Yong-Kui Chang*, Rodrigo Ponce and Xu-Sheng Yang

Solvability of fractional differential inclusions with nonlocal initial conditions via resolvent family of operators

<https://doi.org/10.1515/ijnsns-2019-0210>

Received August 20, 2019; accepted June 26, 2020; published online September 18, 2020

Abstract: In this paper, we consider mild solutions to fractional differential inclusions with nonlocal initial conditions. The main results are proved under conditions that (i) the multivalued term takes convex values with compactness of resolvent family of operators; (ii) the multivalued term takes nonconvex values with compactness of resolvent family of operators and (iii) the multivalued term takes nonconvex values without compactness of resolvent family of operators, respectively.

Keywords: fractional differential inclusions; fractional resolvent family; mild solutions; nonlocal initial condition.

Mathematics Subject Classification(2000): 34A08; 34A60; 34K30.

1 Introduction

A differential inclusion is a generalization of the notion of an ordinary differential equation, which is often used to deal with differential equations with a discontinuous right-hand side or an inaccurately known right-hand side [1, 2]. Differential inclusions are also from the control problem, for instance, for a control problem $x' = f(x, u)$, $u \in U$, where u denotes a control parameter. It is founded that the aforementioned control system has the same trajectories as the differential inclusion $x' \in f(x, U) = \bigcup_{u \in U} f(x, u)$. If the set of controls is dependent upon the state x , i.e. $U = U(x)$, then the differential inclusion $x' \in F(x, U(x))$ is also achieved. This equivalence between control systems and differential

inclusions plays a key role in proving existence theorems in optimal control theory. Differential inclusion has wide applications to models in economics, sociology and bioecology etc., and thus, it has been considerably investigated in last decades, see, for instance, [2–8] and references therein.

The concept of nonlocal initial condition has been introduced to extend the study of classical initial-valued problems. As indicated in [9], this notion can be more natural and more precise in describing nature phenomena than the classical notion since some additional information is taken into account. For nonlocal initial conditions of abstract differential inclusions, we can refer to [4, 6, 10, 11] and references therein.

The concept of fractional calculus can be seen a generalization of the ordinary differentiation and integration to arbitrary noninteger order and is regarded as one of the most powerful tools to describe long-memory processes in the last decades. Many phenomena from physics, chemistry, mechanics, electricity etc. can be modelled by ordinary and partial differential equations involving fractional derivatives, and for this, we refer to [12–20] and references therein for more details. We also note that fractional differential inclusions have also been increasingly concerned [21–34].

In a recent paper, some new properties on the compactness of resolvent family of operators related to fractional differential equations have been established [35]. This new characterization of compactness of resolvent family of operators provides a new way to consider mild solutions of abstract fractional differential equations.

Let $(\mathbb{X}, \|\cdot\|)$ be a real Banach space and A be a closed and linear operator defined in Banach space $\mathbb{X}$. Let $\mathcal{P}(\mathbb{X}) = \{Y \subseteq \mathbb{X} : Y \neq \emptyset\}$. The notation $L^1(J, \mathbb{X}) = \{v : J \rightarrow \mathbb{X} | v \text{ is Bochner integrable}\}$ on a compact interval J of $\mathbb{R}$. In this paper, we consider the following abstract fractional differential inclusions with nonlocal initial conditions:

$$D_t^\alpha x(t) \in Ax(t) + J_t^{1-\alpha} F(t, x(t)), \quad t \in J := [0, b] \quad (1.1)$$

$$x(0) = x_0 + p(x), \quad (1.2)$$

*Corresponding author: **Yong-Kui Chang**, School of Mathematics and Statistics, Xidian University, Xi'an 710071, Shaanxi, China, E-mail: lzchangyk@163.com, ykchang@xidian.edu.cn. <https://orcid.org/0000-0001-5500-826X>

Rodrigo Ponce, Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca Chile, E-mail: rponce@inst-mat.utalca.cl

Xu-Sheng Yang, Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, Gansu, China, E-mail: lzyangxs@163.com

where $0 < \alpha < 1$, $J_t^\beta v(t) = \int_0^t \mathcal{G}_\beta(t-s)v(s)ds$ for $v \in L^1(J, \mathbb{X})$, $\mathcal{G}_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $\beta > 0, t > 0$ and $\Gamma(\cdot)$ stands for the Gamma function and

$$D_t^\alpha x(t) \in Ax(t) + F(t, x(t)), \quad t \in J \quad (1.3)$$

$$x(0) = x_0 + p(x), \quad x'(0) = x_1 + q(x) \quad (1.4)$$

where $1 < \alpha < 2$, D_t^α is understood in Caputo sense, $x_0, x_1 \in \mathbb{X}$, $F: J \times \mathbb{X} \rightarrow \mathcal{P}(\mathbb{X})$, p, q are suitable continuous functions specified later.

We shall establish existence results of mild solutions to the above problems (1.1)–(1.2) and (1.3)–(1.4) under different cases: (i) the multivalued term F takes convex values with compactness of resolvent family of operators; (ii) the multivalued term F takes nonconvex values with compactness of resolvent family of operators and (iii) the multivalued term F has nonconvex values without compactness of resolvent family of operators.

The rest of this paper is organized as follows. Section 2 is preliminaries. Section 3 is devoted to investigate mild solutions to problems (1.1)–(1.2) and (1.3)–(1.4). And Section 4 is conclusions.

2 Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. We denote $\mathcal{P}_{cl}(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{ closed}\}$, $\mathcal{P}_b(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{ compact}\}$ and $\mathcal{P}_{cv}(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{ convex}\}$. We also denote by $\mathcal{L}(\mathbb{X})$ the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$.

A multivalued map $G: \mathbb{X} \rightarrow \mathcal{P}(\mathbb{X})$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in \mathbb{X}$. G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in $\mathbb{X}$ for all $B \in \mathcal{P}_b(\mathbb{X})$, i.e. $\sup_{x \in B} \{\sup\{y : y \in G(x)\}\} < \infty$.

The multivalued map $G: \mathbb{X} \rightarrow \mathcal{P}(\mathbb{X})$ is called upper semicontinuous (u.s.c.) on $\mathbb{X}$ if for each $x_0 \in \mathbb{X}$, the set $G(x_0)$ is a nonempty, closed subset of $\mathbb{X}$, and if for each open set $\mathbb{N}$ of $\mathbb{X}$ containing $G(x_0)$, there exists an open neighbourhood $\mathbb{N}_0$ of x_0 such that $G(\mathbb{N}_0) \subseteq \mathbb{N}$. G is called lower semicontinuous (l.s.c.) if the set $\{x \in \mathbb{X} : G(x) \cap \mathcal{A}\}$ is open for any open subset $\mathcal{A} \subseteq \mathbb{X}$. Also, G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in \mathcal{P}_b(\mathbb{X})$. G has a fixed point if there exists $x \in \mathbb{X}$ such that $x \in G(x)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e.,

$$x_n \rightarrow x_*, y_n \rightarrow y_*, y_n \in G(x_n) \text{ imply } y_* \in G(x_*).$$

The u.s.c. multivalued map G is said to be condensing if for any $B \in \mathcal{P}_b(\mathbb{X})$ with $v(B) \neq 0$, we have $v(G(B)) < v(B)$, where v denotes the Kuratowski measure of noncompactness.

Definition 2.1. The multivalued map $G: J \times \mathbb{X} \rightarrow \mathcal{P}(\mathbb{X})$ is said to be L^1 -Carathéodory if

- (i) $t \mapsto G(t, x)$ is measurable for each $x \in \mathbb{X}$;
- (ii) $u \mapsto G(t, u)$ is u.s.c. on $\mathbb{X}$ for almost all $t \in J$;
- (iii) For each $r > 0$, there exists $\varphi_r \in L^1(J, \mathbb{R}_+)$ such that

$$\|G(t, x)\|_{\mathcal{P}(\mathbb{X})} = \sup\{\|v\| : v \in G(t, x)\} \leq \varphi_r(t),$$

for all $\|x\| \leq r$ and for a.e. $t \in J$.

Lemma 2.1. Let $\mathbb{X}$ be a Banach space. Let $G: J \times \mathbb{X} \rightarrow \mathcal{P}_{cp, cv}(\mathbb{X})$ be an L^1 -Carathéodory multivalued map with

$$S_{G,x} = \{f \in L^1(J, \mathbb{X}) : f(t) \in G(t, x(t)), \text{ for a.e. } t \in J\} \neq \emptyset,$$

and let Γ be a linear continuous mapping from $L^1(J, \mathbb{X})$ to $C(J, \mathbb{X})$, then the operator

$$\Gamma \circ S_G : C(J, \mathbb{X}) \rightarrow \mathcal{P}_{cp, cv}(C(J, \mathbb{X})), x \mapsto (\Gamma \circ S_G)(x) := \Gamma(S_{G,x})$$

is a closed graph operator in $C(J, \mathbb{X}) \times C(J, \mathbb{X})$.

