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Positive periodic solution for inertial neural networks with time-varying delays

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Abstract: In this paper the problems of the existence and stability of positive periodic solutions of inertial neural networks with time-varying delays are discussed by the use of Mawhin's continuation theorem and Lyapunov functional method. Some sufficient conditions are obtained for guaranteeing the existence and stability of positive periodic solutions of the considered system. Finally, a numerical example is given to illustrate the effectiveness of the obtained results.

Keywords: existence; inertial neural networks; positive periodic solution; stability.

1 Introduction

Inertial neural networks (INNs) are represented by second-order differential system. In INNs, inertial terms are described by the first-order derivative terms which have important meaning in biology, engineering technology and information system, for more details, see e. g. [1–3]. Due to the inertial terms, it is very difficult to study the dynamic properties of the network system. Over the past years, many researchers have used different methods and techniques to study INNs in depth and obtained a large number of results. Tu, Cao and Hayat [4] investigated the global dissipativity for INNs with time-varying delays and parameter uncertainties by using generalized Halanay inequality, matrix measure and matrix-norm inequality. Wang and Jiang [5] considered a class of impulsive INNs with time-varying delays. The global exponential stability in Lagrange sense for INNs with delays have been discussed in [6, 7]. Draye, Winters and Cheron [8] studied a class of self-selected modular recurrent neural networks with postural and inertial subnetworks.

Positive solutions or positive equilibrium points of neural networks usually represent special properties of different practical models [9]. Some results have been obtained on the positive periodic solutions of neural networks. Lu and Chen [10] obtained the global stability of nonnegative equilibria for a Cohen-Grossberg neural network system. Ding, Liu and Nieto [11] obtained existence of positive almost periodic solutions to a class of hematopoiesis model. In very recent years, Hien and Hai-An [12] considered the problems of positive solutions and exponential stability of positive equilibrium of INNs with multiple time-varying delays as follows:

$$\frac{d^2 x_i(t)}{dt^2} = -a_i \frac{dx_i(t)}{dt} - b_i x_i(t) + \sum_{j=1}^n c_{ij} f_j(x_j(t)) + \sum_{j=1}^n d_{ij} f_j(x_j(t - \tau_j(t))) + I_i(t), \quad (1.1)$$

where $t \geq 0$, $i = 1, \dots, n$. For the meanings of parameters in (1.1), see them in [12]. Using the comparison principle and homeomorphisms, the authors obtained some dynamic properties of positive solution of system (1.1).

In general, periodic solutions of network systems have many important applications in the real world. Thus, over the past few decades, periodic solutions of network systems have been widely studied and obtained many results. For example, in [13], existence and global exponential stability of periodic solution for discrete-

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time BAM neural networks have been considered. Furthermore, using suitable Lyapunov function and coincidence degree theory, Zhou etc. [14] studied a class of BAM neural network with periodic coefficients and continuously distributed delays. For more results of periodic solutions of network systems, see e. g. [15–22].

The innovation of this paper is mainly reflected in the following two aspects. (1) For the first time, we study the dynamic properties of positive periodic solutions of inertial neural networks with time-varying delays. (2) We find a method of variable substitution, which can change the original system into an equivalent new system, so it is convenient to study the positive periodic solution of the original system.

The rest of the paper is organized as follows. In Section 2, we introduce preliminaries and problem formulation. In Section 3, we establish the existence and uniqueness of positive periodic solutions of system (1.1) by using the method of coincidence degree theory. In Section 4, asymptotic stability result of system (1.1) is obtained. Section 5 gives a numerical example to verify the theoretical results. A brief conclusion is drawn in Section 6.

2 Preliminaries and problem formulation

Denote $[n] = \{1, 2, \dots, n\}$, $C_T = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t+T) \equiv x(t)\}$, T is a given positive constant. Motivated by the above work, in this paper we study a class of INNs with time-varying delays as follows:

$$\frac{d^2 x_i(t)}{dt^2} = -a_i(t) \frac{dx_i(t)}{dt} - b_i(t)x_i(t) + \sum_{j=1}^n c_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n d_{ij}(t)f_j(x_j(t-\tau_j(t))) + I_i(t), \quad (2.1)$$

