider the three species prey-predator model of fractional order on time scales

$$\begin{aligned} {}^{c,\mathbb{T}}D^\alpha p(t) &= Rp\left(1 - \frac{p}{k}\right) - \frac{bps}{a+p} - \gamma pI + c_1 u(t), \alpha \in (0, 1], t \in I = [0, T]_{\mathbb{T}}, \\ {}^{c,\mathbb{T}}D^\alpha s(t) &= \frac{cbps}{a+p} - dsI - ms + c_2 v(t), t \in I, \\ {}^{c,\mathbb{T}}D^\alpha I(t) &= -nI + dsI + cypI + c_3 w(t), t \in I, \end{aligned} \quad (7.56)$$

with the initial conditions $p(0) = p_0$, $s(0) = s_0$, $I(0) = I_0$, where I is infected predator, s is susceptible predator, p is the population density of prey, d the disease transmission coefficient, k the carrying capacities of prey population, n is the death rate of infected predator, R is the growth rate of prey population, m is the death rate of susceptible predator, b is the search rate of the prey toward susceptible predator, a is saturation constant while susceptible predators attack the prey, c is the conversion rate of susceptible predator due to prey. $u(t)$, $v(t)$ and $w(t)$ are some control variables with the control coefficients c_1 , c_2 , c_3 , respectively. This model can be formulated into the fractional system (1.1) with

$$\begin{aligned} \Phi(t, p, q, r) &= Rp\left(1 - \frac{p}{k}\right) - \frac{bpq}{a+p} - \gamma pr, \\ \Psi(t, p, q, r) &= \frac{cbpq}{a+p} - dqr - mq, \\ \Theta(t, p, q, r) &= -nr + dqr + cypr. \end{aligned}$$

Also, we can prove that all the conditions of the Theorems 1.1, 4.6 and 5.4 are satisfied. Hence, the system (7.56) has unique HU stable solution which is controllable.

Figure 8: Trajectory of the controlled system (6.55), when $\mathbb{T} = D$, $p(T) = 3$, $q(T) = 1$, $r(T) = 2$.

7.2 Consider the following fractional order GLV model on time scales

$$\begin{aligned} {}^{c,\mathbb{T}}D^\alpha p(t) &= p - pq + cp^2 - ap^2r + c_1u(t), \alpha \in (0, 1], t \in I = [0, T]_{\mathbb{T}}, \\ {}^{c,\mathbb{T}}D^\alpha q(t) &= -q + pq + c_2v(t), t \in I, \\ {}^{c,\mathbb{T}}D^\alpha r(t) &= -br + ap^2r + c_3w(t), t \in I, \end{aligned}$$

with $p(0) = p_0 \geq 0$, $q(0) = q_0 \geq 0$, $r(0) = r_0 \geq 0$. This model can be formulated into the fractional system (1.1) with

$$\begin{aligned} \Phi(t, p, q, r) &= p - pq + cp^2 - ap^2r, \\ \Psi(t, p, q, r) &= -q + pq, \\ \Theta(t, p, q, r) &= -br + ap^2r. \end{aligned}$$

Also, we can prove that all the conditions of the Theorems 1.1, 4.6 and 5.4 are satisfied. Hence, the system (7.56) has unique HU stable solution which is controllable.

8 Conclusion

We have studied the existence of solutions and HU stability analysis of a fractional dynamic system (1.1) on time scales. For the existence of a unique solution, we used the Banach contraction technique and for the existence of at least one solution, we used the Leray-Schauder alternative-type fixed point theorem under the weaker condition (B). In addition, we have studied the controllability results for the fractional system (1.2). With the help of MATLAB, we have given an example with simulation for the different time scales including $\mathbb{T} = \mathbb{R}$ (case 1 of example 6.1), $\mathbb{T} = \mathbb{Z}$ (case 2 of example 6.1), $\mathbb{T} = \frac{1}{4}\mathbb{Z}$ (case 3 of example 6.1) and $\mathbb{T} = [0, 6]_{\frac{1}{2}\mathbb{Z}} \cup [7, 10]_{\mathbb{R}}$ (case 4 of example 6.1). Also, we have given two applications to illustrate the obtained analytical results.