Let $\mathcal{A}$ be a subset of $J \times B$. $\mathcal{A}$ is $\mathcal{L} \times \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ -algebra generated by all sets of the form $\mathcal{N} \times \mathcal{D}$, where $\mathcal{N}$ is Lebesgue measurable in J and $\mathcal{D}$ is Borel measurable in B . A subset $\mathcal{A}$ of $L^1(J, \mathbb{X})$ is decomposable if, for all $u, v \in \mathcal{A}$ and all measurable subsets $\mathcal{N}$ of J , the function $u\chi_{\mathcal{N}} + v\chi_{J-\mathcal{N}} \in \mathcal{A}$, where χ denotes the characteristic function.

Let $F: J \times \mathbb{X} \rightarrow \mathcal{P}_{cp}(\mathbb{X})$. Assign to F the multivalued operator

$$\mathcal{F}: C(J, \mathbb{X}) \rightarrow \mathcal{P}(L^1(J, \mathbb{X}))$$

by letting

$$\mathcal{F}(x) = S_{F,x} = \{v \in L^1(J, \mathbb{X}) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in J\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated to F .

Definition 2.2. [4] Let $\mathbb{Y}$ be a separable metric space and let $N: \mathbb{Y} \rightarrow \mathcal{P}(L^1(J, \mathbb{X}))$ be a multivalued operator. We say that N has property (BC) if

- (1) N is l.s.c.;
- (2) N has nonempty closed and decomposable values.

Definition 2.3. [4] Let $F: J \times \mathbb{X} \rightarrow \mathcal{P}_{cp}(\mathbb{X})$. F is called to be of lower semicontinuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is l.s.c. and has nonempty closed and decomposable values.

Lemma 2.2. [36] Let $\mathbb{Y}$ be a separable metric space and let $N: \mathbb{Y} \rightarrow \mathcal{P}(L^1(J, \mathbb{X}))$ be a multivalued operator with property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $f: \mathbb{Y} \rightarrow L^1(J, \mathbb{X})$ such that $f(x) \in N(x)$ for every $x \in \mathbb{Y}$.

Let $(\mathbb{X}, d)$ be a metric space induced by the normed space $(\mathbb{X}, \|\cdot\|)$. Let $H_d: \mathcal{P}(\mathbb{X}) \times \mathcal{P}(\mathbb{X}) \rightarrow \mathbb{R}_+ \cup \{\infty\}$ be defined as

$$H_d(C, D) = \max \left\{ \sup_{c \in C} d(c, D), \sup_{d \in D} d(C, d) \right\},$$

where $d(c, D) = \inf_{d \in D} d(c, d)$, $d(C, d) = \inf_{c \in C} d(c, d)$. Then $(\mathcal{P}_{b,cl}(\mathbb{X}), H_d)$ is a metric space and $(\mathcal{P}_{cl}(\mathbb{X}), H_d)$ is a generalized (complete) metric space.

Definition 2.4. [4] A multivalued operator $G: \mathbb{X} \rightarrow \mathcal{P}_{cl}(\mathbb{X})$ is called

(i) γ -Lipschitz if there exists $\gamma > 0$ such that

$$H_d(G(x), G(y)) \leq \gamma d(x, y), \quad \text{for each } x, y \in \mathbb{X};$$

(ii) a contraction if it is γ -Lipschitz with $\gamma < 1$.

For more detailed results on multivalued maps and differential inclusions, we refer to [1, 2, 4, 7, 8]. We now give some important properties of resolvent family of operators.

Definition 2.5. [20] Let $\alpha > 0$. The α -order Caputo fractional derivative of v is defined as follows:

$$D_t^\alpha v(t) := \int_0^t \mathcal{G}_{m-\alpha}(t-s) v^{(m)}(s) ds,$$

where $m = [\alpha]$.

Definition 2.6. [35] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space $\mathbb{X}$ and $\alpha > 0$. We call A the generator of an $(\alpha, 1)$ -resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha: \mathbb{R}_+ \rightarrow \mathcal{L}(\mathbb{X})$ such that $\{\lambda^\alpha: \operatorname{Re} \lambda > \omega\} \subseteq \rho(A)$ and

$$\lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad \operatorname{Re} \lambda > \omega, x \in \mathbb{X}.$$

In this case, the family $\{S_\alpha(t)\}_{t \geq 0}$ is called an $(\alpha, 1)$ -resolvent family generated by A .

Definition 2.7. [35] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space $\mathbb{X}$ and $1 \leq \alpha \leq 2$. We say that A is the generator of an (α, α) -resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $R_\alpha: \mathbb{R}_+ \rightarrow \mathcal{L}(\mathbb{X})$ such that $\{\lambda^\alpha: \operatorname{Re} \lambda > \omega\} \subseteq \rho(A)$ and

$$(\lambda^\alpha - A)^{-1} x = \int_0^\infty e^{-\lambda t} R_\alpha(t) x dt, \quad \operatorname{Re} \lambda > \omega, x \in \mathbb{X}.$$

In this case, the family $\{R_\alpha(t)\}_{t \geq 0}$ is called an (α, α) -resolvent family generated by A .

Recall that a strongly continuous family $\{T(t)\}_{t \geq 0} \subseteq \mathcal{L}(\mathbb{X})$ is said to be of type (M, ω) if there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq M e^{\omega t}$ for all $t \geq 0$.

Lemma 2.3. [35, Theorem 3.1] Let $0 < \alpha \leq 1$ and $\{S_\alpha(t)\}_{t \geq 0}$ be an $(\alpha, 1)$ -resolvent family of type (M, ω) generated by A . Suppose that $S_\alpha(t)$ is continuous in the uniform operator topology for all $t > 0$, then the following assertions are equivalent:

- (i) $S_\alpha(t)$ is a compact operator for all $t > 0$.
- (ii) $(\mu - A)^{-1}$ is a compact operator for all $\mu > \omega^{\frac{1}{\alpha}}$.

Lemma 2.4. [35, Theorem 3.5] Let $1 < \alpha \leq 2$ and A be the generator of an $(\alpha, 1)$ -resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ of type (M, ω) . Then A generates an (α, α) -resolvent family $\{R_\alpha(t)\}_{t \geq 0}$ of type $(\frac{M}{\alpha-1}, \omega)$ and the following assertions are equivalent:

- (i) $R_\alpha(t)$ is a compact operator for all $t > 0$.
- (ii) $(\mu - A)^{-1}$ is a compact operator for all $\mu > \omega^{\frac{1}{\alpha}}$.

Next, we list some well-known fixed point theorems.

Lemma 2.5. [1] Let Ξ be a bounded, convex and closed subsets of a Banach space $\mathbb{X}$ and let $Y: \Xi \rightarrow \Xi$ be a condensing map. Then, Y has a fixed point in Ξ .

Lemma 2.6. [1] Let Ξ be a bounded and convex set in Banach space $\mathbb{X}$. $Y: \Xi \rightarrow \mathcal{P}(\Xi)$ is an u.s.c., condensing multivalued map. If for every $x \in \Xi$, $Y(x)$ is a closed and convex set in Ξ , then Y has a fixed point in Ξ .

Lemma 2.7. (see [4, Theorem 1.11]) Let $(\mathbb{X}, d)$ be a metric space. If $G: \mathbb{X} \rightarrow \mathcal{P}_{cl}(\mathbb{X})$ is a contraction, then $\operatorname{Fix}(G) \neq \emptyset$, where $\operatorname{Fix}(G)$ denotes the fixed point set of G .

3 Existence results

In this section, we shall investigate some existence results for mild solutions to Eqs. (1.1)–(1.2) and Eqs. (1.3)–(1.4). We shall prove our main results under conditions that (i) the multivalued term takes convex values with compactness of resolvent family of operators; (ii) the multivalued term takes nonconvex values with compactness of resolvent family of operators and (iii) the multivalued term takes nonconvex values without compactness of resolvent family of operators.

For the problem (1.1)–(1.2), according to [35], we have the following definition.