where $t \geq 0$, $i \in [n]$, $x_i(t)$ denotes the state of i th neuron at time t , $a_i(t)$ is the damping coefficient, $b_i(t)$ denotes the strength of different neuron at time t , $c_{ij}(t)$ and $d_{ij}(t)$ are the neuron connection weights at time t , $f_j(\cdot)$ is the activation function which is a continuous function, $\tau_j(t)$ is a delay function with $0 \leq \tau_j \leq \hat{\tau}$, $\hat{\tau}$ is a constant, $I_i(t)$ is an external input of i th neuron at time t . Throughout the present paper, assume that $a_i(t)$, $b_i(t)$, $c_{ij}(t)$ and $d_{ij}(t)$ are continuous positive T -periodic functions. Let

$$y_i(t) = \frac{dx_i(t)}{dt} + \xi_i x_i(t), i \in [n], \quad (2.2)$$

where ξ_i is a constant. Then system (2.1) is changed into the following form:

$$\begin{cases} x'_i(t) = -\xi_i x_i(t) + y_i(t), \\ y'_i(t) = -(a_i(t) - \xi_i)y_i(t) + [(a_i(t) - \xi_i)\xi_i - b_i(t)]x_i(t) \\ + \sum_{j=1}^n c_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n d_{ij}(t)f_j(x_j(t-\tau_j(t))) + I_i(t). \end{cases} \quad (2.3)$$

Let $x_i(t) = e^{z_i(t)}$. Then system (2.3) is changed into the following system:

$$\begin{cases} z'_i(t) = -\xi_i + y_i(t)e^{-z_i(t)}, \\ y'_i(t) = -(a_i(t) - \xi_i)y_i(t) + [(a_i(t) - \xi_i)\xi_i - b_i(t)]e^{z_i(t)} \\ + \sum_{j=1}^n c_{ij}(t)f_j(e^{z_j(t)}) + \sum_{j=1}^n d_{ij}(t)f_j(e^{z_j(t-\tau_j(t))})I_i(t). \end{cases} \quad (2.4)$$

Obviously, under the above transforms, periodic solutions of system (2.4) are positive periodic solutions of system (2.1).

Lemma 2.1. [23] Assume that $\mathcal{X}$ and $\mathcal{Y}$ are two Banach spaces, and $L : D(L) \subset \mathcal{X} \rightarrow \mathcal{Y}$, is a Fredholm operator with index zero. Furthermore, $\Omega \subset \mathcal{X}$ is an open bounded set and $N : \overline{\Omega} \rightarrow \mathcal{Y}$ is L -compact on $\overline{\Omega}$. if all the following conditions hold:

- (1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \forall \lambda \in (0, 1)$,
- (2) $Nx \notin \text{Im}L, \forall x \in \partial\Omega \cap \text{Ker}L$,

(3) $\deg\{JQN, \Omega \cap \text{Ker}L, 0\} \neq 0$,

where $J : \text{Im}Q \rightarrow \text{Ker}L$ is an isomorphism. Then equation $Lx = Nx$ has a solution on $\overline{\Omega} \cap D(L)$.

Lemma 2.2. [24] Let $g \in C_T, \tau \in C_T^1$ with $\tau'(t) < 1 \forall t \in [0, T]$. Then $g(\mu(t)) \in C_T$, where $\mu(t)$ is the inverse function of $t - \tau(t)$.

Throughout the paper, the following assumptions hold. There exist constants $L_j \geq 0, K_i \geq 0$ and $\tilde{L}_j \geq 0$ such that

(H₁)

$$|f_j(x)| \leq L_j, j \in [n], \forall x \in \mathbb{R},$$

(H₂)

$$f_i(e^x) > 0 \text{ for } x \in (-\infty, -K_i) \cup (K_i, +\infty), i \in [n].$$

(H₃)

$$|f_j(x) - f_j(y)| \leq \tilde{L}_j |x - y|, j \in [n], \forall x, y \in \mathbb{R},$$

3 Existence and uniqueness of positive periodic solution

Let $z(t) = (z_1(t), \dots, z_n(t))^T, y(t) = (y_1(t), \dots, y_n(t))^T$. Set

$$\mathcal{X} = \mathcal{Y} = \{w(t) = (z(t), y(t))^T \in C(\mathbb{R}, \mathbb{R}^{2n}), w(t+T) = w(t)\}$$

with the norm $\|w\| = \max\{|z|_\infty, |y|_\infty\}$, where $|\phi|_\infty = \max_{i \in [n], t \in \mathbb{R}} |\phi_i(t)| \forall \phi \in \mathbb{R}^n$. It is easy to see that $\mathcal{X}$ and $\mathcal{Y}$ are two Banach space. Let