Acknowledgement: The authors would like to express their sincere thanks to the editor and anonymous reviewers for constructive comments and suggestions to improve the quality of this manuscript.

Author contribution: All the authors have accepted responsibility for the entire content of this submitted manuscript and approved submission.

Research funding: The research of “Vipin Kumar” is funded by the University Grants Commission (UGC) of India under the research fellowship number 2121540900.

Conflict of interest statement: The authors declare no conflicts of interest.

References

- [1] R. L. Magin, *Fractional Calculus in Bioengineering*, vol. 2, Connecticut, Begell House, 2006, 6.
- [2] H. Rudolf, Ed. in *Applications of Fractional Calculus in Physics*, vol. 128, Singapore, World Scientific, 2000.
- [3] R. L. Bagley and P. J. Torvik, “A theoretical basis for the application of fractional calculus to viscoelasticity,” *J. Rheol.*, vol. 27, no. 3, pp. 201–210, 1983.
- [4] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, HEP, Springer, 2011.
- [5] H. A. A. El-Saka, A. A. M. Arafa, and M. I. Gouda, “Dynamical analysis of a fractional SIRS model on homogenous networks,” *Adv. Differ. Equ.*, vol. 2019, no. 1, p. 144, 2019.
- [6] G. M. Zaslavsky, “Fractional kinetic equation for Hamiltonian chaos,” *Phys. Nonlinear Phenom.*, vol. 76, no. 1–3, pp. 110–122, 1994.
- [7] K. Shah and R. A. Khan, “Iterative scheme for a coupled system of fractional-order differential equations with three-point boundary conditions,” *Math. Method Appl. Sci.*, vol. 41, no. 3, pp. 1047–1053, 2018.
- [8] K. Shah, J. R. Wang, H. Khalil, and R. A. Khan, “Existence and numerical solutions of a coupled system of integral BVP for fractional differential equations,” *Adv. Differ. Equ.*, vol. 2018, no. 1, p. 149, 2018.