Definition 3.1. Let A be the generator of an $(\alpha, 1)$ -resolvent family $S_\alpha(t)$; the mild solutions of the problem (1.1)–(1.2) are defined as follows:

$$x(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v(s)ds, v \in S_{F,x}, t \in J.$$

We list the following assumptions:

- (A1) A generates an $(\alpha, 1)$ -resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ of type (M, ω) . $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega$, and $S_\alpha(t)$ is continuous in the uniform operator topology for all $t > 0$.
- (A2) $F: J \times \mathbb{X} \rightarrow \mathcal{P}_{cp, cv}(\mathbb{X})$ satisfies the following conditions:
- (a) For a.e. $t \in J$, $F(t, \cdot)$ is u.s.c, and for each $x \in \mathbb{X}$, $F(\cdot, x)$ is measurable. And for each $x \in C(J, \mathbb{X})$, SF, x is nonempty;
- (b) There exists a function $\phi \in L^1(J, \mathbb{R}_+)$ such that

$$F(t, x)_p \leq \phi(t)x, \quad \forall t \in J, x \in \mathbb{X}.$$

- (A3) $p: C(J, \mathbb{X}) \rightarrow C(J, \mathbb{X})$ is continuous and there exists $L_p > 0$ such that

$$\|p(x) - p(y)\| < L_p \|x - y\|, \quad \forall x, y \in C(J, \mathbb{X}).$$

Remark 3.1. (i) Of concern, for useful criteria for the continuity of $S_\alpha(t)$ in the uniform operator topology, one can refer to the work [37]. For instance, this property holds true for the class of analytic resolvent.

(ii) According to Lemma 2.3, the condition (A1) implies $S_\alpha(t)$ is compact for all $t > 0$.

Theorem 3.1. If conditions (A1)–(A3) hold, then the problem (1.1)–(1.2) admits at least one mild solution on J provided that

$$\tilde{M}(L_p + \|\phi\|_{L^1}) < 1, \quad (3.1)$$

where $\tilde{M} = \max\{M, Me^{\omega b}\}$.

Proof. Consider the operator $N: C(J, \mathbb{X}) \rightarrow \mathcal{P}(C(J, \mathbb{X}))$ defined by

$$N(x) = \left\{ h \in C(J, \mathbb{X}) : h(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v(s)ds, t \in J \right\},$$

where $v \in S_{F,x}$. Clearly, the fixed points of N are mild solutions to (1.1)–(1.2). We shall show that N satisfies all the hypothesis of Lemma 2.6. The proof will be given in several steps.

Step 1. There exists a positive number r such that $N(B_r) \subseteq B_r$, where $B_r = \{x \in C(J, \mathbb{X}) : \|x\|_\infty \leq r\}$. If it is not true, then for each positive number r , there exists a function x^r such that $h^r \in N(x^r)$ but $\|h^r(t)\| > r$ for some $t \in J$,

$$h^r(t) = S_\alpha(t)[x_0 + p(x^r)] + \int_0^t S_\alpha(t-s)v^r(s)ds,$$

where $v^r \in S_{F,x^r}$. However, on the other hand, we have

$$\begin{aligned} r &< \|S_\alpha(t)[x_0 + p(x^r)] + \int_0^t S_\alpha(t-s)v^r(s)ds\| \\ &\leq \tilde{M}(\|x_0\| + \|p(x^r)\|) + \tilde{M} \int_0^t \|\phi(s)\| \|x^r\| ds \\ &\leq \tilde{M}\|x_0\| + \tilde{M}(L_p\|x^r\| + p(0)) + \tilde{M}\phi_{L^1}r \\ &\leq \tilde{M}(x_0 + \|p(0)\|) + \tilde{M}(L_p + \|\phi\|_{L^1})r. \end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \rightarrow \infty$, we obtain the following equation:

$$1 \leq \tilde{M}(L_p + \phi_{L^1}),$$

which contradicts the relation (3.1).

Step 2. $N(x)$ is convex for each $x \in C(J, \mathbb{X})$.

Indeed, if $h_1, h_2 \in N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$h_i(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v_i(s)ds, i = 1, 2.$$

Let $\theta \in (0, 1)$. Then for each $t \in J$, we have

$$\begin{aligned} (\theta h_1 + (1-\theta)h_2)(t) &= S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)[\theta v_1(s) \\ &\quad + (1-\theta)v_2(s)]ds. \end{aligned}$$

Because $S_{F,x}$ is convex (since F has convex values), $\theta h_1 + (1-\theta)h_2 \in N(x)$.

Step 3. $N(x)$ is closed for each $x \in C(J, \mathbb{X})$.

Let $\{h_n\}_{n \geq 0} \in N(x)$ such that $h_n \rightarrow h$ in $C(J, \mathbb{X})$. Then $h \in C(J, \mathbb{X})$ and there exist $\{v_n\} \in S_{F,x}$ such that for each $t \in J$

$$h_n(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v_n(s)ds.$$

Due to the fact that F has compact values, we may pass to a subsequence if necessary to get that v_n converges to v in $L^1(J, X)$ and hence $v \in S_{F,x}$. Then for each $t \in J$,

$$h_n(t) \rightarrow h(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v(s)ds.$$

Thus, $h \in N(x)$.

Step 4. N is u.s.c. and condensing.

Now, we decompose N as $N_1 + N_2$ as

$$(N_1 x)(t) = S_\alpha(t)[x_0 + p(x)]$$

$$N_2(x) = \left\{ m \in C(J, \mathbb{X}) : m(t) = \int_0^t S_\alpha(t-s)v(s)ds, t \in J \right\}.$$

We only need to prove that N_1 is a contraction and N_2 is completely continuous.

To show that N_1 is a contraction, for arbitrary $x_1, x_2 \in B_r$ and each $t \in J$, we have from (A3)

$$\|N_1(x_1)(t) - N_1(x_2)(t)\| = \|S_\alpha(t)[p(x_1) - p(x_2)]\|$$

$$\leq \tilde{M}L_p \|x_1 - x_2\|_\infty.$$

Thus

$$\|N_1(x_1) - N_1(x_2)\|_\infty \leq \tilde{M}L_p \|x_1 - x_2\|_\infty.$$

From the relation (3.1), we conclude that N_1 is a contraction.

Next, we show that N_2 is u.s.c. and condensing.

(i) $N_2(B_r)$ is obviously bounded.

(ii) $N_2(B_r)$ is equicontinuous.

Indeed, let $x \in B_r$, $m \in N_2(x)$ and take $t_1, t_2 \in J$ with $t_2 < t_1$. Then there exists a selection $v \in S_{F,x}$ such that

$$\|m(t)\| = \int_0^t \|S_\alpha(t-s)v(s)\| ds, \quad t \in J.$$

Then,

$$\|m(t_1) - m(t_2)\| \leq \int_{t_2}^{t_1} \|S_\alpha(t_1-s)v(s)\| ds + \int_0^{t_2} \|[S_\alpha(t_1-s) - S_\alpha(t_2-s)]v(s)\| ds$$

$$= I_1 + I_2.$$

For the term I_1 , as $t_1 \rightarrow t_2$, we have

$$I_1 \leq \int_{t_2}^{t_1} \tilde{M}\phi(s) \|x(s)\| ds \leq \tilde{M}r \int_{t_2}^{t_1} \phi(s) ds \rightarrow 0.$$

Next for the term I_2 , we have

$$I_2 \leq \int_0^{t_2} \|[S_\alpha(t_1-s) - S_\alpha(t_2-s)]\| \|v(s)\| ds$$

$$\leq \int_0^{t_2} \|[S_\alpha(t_1-s) - S_\alpha(t_2-s)]\| \phi(s) \|x(s)\| ds$$

$$\leq r \int_0^{t_2} \|[S_\alpha(t_1-s) - S_\alpha(t_2-s)]\| \phi(s) ds.$$

Now take into account that

$$\|[S_\alpha(t_1 - \cdot) - S_\alpha(t_2 - \cdot)]\| \phi(s) \leq 2\tilde{M}\phi(s) \in L^1(J, \mathbb{R}_+),$$

and $S_\alpha(t_1 - s) - S_\alpha(t_2 - s) \rightarrow 0$ in $\mathcal{L}(\mathbb{X})$, as $t_1 \rightarrow t_2$ (see (A1)). By the Lebesgue's dominated convergence theorem, $I_2 \rightarrow 0$

as $t_1 \rightarrow t_2$.

(iii) $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$.

For $t = 0$, the conclusion obviously holds. Let $0 < t \leq b$ and ε be a real number satisfying $0 < \varepsilon < t$. For $x \in B_r$ and $v \in S_{F,x}$ such that

$$m(t) = \int_0^t S_\alpha(t-s)v(s)ds, \quad t \in J.$$

Define

$$m_\varepsilon(t) = \int_0^{t-\varepsilon} S_\alpha(t-s)v(s)ds, \quad t \in J.$$

In view of (A1) and Lemma 2.3, we have $S_\alpha(t)$ which is compact for $t > 0$. Therefore, the set $\mathcal{K} = \{S_\alpha(t-s)v(s), 0 \leq s \leq t-\varepsilon\}$ is relatively compact. Then, $\overline{\text{conv } \mathcal{K}}$ is compact. Considering $m_\varepsilon(t) \in \overline{\text{conv } \mathcal{K}}$ for all $t \in J$, the set $V_\varepsilon(t) = \{m_\varepsilon(t) : m_\varepsilon(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$ for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for $m \in N(B_r)$,

$$\|m(t) - m_\varepsilon(t)\| \leq \left\| \int_{t-\varepsilon}^t S_\alpha(t-s)v(s)ds \right\|$$

$$\leq \tilde{M}r \int_{t-\varepsilon}^t \phi(s)ds.$$

Therefore, let $\varepsilon \rightarrow 0$, we see that there are relatively compact sets arbitrarily close to the set $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$. Hence, the set $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$.

As a consequence of the above steps and the Arzela-Ascoli theorem, we can deduce that N_2 is completely continuous.

(iv) N_2 has a closed graph.

Let $x_n \rightarrow x_*$, $m_n \in N_2(x_n)$ and $m_n \rightarrow m_*$. We shall show that $m_* \in N_2(x_*)$. Now $m_n \in N_2(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that

$$m_n(t) = \int_0^t S_\alpha(t-s)v_n(s)ds, \quad t \in J.$$

We must prove that there exists $v_* \in S_{F,x_*}$ such that

$$m_*(t) = \int_0^t S_\alpha(t-s)v_*(s)ds, \quad t \in J.$$

Consider the linear continuous operator defined by

$$\Gamma : L^1(J, \mathbb{X}) \rightarrow C(J, \mathbb{X}) \quad v \mapsto (\Gamma v)(t) = \int_0^t R_\alpha(t-s)v(s)ds.$$

From Lemma 2.1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have $m_n(t) \in \Gamma(S_{F,x_n})$.

Since $x_n \rightarrow x_*$ and $m_n \rightarrow m_*$, it follows again from Lemma 2.1 that $m_*(t) \in \Gamma(S_{F,x_*})$. That is, there must exist $v_* \in S_{F,x_*}$ such that

$$m_*(t) = \int_0^t S_\alpha(t-s)v_*(s)ds, \quad t \in J.$$

Therefore, N_2 is u.s.c. On the other hand, N_1 is a contraction, hence $N = N_1 + N_2$ is u.s.c. and condensing. By the fixed point theorem Lemma 2.6, there exists a fixed point $x(\cdot)$ for N on B_r . Thus, the problem (1.1)–(1.2) admits a mild solution. $\square$

Replace the condition (A2)(b) by (b'). There exists a constant $\tau \in (0, 1)$ and a function $\phi \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, x)\|_{\mathcal{P}} \leq \phi(t)(\|x\|)^\tau, \quad \forall t \in J, x \in C(J, \mathbb{X}).$$

From the above proof of Theorem 3.1, we can obtain the following result.

Corollary 3.1. If conditions (A1)–(A2)(a) and (A2)(b')–(A3) hold, then the problem (1.1)–(1.2) admits at least one mild solution on J provided that

$$\tilde{M}L_p < 1. \quad (3.2)$$

For the problem (1.3)–(1.4), we first consider the following equation:

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t) + v(t), \quad t \in J, \\ x(0) &= x_0, \quad x'(0) = x_1, \end{aligned}$$

where $1 < \alpha < 2, v \in L^1(J, \mathbb{X})$. By Laplace transform, we have

$$\lambda^\alpha \hat{x}(\lambda) - \lambda^{\alpha-1}x(0) - \lambda^{\alpha-2}x'(0) = A\hat{x}(\lambda) + \hat{v}(\lambda).$$

that is

$$\begin{aligned} \hat{x}(\lambda) &= \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x_0 + \lambda^{\alpha-2}(\lambda^\alpha - A)^{-1}x_1 + (\lambda^\alpha - A)^{-1}\hat{v}(\lambda) \\ &= \widehat{S_\alpha}(\lambda)x_0 + (\widehat{G_1 * S_\alpha})(\lambda)x_1 + (\widehat{R_\alpha * v})(\lambda). \end{aligned}$$

Thus, we have

$$x(t) = S_\alpha(t)x_0 + \int_0^t S_\alpha(\theta)x_1 d\theta + \int_0^t R_\alpha(t-s)v(s)ds.$$

Now, we can give the following definition.

Definition 3.2. Let $1 < \alpha < 2$ and A be the generator of an $(\alpha, 1)$ -resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ of type (M, ω) . Then A generates an (α, α) -resolvent family $\{R_\alpha(t)\}_{t \geq 0}$ of type

$(\frac{M}{\alpha-1}, \omega)$ and the mild solution of the problem (1.3)–(1.4) can be given as follows:

$$\begin{aligned} x(t) &= S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta \\ &\quad + \int_0^t R_\alpha(t-s)v(s)ds, \quad v \in S_{F,x}, t \in J. \end{aligned}$$

Let us list the following basic assumptions:

(A4) Let $1 < \alpha < 2$ and A generates an $(\alpha, 1)$ -resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ of type (M, ω) . $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega$.

(A5) $q : C(J, \mathbb{X}) \rightarrow C(J, \mathbb{X})$ is continuous and there exists $L_q > 0$ such that

$$\|q(x) - q(y)\| < L_q\|x - y\|, \quad \forall x, y \in C(J, \mathbb{X}).$$

Remark 3.2. If (A4) holds, according to Lemma 2.4, A generates an (α, α) -resolvent family $\{R_\alpha(t)\}_{t \geq 0}$ of type $(\frac{M}{\alpha-1}, \omega)$ and $R_\alpha(t)$ is a compact operator for all $t > 0$. And from the proof of [35, Theorem 3.5], $R_\alpha(t)$ is continuous in the uniform operator topology for all $t > 0$.

Theorem 3.2. If conditions (A2)–(A5) hold, then the problem (1.3)–(1.4) admits at least one mild solution on J provided that

$$\tilde{M}(L_p + bL_q + \|\phi\|_{L^1}) < 1, \quad (3.3)$$

where $\tilde{M} = \max\{\frac{M}{\alpha-1}, \frac{M}{\alpha-1}e^{\omega b}\}$.

Proof. Consider the operator $N : C(J, \mathbb{X}) \rightarrow \mathcal{P}(C(J, \mathbb{X}))$ defined by

$$\begin{aligned} N(x) &= \left\{ h \in C(J, \mathbb{X}) : h(t) = S_\alpha(t)[x_0 + p(x)] \right. \\ &\quad \left. + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta + \int_0^t R_\alpha(t-s)v(s)ds, t \in J \right\} \end{aligned}$$

where $v \in S_{F,x}$. Clearly, the fixed points of N are mild solutions to (1.1)–(1.2). We shall show that N satisfies all the hypothesis of Lemma 2.6. The proof will be given in several steps.

Step 1. There exists a positive number r such that $N(B_r) \subseteq B_r$, where $B_r = \{x \in C(J, \mathbb{X}) : \|x\|_\infty \leq r\}$. If it is not true, then for each positive number r , there exists a function x^r such that $h^r \in N(x^r)$ but $\|h^r(t)\| > r$ for some $t \in J$,

$$\begin{aligned} h^r(t) &= S_\alpha(t)[x_0 + p(x^r)] + \int_0^t S_\alpha(\theta)[x_1 + q(x^r)]d\theta \\ &\quad + \int_0^t R_\alpha(t-s)v^r(s)ds, \end{aligned}$$

where $v^r \in S_{F,x^r}$. However, on the other hand, we have

$$\begin{aligned}
r &< \left\| S_\alpha(t)[x_0 + p(x')] + \int_0^t S_\alpha(\theta)[x_1 + q(x')]d\theta \right\| \\
&\quad + \left| \int_0^t R_\alpha(t-s)v^r(s)ds \right| \\
&\leq \tilde{M}(\|x_0\| + \|p(x')\|) + b\tilde{M}(\|x_1\| + \|q(x')\|) \\
&\quad + \tilde{M} \int_0^t |\phi(s)|\|x\|ds \\
&\leq \tilde{M}\|x_0\| + \tilde{M}(L_p\|x^r\| + \|p(0)\|) + b\tilde{M}\|x_1\| \\
&\quad + b\tilde{M}(L_q\|x^r\| + \|q(0)\|) + \tilde{M}\|\phi\|_{L^1} r \\
&\leq \tilde{M}(x_0 + \|p(0)\| + b\|x_1\| + b\|q(0)\|) \\
&\quad + \tilde{M}(L_p + bL_q + \|\phi\|_{L^1})r.
\end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \rightarrow \infty$, we obtain

$$1 \leq \tilde{M}(L_p + bL_q + \|\phi\|_{L^1}),$$

which contradicts the relation (3.3).