- [9] J. R. Wang, K. Shah, and A. Ali, "Existence and Hyers-Ulam stability of fractional nonlinear impulsive switched coupled evolution equations," *Math. Method Appl. Sci.*, vol. 41, no. 6, pp. 2392–2402, 2018.
- [10] K. Shah, R. A. Khan, and D. Baleanu, "Study of implicit type coupled system of non-integer order differential equations with antiperiodic boundary conditions," *Math. Method Appl. Sci.*, vol. 42, no. 6, pp. 2033–2042, 2019.
- [11] A. Ali, K. Shah, F. Jarad, V. Gupta, and T. Abdeljawad, "Existence and stability analysis to a coupled system of implicit type impulsive boundary value problems of fractional-order differential equations," *Adv. Differ. Equ.*, vol. 2019, no. 1, p. 101, 2019.
- [12] J. R. Wang, M. Fečkan, and Y. Zhou, "Ulam's type stability of impulsive ordinary differential equations," *J. Math. Anal. Appl.*, vol. 395, no. 1, pp. 258–264, 2012.
- [13] D. Popa and I. Raşa, "On the Hyers-Ulam stability of the linear differential equation," *J. Math. Anal. Appl.*, vol. 381, no. 2, pp. 530–537, 2011.
- [14] J. R. Wang, L. Lv, and Y. Zhou, "Ulam stability and data dependence for fractional differential equations with Caputo derivative," *Electron. J. Qual. Theor. Differ. Equ.*, vol. 2011, no. 63, pp. 1–10, 2011.
- [15] J. R. Wang and X. Li, "A uniform method to Ulam-Hyers's stability for some linear fractional equations," *Mediterr. J. Math.*, vol. 13, pp. 625–635, 2016.
- [16] Y. Pan, Z. Han, S. Sun, and Y. Zhao, "The existence of solutions to a system of discrete fractional boundary value problems," in *Abstract and Applied Analysis 2012*, 2012, p. 15, Art no. 707631.
- [17] S. Hilger, "Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten," Ph.D. Thesis, German, Würzburg, Universität, 1988.
- [18] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Boston, MA, Birkhäuser Boston, Inc., 2001.
- [19] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Boston, MA, Birkhäuser Boston, Inc., 2003.
- [20] S. Dhama and S. Abbas, "Existence and stability of square-mean almost automorphic solution for neutral stochastic evolution equations with Stepanov-like terms on time scales," *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math.*, vol. 113, no. 2, pp. 1231–1250, 2019.
- [21] R. P. Agarwal, C. Wang, and D. O'Regan, "Recent development of time scales and related topics on dynamic equations," *Memoir. Differ. Equ. Math. Phys.*, vol. 67, pp. 131–135, 2016.
- [22] R. P. Agarwal and M. Bohner, "Basic calculus on time scales and some of its applications," *Results Math.*, vol. 35, no. 1, pp. 3–22, 1999.
- [23] C. Wang, R. P. Agarwal, and D. O'Regan, "Calculus of fuzzy vector-valued functions and almost periodic fuzzy vector-valued functions on time scales," *Fuzzy Set Syst.*, vol. 375, pp. 1–52, 2019.
- [24] C. Wang and R. P. Agarwal, "Almost automorphic functions on semigroups induced by complete-closed time scales and application to dynamic equations," *Discrete Cont. Dyn. Syst. - Ser. B*, vol. 25, no. 2, p. 781, 2020.
- [25] C. Wang, R. P. Agarwal, and R. Sakthivel, "Almost periodic oscillations for delay impulsive stochastic Nicholson's blowflies timescale model," *Comput. Appl. Math.*, vol. 37, no. 3, pp. 3005–3026, 2018.
- [26] Y. Shen and D. He, "The general solution and Ulam stability of inhomogeneous Euler-Cauchy dynamic equations on time scales," *J. Comput. Anal. Appl.*, vol. 26, no. 2, pp. 234–241, 2019.
- [27] D. R. Anderson and M. Onitsuka, "Hyers-Ulam stability of first-order homogeneous linear dynamic equations on time scales," *Demonstr. Math.*, vol. 51, no. 1, pp. 198–210, 2018.
- [28] Y. Shen, "The Ulam stability of first order linear dynamic equations on time scales," *Results Math.*, vol. 72, no. 4, pp. 1881–1895, 2017.
- [29] A. Ahmadkhanlu and M. Jahanshahi, "On the existence and uniqueness of solution of initial value problem for fractional order differential equations on time scales," *Bull. Iran. Math. Soc.*, vol. 38, no. 1, pp. 241–252, 2012.
- [30] N. Benkhetto, A. Hammoudi, and D. F. M. Torres, "Existence and uniqueness of solution for a fractional Riemann-Liouville initial value problem on time scales," *J. King Saud Univ. Sci.*, vol. 28, no. 1, pp. 87–92, 2016.
- [31] N. R. O. Bastos, D. Mozyrska, and D. F. M. Torres, "Fractional derivatives and integrals on time scales via the inverse generalized Laplace transform," *Int. J. Math. Comput.*, vol. 11, no. J11, pp. 1–9, 2011.
- [32] N. Benkhetto, A. M. C. Brito da Cruz, and D. F. M. Torres, "A fractional calculus on arbitrary time scales: fractional differentiation and fractional integration," *Signal Process.*, vol. 107, pp. 230–237, 2015.
- [33] V. Kumar and M. Malik, "Existence and stability of fractional integro differential equation with non-instantaneous integrable impulses and periodic boundary condition on time scales," *J. King Saud Univ. Sci.*, vol. 31, no. 4, pp. 1311–1317, 2019.
- [34] V. Kumar and M. Malik, "Existence, uniqueness and stability of nonlinear implicit fractional dynamical equation with impulsive condition on time scales," *Nonaut. Dyn. Syst.*, vol. 6, no. 1, pp. 65–80, 2019.
- [35] A. Kumar, M. Malik, and R. Sakthivel, "Controllability of the second-order nonlinear differential equations with non-instantaneous impulses," *J. Dyn. Contr. Syst.*, vol. 24, no. 2, pp. 325–342, 2018.
- [36] M. Malik, A. Kumar, and R. Sakthivel, "Exact and trajectory controllability of second-order evolution systems with impulses and deviated arguments," *Math. Method Appl. Sci.*, vol. 41, no. 11, pp. 4259–4272, 2018.
- [37] R. Sakthivel, N. I. Mahmudov, J. J. Nieto, and J. H. Kim, "On controllability of nonlinear impulsive integrodifferential systems," *Dyn. Cont. Discrete Impuls. Syst. Ser. A*, vol. 15, no. 3, pp. 333–343, 2008.
- [38] R. Sakthivel, N. I. Mahmudov, and S. G. Lee, "Controllability of non-linear impulsive stochastic systems," *Int. J. Contr.*, vol. 82, no. 5, pp. 801–807, 2009.