Step 2. $N(x)$ is convex for each $x \in C(J, \mathbb{X})$.

Indeed, if $h_1, h_2 \in N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$\begin{aligned}
h_i(t) &= S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta \\
&\quad + \int_0^t R_\alpha(t-s)v_i(s)ds, \quad i = 1, 2.
\end{aligned}$$

Let $\delta \in (0, 1)$. Then for each $t \in J$, we have

$$\begin{aligned}
(\delta h_1 + (1-\delta)h_2)(t) &= S_\alpha(t)[x_0 + p(x)] \\
&\quad + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta \\
&\quad + \int_0^t R_\alpha(t-s)[\delta v_1(s) + (1-\delta)v_2(s)]ds.
\end{aligned}$$

Because $S_{F,x}$ is convex (since F has convex values), $\delta h_1 + (1-\delta)h_2 \in N(x)$.

Step 3. $N(x)$ is closed for each $x \in C(J, \mathbb{X})$.

Let $\{h_n\}_{n \geq 0} \in N(x)$ such that $h_n \rightarrow h$ in $C(J, \mathbb{X})$. Then $h \in C(J, \mathbb{X})$ and there exist $\{v_n\} \in S_{F,x}$ such that for each $t \in J$

$$\begin{aligned}
h_n(t) &= S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta \\
&\quad + \int_0^t R_\alpha(t-s)v_n(s)ds.
\end{aligned}$$

Due to the fact that F has compact values, we may pass to a subsequence if necessary to get that v_n converges to v in $L^1(J, X)$ and hence $v \in S_{F,x}$. Then for each $t \in J$,

$$\begin{aligned}
h_n(t) &\rightarrow h(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta \\
&\quad + \int_0^t R_\alpha(t-s)v(s)ds.
\end{aligned}$$

Thus, $h \in N(x)$.

Step 4. N is u.s.c. and condensing.

Now, we decompose N as $N_1 + N_2$ as

$$\begin{aligned}
(N_1 x)(t) &= S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta \\
N_2(x) &= \left\{ m \in C(J, \mathbb{X}) : m(t) = \int_0^t R_\alpha(t-s)v(s)ds, t \in J \right\}.
\end{aligned}$$

We only need to prove that N_1 is a contraction and N_2 is completely continuous.

To show that N_1 is a contraction, for arbitrary $x_1, x_2 \in B_r$ and each $t \in J$, we have from (A3) and (A5)

$$\begin{aligned}
\|N_1(x_1)(t) - N_1(x_2)(t)\| &\leq \|S_\alpha(t)[p(x_1) - p(x_2)]\| \\
&\quad + \left\| \int_0^t S_\alpha(\theta)[q(x_1) - q(x_2)]d\theta \right\| \\
&\leq \tilde{M}L_p\|x_1 - x_2\|_\infty + b\tilde{M}L_q\|x_1 - x_2\|_\infty,
\end{aligned}$$

Thus

$$\|N_1(x_1) - N_1(x_2)\|_\infty \leq \tilde{M}(L_p + bL_q)\|x_1 - x_2\|_\infty.$$

From the relation (3.3), we conclude that N_1 is a contraction.

Next, we show that N_2 is u.s.c. and condensing.

(i) $N_2(B_r)$ is obviously bounded.

(ii) $N_2(B_r)$ is equicontinuous.

Indeed, let $x \in B_r$, $m \in N_2(x)$ and take $t_1, t_2 \in J$ with $t_2 < t_1$. Then, there exists a selection $v \in S_{F,x}$ such that

$$m(t) = \int_0^t R_\alpha(t-s)v(s)ds, \quad t \in J.$$

Then

$$\begin{aligned}
\|m(t_1) - m(t_2)\| &\leq \int_{t_2}^{t_1} \|R_\alpha(t_1-s)v(s)\|ds + \int_0^{t_2} \|R_\alpha(t_1-s) \\
&\quad - R_\alpha(t_2-s)\|v(s)\|ds \\
&= I_1 + I_2.
\end{aligned}$$

For the term I_1 , as $t_1 \rightarrow t_2$, we have

$$I_1 \leq \int_{t_2}^{t_1} \tilde{M}\phi(s)\|x(s)\|ds \leq \tilde{M}r \int_{t_2}^{t_1} \phi(s)ds \rightarrow 0.$$

Next for the term I_2 , we have

$$\begin{aligned}
I_2 &\leq \int_0^{t_2} \| [R_\alpha(t_1 - s) - R_\alpha(t_2 - s)] \| \| v(s) \| ds \\
&\leq \int_0^{t_2} \| [R_\alpha(t_1 - s) - R_\alpha(t_2 - s)] \| \phi(s) \| x(s) \| ds \\
&\leq r \int_0^{t_2} \| [R_\alpha(t_1 - s) - R_\alpha(t_2 - s)] \| \phi(s) ds.
\end{aligned}$$

Now take into account that

$$\| [R_\alpha(t_1 - \cdot) - R_\alpha(t_2 - \cdot)] \| \phi(s) \leq 2\tilde{M}\phi(s) \in L^1(J, \mathbb{R}_+),$$

and $R_\alpha(t_1 - s) - R_\alpha(t_2 - s) \rightarrow 0$ in $\mathcal{L}(\mathbb{X})$, as $t_1 \rightarrow t_2$ (see (A4)). By the Lebesgue's dominated convergence theorem, $I_2 \rightarrow 0$ as $t_1 \rightarrow t_2$.

(iii) $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$.

For $t = 0$, the conclusion obviously holds. Let $0 < t \leq b$ and ε be a real number satisfying $0 < \varepsilon < t$. For $x \in B_r$ and $v \in S_{F,x}$ such that

$$m(t) = \int_0^t R_\alpha(t-s)v(s)ds, \quad t \in J.$$

Define

$$m_\varepsilon(t) = \int_0^{t-\varepsilon} R_\alpha(t-s)v(s)ds, \quad t \in J.$$

In view of (A4) and Lemma 2.4, we have $R_\alpha(t)$ which is compact for $t > 0$. Therefore, the set $\mathcal{K} = \{R_\alpha(t-s)v(s), 0 \leq s \leq t-\varepsilon\}$ is relatively compact. Then, $\overline{\text{conv}} \mathcal{K}$ is compact. Considering $m_\varepsilon(t) \in \overline{\text{conv}} \mathcal{K}$ for all $t \in J$, the set $V_\varepsilon(t) = \{m_\varepsilon(t) : m_\varepsilon(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$ for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for $m \in N(B_r)$,

$$\begin{aligned}
m(t) - m_\varepsilon(t) &\leq \left\| \int_{t-\varepsilon}^t R_\alpha(t-s)v(s)ds \right\| \\
&\leq \tilde{M}r \int_{t-\varepsilon}^t \phi(s)ds.
\end{aligned}$$

Therefore, let $\varepsilon \rightarrow 0$, we see that there are relatively compact sets arbitrarily close to the set $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$. Hence, the set $V(t) = \{m(t) : m(t) \in N_2(B_r)\}$ is relatively compact in $\mathbb{X}$.

As a consequence of the above steps and the Arzela–Ascoli theorem, we can deduce that N_2 is completely continuous.

(iv) N_2 has a closed graph.

Let $x_n \rightarrow x_*$, $m_n \in N_2(x_n)$ and $m_n \rightarrow m_*$. We shall show that $m_* \in N_2(x_*)$. Now $m_n \in N_2(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that

$$m_n(t) = \int_0^t R_\alpha(t-s)v_n(s)ds, \quad t \in J.$$

We must prove that there exists $v_* \in S_{F,x_*}$ such that

$$m_*(t) = \int_0^t R_\alpha(t-s)v_*(s)ds, \quad t \in J.$$

Consider the linear continuous operator defined by

$$\Gamma : L^1(J, \mathbb{X}) \rightarrow C(J, \mathbb{X})$$

From Lemma 2.1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have $m_n(t) \in \Gamma(S_{F,x_n})$.

Since $x_n \rightarrow x_*$ and $m_n \rightarrow m_*$, it follows again from Lemma 2.1 that $m_*(t) \in \Gamma(S_{F,x_*})$. That is, there must exist $v_* \in S_{F,x_*}$ such that

$$m_*(t) = \int_0^t R_\alpha(t-s)v_*(s)ds, \quad t \in J.$$

Therefore, N_2 is u.s.c. On the other hand, N_1 is a contraction, hence $N = N_1 + N_2$ is u.s.c. and condensing. By the fixed point theorem Lemma 2.6, there exists a fixed point $x(\cdot)$ for N on B_r . Thus, the problem (1.1)–(1.2) admits a mild solution. $\square$

According to the above proof of Theorem 3.2, we can also have the following result.