- [39] R. Sakthivel, N. I. Mahmudov, and J. H. Kim, "On controllability of second order nonlinear impulsive differential systems," *Nonlinear Anal. Theor. Method Appl.*, vol. 71, no. 1–2, pp. 45–52, 2009.
- [40] M. Bohner and N. Wintz, "Controllability and observability of time-invariant linear dynamic systems," *Math. Bohem.*, vol. 137, no. 2, pp. 149–163, 2012.
- [41] M. J. Davis, I. A. Gravagne, B. J. Jackson, I. I. Marks, and J. Robert, "Controllability, observability, realizability and stability of dynamic linear systems," *Electron. J. Differ. Equ.*, vol. 2009, no. 37, pp. 1–32, 2009.
- [42] M. Malik and V. Kumar, "Controllability of neutral differential equation with impulses on time scales," *Differ. Equ. Dyn. Equ.*, 2019, <https://doi.org/10.1007/s12591-019-00454-2>.
- [43] M. Malik and V. Kumar, "Existence, stability and controllability results of a Volterra integro-dynamic system with non-instantaneous impulses on time scales," *IMA J. Math. Contr. Inf.*, vol. 37, no. 1, pp. 276–299, 2020.
- [44] V. Lupulescu and A. Younus, "On controllability and observability for a class of linear impulsive dynamic systems on time scales," *Math. Comput. Model.*, vol. 54, no. 5, pp. 1300–1310, 2011.
- [45] M. Sambath, P. Ramesh, and K. Balachandran, "Asymptotic behavior of the fractional order three species prey-predator model," *Int. J. Nonlinear Sci. Numer. Stimul.*, vol. 19, no. 7–8, pp. 721–733, 2018.
- [46] A. E. Matouk and A. A. Elsadany, "Dynamical analysis, stabilization and discretization of a chaotic fractional-order GLV model," *Nonlinear Dynam.*, vol. 85, no. 3, pp. 1597–1612, 2016.
- [47] J. Park, "Multi stability of extinction states in the toy model for three species," *Chaos, Solitons & Fractals*, vol. 114, pp. 92–98, 2018.
- [48] Y. Yan and C. Kou, "Stability analysis for a fractional differential model of HIV infection of CD4+ T-cells with time delay," *Math. Comput. Simulat.*, vol. 82, no. 9, pp. 1572–1585, 2012.
- [49] Y. Ding and H. Ye, "A fractional-order differential equation model of HIV infection of CD4+ T-cells," *Math. Comput. Model.*, vol. 50, no. 3–4, pp. 386–392, 2009.