Corollary 3.2. If conditions (A2)(a), (A2)(b') and (A3)–(A5) hold, then the problem (1.1)–(1.2) admits at least one mild solution on J provided that

$$\tilde{M}(L_p + bL_q) < 1. \quad (3.4)$$

Next we consider problems (1.1)–(1.2) and (1.3)–(1.4) when the multivalued map F takes nonconvex values with compactness of resolvent family of operators. Let $\mathbb{X}$ be a separable Banach space $\mathbb{X}$. We list the following condition:

(C1) $F : J \times \mathbb{X} \rightarrow \mathcal{P}_{cp}(\mathbb{X})$ satisfies

(I) $(t, x) \mapsto F(t, x)$ is $\mathcal{L} \times \mathcal{B}$ measurable;

(II) $x \mapsto F(t, x)$ is l.s.c for a.e. $t \in J$.

Theorem 3.3. Suppose hypotheses (A1), (A2)(b'), (C1) and (A3) are satisfied, then the problem (1.1)–(1.2) admits at least one mild solution on J if condition (3.1) holds.

Proof. Hypotheses (A2)(b) and (C1) imply that F is of l.s.c. type. In view of Lemma 2.2, there exists a continuous function $f : C(J, \mathbb{X}) \rightarrow L^1(J, \mathbb{X})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C(J, \mathbb{X})$. Now consider the following equation:

$$D_t^\alpha x(t) = Ax(t) + J_t^{1-\alpha} f(x)(t), \quad t \in J \quad (3.5)$$

$$x(0) = x_0 + P(x), \quad (3.6)$$

Notice that if $x \in C(J, \mathbb{X})$ is a solution of the problem (3.5)–(3.6), then x is also a solution of the problem (1.1)–(1.2). Next, we transform the problem (3.5)–(3.6) into a fixed point problem by defining $N : C(J, \mathbb{X}) \rightarrow C(J, \mathbb{X})$ as

$$N(x) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)f(x)(s)ds, \quad t \in J.$$

We shall show that N satisfies all the hypothesis of Lemma 2.5. The proof will be given in several steps.

Step 1. There exists a positive number r such that $N(B_r) \subseteq B_r$, where $B_r = \{x \in C(J, \mathbb{X}) : \|x\|_\infty \leq r\}$.

This can be conducted similarly as **Step 1.** in the proof of Theorem 3.1.

We decompose N as $N_1 + N_2$ as

$$N_1(x)(t) = S_\alpha(t)[x_0 + p(x)]$$

$$N_2(x)(t) = \int_0^t S_\alpha(t-s)f(x)(s)ds.$$

Step 2. N_2 is continuous on B_r .

Let $\{x_n\}$ be a sequence such that $x_n \rightarrow x$ in B_r . Then

$$\begin{aligned} & \|N_2(x_n)(t) - N_2(x)(t)\| \\ & \leq \int_0^t \|S_\alpha(t-s)\| \|f(x_n)(s) - f(x)(s)\| ds \\ & \leq \tilde{M} \int_0^t \phi(s) (\|x_n(s)\| + \|x(s)\|) ds \\ & \leq 2r\tilde{M} \int_0^t \phi(s) ds. \end{aligned}$$

Note that $\phi \in L^1(J, \mathbb{R}_+)$, $\int_0^t \|f(x_n)(s) - f(x)(s)\| ds \rightarrow 0, n \rightarrow \infty$ by the Lebesgue's dominated convergence theorem. Hence, N_2 is continuous.

Step 3. N is condensing.

Similarly conducted as the proof of Theorem 3.1, we can prove that N_1 is a contraction and N_2 is completely continuous.

From the above three steps, we can complete the proof via Lemma 2.5.

Theorem 3.4. Suppose hypotheses (C1), (A2)(b) and (A3)–(A5) are satisfied, then the problem (1.3)–(1.4) admits at least one mild solution on J if condition (3.3) holds.

Proof. Deduced as the proof of Theorem 3.3, we can transform the problem (1.3)–(1.4) into a single-valued problem. We define $N = N_1 + N_2 : C(J, \mathbb{X}) \rightarrow C(J, \mathbb{X})$ as

$$N_1(x)(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta$$

$$N_2(x)(t) = \int_0^t R_\alpha(t-s)f(x)(s)ds.$$

Similarly conducted as the proof of Theorems 3.2 and 3.3, we can prove that N_1 is a contraction and N_2 is completely continuous. Thus, Lemma 2.5 can be applied to complete the proof. $\square$

Similarly, from proofs of Theorems 3.3 and 3.4, we have the following results.

Corollary 3.3. Suppose hypotheses (A1), (A2)(b'), (C1) and (A3) are satisfied, then the problem (1.1)–(1.2) admits at least one mild solution on J if condition (3.2) holds.

Corollary 3.4. Suppose hypotheses (C1), (A2)(b') and (A3)–(A5) are satisfied, then the problem (1.3)–(1.4) admits at least one mild solution on J if condition (3.4) holds.

In the following, we give some results when the multivalued map F has nonconvex values without compactness of resolvent family of operators. Let us list the following assumptions:

(A6) $F : J \times \mathbb{X} \rightarrow \mathcal{P}_{cp}(\mathbb{X})$ satisfies the following conditions:

- (1) $F : J \times \mathbb{X} \rightarrow \mathcal{P}_{cp}(\mathbb{X}) : (\cdot, x) \mapsto F(\cdot, x)$ is measurable for each $x \in \mathbb{X}$;
- (2) There exists a function $l \in L^1(J, \mathbb{R}_+)$ such that

$$H_d(F(t, x_1), F(t, x_2)) \leq l(t)\|x_1 - x_2\|, \quad \text{for a.e. } t \in J, \forall x_1, x_2 \in \mathbb{X},$$

$$d(0, F(t, 0)) \leq l(t), \quad \text{for a.e. } t \in J.$$

Remark 3.3. [4] Owing to (A6)(1), for each $x \in C(J, \mathbb{X})$, F has a measurable selection, thus $S_{F,x} \neq \emptyset$.

Theorem 3.5. Let A be the generator of an $(\alpha, 1)$ -resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ of type (M, ω) . Assume that conditions (A3) and (A6) are satisfied, then the problem (1.1)–(1.2) admits at least one mild solution on J provided that

$$\tilde{M}(L_p + \|l\|_{L^1}) < 1, \quad (3.7)$$

where $\tilde{M} = \max\{M, Me^{\omega b}\}$.

Proof. Transform the problem (1.1)–(1.2) into a fixed point problem. Let the multivalued operator $N : C(J, \mathbb{X}) \rightarrow \mathcal{P}(C(J, \mathbb{X}))$ be defined as in Theorem 3.1. We shall prove that N admits at least one fixed point. We divide the proof into two steps.

Step 1. For each $x \in C(J, \mathbb{X})$, $N(x) \in \mathcal{P}_{cl}(C(J, \mathbb{X}))$.

This can be proved just as Step 3 in the proof of Theorem 3.1.

Step 2. For each $x, \tilde{x} \in C(J, \mathbb{X})$, there exists a constant $0 < \gamma < 1$ such that $H_d(N(x), N(\tilde{x})) \leq \gamma \|x - \tilde{x}\|_\infty$.

Let $x, \tilde{x} \in C(J, \mathbb{X})$ and $h \in N(x)$. Then there exists $v \in S_{F,x}$ such that for each $t \in J$

$$h(t) = S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(t-s)v(s)ds.$$

From (A6)(2), we have

$$H_d(F(t, x(t)), F(t, \tilde{x}(t))) \leq l(t)\|x(t) - \tilde{x}(t)\|.$$

Thus there exists $w \in S_{F,\tilde{x}}$ such that

$$\|v(t) - w(t)\| \leq l(t)\|x(t) - \tilde{x}(t)\|, t \in J.$$

Consider $U: J \rightarrow \mathcal{P}(\mathbb{X})$ defined as

$$W(t) = \{w \in \mathbb{X} : \|v(t) - w(t)\| \leq l(t)\|x(t) - \tilde{x}(t)\|\}.$$

Because $U(t) = W(t) \cap F(t, \tilde{x})$ is measurable (see [38, Proposition III.4]), there exists a function $\tilde{v}(t)$, which is a measurable selection for U . Hence, $\tilde{v}(t) \in F(t, \tilde{x}(t))$ and

$$\|v(t) - \tilde{v}(t)\| \leq l(t)\|x(t) - \tilde{x}(t)\|, t \in J.$$

For each $t \in J$, we now define

$$\tilde{h}(t) = S_\alpha(t)[x_0 + p(\tilde{x})] + \int_0^t S_\alpha(t-s)\tilde{v}(s)ds.$$

Then for each $t \in J$, we have

$$\begin{aligned} \|h(t) - \tilde{h}(t)\| &\leq \|S_\alpha(t)[p(x(t)) - p(\tilde{x}(t))]\| + \int_0^t S_\alpha(t-s)\|v(s) \\ &\quad - \tilde{v}(s)\|ds \leq \tilde{M}L_p\|x - \tilde{x}\|_\infty + \tilde{M} \int_0^t l(s)ds\|x - \tilde{x}\|_\infty \\ &\leq \tilde{M}[L_p + \|l\|_{L^1}]\|x - \tilde{x}\|_\infty. \end{aligned}$$

Thus,

$$\|h - \tilde{h}\|_\infty \leq \tilde{M}[L_p + \|l\|_{L^1}]\|x - \tilde{x}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of $\tilde{x}$ and x , we can obtain

$$H_d(N(x), N(\tilde{x})) \leq \tilde{M}[L_p + \|l\|_{L^1}]\|x - \tilde{x}\|_\infty.$$

Owing to relation (3.7), we conclude that N is a contraction. Thus, by Lemma 2.7, N admits a fixed point, which just is one mild solution to the problem (1.1)–(1.2). $\square$

Theorem 3.6. Let $1 < \alpha < 2$ and A generates an $(\alpha, 1)$ -resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ of type (M, ω) . Suppose that conditions (A3), (A5) and (A6) are satisfied, then the problem (1.3)–(1.4) has at least one mild solution on J provided that

$$\tilde{M}(L_p + bL_q + \|l\|_{L^1}) < 1, \quad (3.8)$$

where $\tilde{M} = \max\left\{\frac{M}{\alpha-1}, \frac{M}{\alpha-1}e^{\omega b}\right\}$.

Proof. Transform the problem (1.3)–(1.4) into a fixed point problem. Let the multivalued operator $N: C(J, \mathbb{X}) \rightarrow \mathcal{P}(C(J, \mathbb{X}))$ be defined as in Theorem 3.2. We shall prove that N admits at least one fixed point. We divide the proof into two steps.

Step 1. For each $x \in C(J, \mathbb{X})$, $N(x) \in \mathcal{P}_{cl}(C(J, \mathbb{X}))$.

This can be proved just as Step 3 in the proof of Theorem 3.2.

Step 2. N is a contraction.

Let $x, \tilde{x} \in C(J, \mathbb{X})$ and $h \in N(x)$. Then, there exists $v \in S_{F,x}$ such that for each $t \in J$

$$\begin{aligned} h(t) &= S_\alpha(t)[x_0 + p(x)] + \int_0^t S_\alpha(\theta)[x_1 + q(x)]d\theta \\ &\quad + \int_0^t R_\alpha(t-s)v(s)ds. \end{aligned}$$

From (A6)(2), we have

$$H_d(F(t, x(t)), F(t, \tilde{x}(t))) \leq l(t)\|x(t) - \tilde{x}(t)\|.$$

Thus, there exists $w \in S_{F,\tilde{x}}$ such that

$$\|v(t) - w(t)\| \leq l(t)\|x(t) - \tilde{x}(t)\|, t \in J.$$

Consider $U: J \rightarrow \mathcal{P}(\mathbb{X})$ defined as

$$W(t) = \{w \in \mathbb{X} : \|v(t) - w(t)\| \leq l(t)\|x(t) - \tilde{x}(t)\|\}.$$

Because $U(t) = W(t) \cap F(t, \tilde{x})$ is measurable (see [38, Proposition III.4]), there exists a function $\tilde{v}(t)$, which is a measurable selection for U . Hence, $\tilde{v}(t) \in F(t, \tilde{x}(t))$ and

$$\|v(t) - \tilde{v}(t)\| \leq l(t)\|x(t) - \tilde{x}(t)\|, t \in J.$$

For each $t \in J$, we now define

$$\begin{aligned} \tilde{h}(t) &= S_\alpha(t)[x_0 + p(\tilde{x})] + \int_0^t S_\alpha(\theta)[x_1 + q(\tilde{x})]d\theta \\ &\quad \times \int_0^t R_\alpha(t-s)\tilde{v}(s)ds. \end{aligned}$$

Then for each $t \in J$, we have

$$\begin{aligned} \|h(t) - \tilde{h}(t)\| &\leq \|S_\alpha(t)[p(x(t)) - p(\tilde{x})(t)]\| \\ &\quad + \left\| \int_0^t S_\alpha(\theta)[q(x(t)) - q(\tilde{x}(t))]d\theta \right\| \\ &\quad + \left\| \int_0^t R_\alpha(t-s)[v(s) - \tilde{v}(s)]ds \right\| \\ &\leq \tilde{M}L_p\|x - \tilde{x}\|_\infty + b\tilde{M}L_q\|x - \tilde{x}\|_\infty \\ &\quad + \tilde{M} \int_0^t l(s)ds\|x - \tilde{x}\|_\infty \\ &\leq \tilde{M}[L_p + bL_q + \|l\|_{L^1}]\|x - \tilde{x}\|_\infty. \end{aligned}$$

Thus,

$$\|h - \tilde{h}\|_\infty \leq \tilde{M}[L_p + bL_q + \|l\|_{L^1}]\|x - \tilde{x}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of $\tilde{x}$ and x , we can obtain

$$H_d(N(x), N(\tilde{x})) \leq \tilde{M}[L_p + bL_q + \|l\|_{L^1}]\|x - \tilde{x}\|_\infty.$$

Owing to relation (3.8), we conclude that N is a contraction. Thus, by Lemma 2.7, N admits a fixed point, which just is one mild solution to the problem (1.3)–(1.4). $\square$

Example 3.1. As a simple application, we consider the following equations:

$$D_t^\alpha u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + g(t, u(t, x)), \quad (t, x) \in [0, 1] \times [0, \pi], \quad (3.9)$$

$$u(t, 0) = u(t, \pi) = 0, \quad \dot{u}_x(t, 0) = \dot{u}_x(t, \pi) = 0, \quad t \in [0, 1], \quad (3.10)$$

$$u(0, x) = \sum_{k=1}^n a_k u(t, x) + u_0(x), \quad u'(0, x) = \sum_{k=1}^n b_k u'(t, x) + u_1(x), \quad (3.11)$$

where $1 < \alpha < 2$, $a_k, b_k \in \mathbb{R}$, $n \in \mathbb{N}$. Let $\mathbb{X} = L^2([0, \pi])$ and consider the operator $A: D(A) \subset \mathbb{X} \rightarrow \mathbb{X}$ defined by $D(A) = \{u \in \mathbb{X} : u \in H^2([0, \pi]), u(0) = u(\pi)\}$ and for $u \in D(A)$, $Au = \frac{\partial^2 u}{\partial x^2}$. Define the functions $g_j: [0, 1] \times D(A) \rightarrow \mathbb{X}$ and $p, q: D(A) \rightarrow \mathbb{X}$ by

$$g_j(t, u(t, x)) = \frac{e^{-t}u(t, x)}{(6j+t)(1+u(t, x))}, \quad j = 1, 2,$$

$$p(u)(x) = \sum_{k=1}^n a_k u(t, x), \quad q(u)(x) = \sum_{k=1}^n b_k u'(t, x).$$

It is well known that A generates a compact and analytic (and hence norm continuous for all $t > 0$) C_0 -semigroup $\{T(t)\}_{t \geq 0}$ on $\mathbb{X}$ such that $\|T(t)\| \leq 1$. Now, we can extract an (α, α) -resolvent family $\{R_\alpha(t)\}_{t \geq 0}$ of type $(1, 1)$ (see [39]). Meanwhile, the compactness of $T(t)$ implies that $(\lambda^\alpha - A)^{-1}$ is compact.

Let $F = \{g_1, g_2\}$, $J = [0, 1]$. We note that the above problem (3.9)–(3.11) can be rewritten in the abstract form (1.3)–(1.4). Furthermore, we assume that $\sum_{k=1}^n |a_k| \leq \frac{1}{6}$, $\sum_{k=1}^n |b_k| \leq \frac{1}{6}$. We also observe that in this case

$$\phi(t) = \frac{e^{-t}}{6+t}, \quad b = \tilde{M} = 1, \quad \text{and } L_p = \frac{1}{6}, L_q = \frac{1}{6}, \quad \|\phi\|_{L^1} \leq \frac{1}{6}.$$

According to Theorem 3.2, the problem (3.9)–(3.11) has at least one mild solution on J .

4 Conclusions

In this paper, we establish some sufficient conditions to guarantee the existence of mild solutions to abstract fractional differential inclusions with nonlocal initial conditions under conditions that (i) the multivalued term takes convex values with compactness of resolvent family of operators; (ii) the multivalued term takes nonconvex values with compactness of resolvent family of operators and (iii) the multivalued term takes nonconvex values without compactness of resolvent family of operators. The main results are based upon theories of resolvent family of operators, multivalued analysis and fixed point approach. It is noted that several partial differential equations arising in physics and applied sciences can be described by fractional differential equations of degenerate type (cf. [40, 41]); we propose to investigate the existence of solutions to fractional stochastic equations of degenerate type via the resolvent family in future works.

Acknowledgments: The second author was partially supported by Fondecyt 11130619.

Author contribution: All the authors have accepted responsibility for the entire content of this submitted manuscript and approved submission.

Research funding: None declared.

Conflict of interest statement: The authors declare no conflicts of interest regarding this article.

References

- [1] K. Deimling, *Multivalued Differential Equations*, Berlin, Walter de Gruyter, 1992.
- [2] A. A. Tolstonogov, *Differential Inclusions in a Banach Space*, Dordrecht, Kluwer Academic, 2000.
- [3] J. Barrios, A. Piétras, A. Marrero, and H. Arazoza, “HIV model described by differential inclusions,” in IWANN 2009, Part I, LNCS 5517, J. Cabestany, et al., Eds., 2009, pp. 909–916.

- [4] M. Benchohra, J. Henderson, and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, New York, Hindawi Publishing Corporation, 2006.
- [5] K. Ezzinbi and M. S. Lalaoui, “Existence and controllability for nondensely defined partial neutral functional differential inclusions,” *Appl. Math.*, vol. 60, pp. 321–340, 2015.
- [6] X. Fu, “Approximate controllability for neutral impulsive differential inclusions with nonlocal conditions,” *J. Dyn. Contr. Syst.*, vol. 17, pp. 359–386, 2011.
- [7] J. R. Graef, J. Henderson, and A. Ouahab, *Impulsive Differential Inclusions: A Fixed Point Approach*, Berlin, Walter de Gruyter, 2013.
- [8] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis Theory, Mathematics and its Applications*, Vol. I, Dordrecht, Kluwer Academic, 1997.
- [9] K. Deng, “Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions,” *J. Math. Anal. Appl.*, vol. 179, pp. 630–637, 1993.
- [10] X. Fu and Y. Zhang, “Exact null controllability of non-autonomous functional evolution systems with nonlocal conditions,” *Acta Math. Sci.*, vol. 33, pp. 747–757, 2013.
- [11] W. S. Li, Y. K. Chang, and J. J. Nieto, “Solvability of impulsive neutral evolution differential inclusions with state-dependent delay,” *Math. Comput. Model.*, vol. 49, pp. 1920–1927, 2009.
- [12] S. Abbas, “Pseudo almost automorphic solutions of fractional order neutral differential equation,” *Semigroup Forum*, vol. 81, pp. 393–404, 2010.
- [13] S. Abbas, M. Banerjee, and S. Momani, “Dynamical analysis of a fractional order modified logistic model,” *Comp. Math. Appl.*, vol. 62, pp. 1098–1104, 2011.
- [14] S. Abbas, M. Benchohra, and G. M. N’Guérékata, *Topics in Fractional Differential Equations*, New York, Springer, 2012.
- [15] K. Diethelm, *The Analysis of Fractional Differential Equations*, Berlin, Springer, 2010.
- [16] A. Kilbas, H. Srivastava, and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Amsterdam, Elsevier Science B.V., 2006.
- [17] X. B. Shu and Q. Wang, “A study on the mild solution of impulsive fractional evolution equations,” *Appl. Math. Comput.*, vol. 273, pp. 465–476, 2016.
- [18] G. Wang, X. Ren, Z. Bai, and W. Hou, “Radial symmetry of standing waves for nonlinear fractional Hardy–Schrödinger equation,” *Appl. Math. Lett.*, vol. 96, pp. 131–137, 2019.
- [19] Y. Zhou, F. Jiao, and J. Pecaric, “Abstract Cauchy problem for fractional functional differential equations,” *Topol. Methods Nonlinear Anal.*, vol. 42, pp. 119–136, 2013.
- [20] Y. Zhou, *Basic Theory of Fractional Differential Equations*, Singapore, World Scientific, 2014.
- [21] B. Ahmad and S. K. Ntouyas, “Fractional differential inclusions with fractional separated boundary conditions,” *Fract. Calc. Appl. Anal.*, vol. 15, pp. 362–382, 2012.
- [22] B. Ahmad and S. K. Ntouyas, “Existence of solutions for fractional differential inclusions with nonlocal strip conditions,” *Arab J. Math. Sci.*, vol. 2, pp. 121–134, 2012.
- [23] A. Cernea, “On a fractional integro-differential inclusion,” *Electron. J. Qual. Theory Diff. Equ.*, vol. 25, pp. 1–11, 2014.
- [24] J. Henderson and A. Ouahab, “Fractional functional differential inclusions with finite delay,” *Nonlinear Anal.*, vol. 70, pp. 2091–2105, 2009.
- [25] R. Kamocki and C. Obczynski, “On fractional differential inclusions with the Jumarie derivative,” *J. Math. Phys.*, vol. 55, p. 10, 2014, Art no. 022902.
- [26] S. Kumar and N. Sukavanam, “Approximate controllability of fractional order semilinear systems with bounded delay,” *J. Differ. Equ.*, vol. 252, pp. 6163–6174, 2012.
- [27] Z. Liu and X. Li, “On the controllability of impulsive fractional evolution inclusions in Banach spaces,” *J. Optim. Theor. Appl.*, vol. 156, pp. 167–182, 2013.
- [28] R. Sakthivel, R. Ganesh, and S. Anthoni, “Approximate controllability of fractional nonlinear differential inclusions,” *Appl. Math. Comput.*, vol. 225, pp. 708–717, 2013.
- [29] R. Sakthivel, Y. Ren, A. Debbouche, and N.I. Mahmudov, “Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions,” *J. Differ. Equ.*, vol. 95, pp. 2361–2382, 2016.
- [30] J. Tariboon, T. Sitthiwirattam, and S. Ntouyas, “Existence results for fractional differential inclusions with multi-point and fractional integral boundary conditions,” *J. Comput. Anal. Appl.*, vol. 17, pp. 343–360, 2014.
- [31] G. Wang, “Twin iterative positive solutions of fractional q -difference Schrödinger equations,” *Appl. Math. Lett.*, vol. 76, pp. 103–109, 2018.
- [32] J. Wang, A. Ibrahim, and M. Feckan, “Nonlocal Cauchy problems for semilinear differential inclusions with fractional order in Banach spaces,” *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 27, pp. 281–293, 2015.
- [33] R. N. Wang, P. X. Zhu, and Q. H. Ma, “Multi-valued nonlinear perturbations of time fractional evolution equations in Banach spaces,” *Nonlinear Dynam.*, vol. 80, pp. 1745–1759, 2015.
- [34] Z. Yan, “Controllability of fractional-order partial neutral functional integrodifferential inclusions with infinite delay,” *J. Franklin Inst.*, vol. 348, pp. 2156–2173, 2011.
- [35] C. Lizama, A. Pereira, and R. Ponce, “On the compactness of fractional resolvent operator functions,” *Semigroup Forum*, vol. 93, pp. 363–374, 2016.
- [36] A. Bressan and G. Colombo, “Extensions and selections of maps with decomposable values,” *Stud. Math.*, vol. 90, pp. 69–86, 1988.
- [37] Z. Fan, “Characterization of compactness for resolvents and its applications,” *Appl. Math. Comput.*, vol. 232, pp. 60–67, 2014.
- [38] C. Castaing, and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Berlin, Springer, 1977.
- [39] L. Abadias and P. J. Miana, “A subordination principle on Wright functions and regularized resolvent families,” *J. Func. Space*, vol. 2015, p. 9, 2015, Article no. 158145.
- [40] A. Debbouche and J. J. Nieto, “Sobolev type fractional abstract evolution equations with nonlocal conditions and optimal multi-controls,” *Appl. Math. Comput.*, vol. 245, pp. 74–85, 2014.
- [41] A. Debbouche and D. F. M. Torres, “Sobolev type fractional dynamic equations and optimal multi-integral controls with fractional nonlocal conditions,” *Fract. Calc. Appl. Anal.*, vol. 18, pp. 95–121, 2015.