$$\begin{aligned} L : D(L) \subset \mathcal{X} &\rightarrow \mathcal{X}, (Lw)(t) = w'(t) = (z'(t), y'(t))^T, t \in \mathbb{R}, \\ (Lw)_i(t) &= z'_i(t), i \in [n], t \in \mathbb{R}, \end{aligned} \quad (3.1)$$

and

$$(Lw)_{n+i}(t) = y'_i(t), i \in [n], t \in \mathbb{R}. \quad (3.2)$$

Let $N : \mathcal{X} \rightarrow \mathcal{X}$ with

$$(Nw)_i(t) = -\xi_i + y_i(t)e^{-z_i(t)} \quad i \in [n], t \in \mathbb{R}, \quad (3.3)$$

and

$$\begin{aligned} (Nw)_{n+i}(t) &= -(a_i(t) - \xi_i)y_i(t) + [(a_i(t) - \xi_i)\xi_i - b_i(t)]e^{z_i(t)} + \sum_{j=1}^n c_{ij}(t)f_j(e^{z_j(t)}) \\ &\quad + \sum_{j=1}^n d_{ij}(t)f_j(e^{z_j(t-\tau_j(t))}) + I_i(t), i \in [n], t \in \mathbb{R}. \end{aligned} \quad (3.4)$$

Obviously, $\text{Ker}L = \mathbb{R}^{2n}, \text{Im}L = \{w : w \in \mathcal{X}, \int_0^T w(s)ds = 0\}$ is closed in $\mathcal{Y}$, $\dim \text{Ker}L = \text{condim} \text{Im}L = 2n$. So L is a Fredholm operator with index zero. Let

$$P : \mathcal{X} \rightarrow \text{Ker}L, Q : \mathcal{Y} \rightarrow \mathcal{Y}/\text{Im}L$$

be defined by

$$Px = \frac{1}{T} \int_0^T w(s) ds, Qy = \frac{1}{T} \int_0^T y(s) ds,$$

and let

$$L_p = L|_{\mathcal{X} \cap \text{Ker} P} : \mathcal{X} \cap \text{Ker} P \rightarrow \text{Im} L.$$

Then L_p has its right inverse L_p^{-1} .

Theorem 3.1. Suppose that $\int_0^T I_i(t) dt = 0$, $\tau'_i(t) < 1$, $i \in [n]$. Furthermore, assumptions (H_1) and (H_2) hold. Then system (2.1) has at least one T -periodic solution, provide that the following conditions hold:

$$(a_i(t) - \xi_i) > \frac{1}{T} b_i(t) - (a_i(t) - \xi_i) \xi_i > 0, \Gamma_i(t) > 0, t \in [0, T], i \in [n], \quad (3.5)$$

where $\xi_i > 0$ is defined by (2.3) and $\Gamma_i(t) = \sum_{j=1}^n \frac{d_{ij}(\mu_j(t))}{1 - \tau'(\mu_j(t))} - \sum_{j=1}^n c_{ij}(t)$, $\mu_j(t)$ is a inverse function of $t - \tau_j(t)$.

Proof. Consider the following operator equation:

$$Lw = \lambda Nw, w \in D(L), \lambda \in (0, 1),$$

where L and N are defined by (3.1)–(3.4). Let $\Omega_1 = \{w : w \in D(L), Lw = \lambda Nw, \lambda \in (0, 1)\}$. In view of (3.1)–(3.4), $\forall x \in \Omega_1$, we have

$$z'_i(t) = \lambda [-\xi_i + y_i(t)e^{-z_i(t)}], i \in [n], t \in \mathbb{R}, \quad (3.6)$$

$$\begin{aligned} y'_i(t) = \lambda & \left[-(a_i(t) - \xi_i)y_i(t) + [(a_i(t) - \xi_i)\xi_i - b_i(t)]e^{z_i(t)} + \sum_{j=1}^n c_{ij}(t)f_j(e^{z_j(t)}) \right. \\ & \left. + \sum_{j=1}^n d_{ij}(t)f_j(e^{z_j(t-\tau_j(t))}) + I_i(t) \right], i \in [n], t \in \mathbb{R}. \end{aligned} \quad (3.7)$$

Integrating two sides of (3.6) and (3.7) on $[0, T]$ respectively, we have

$$\bar{\xi}_i T = \int_0^T y_i(s)e^{-z_i(s)} ds, i \in [n], \quad (3.8)$$

$$\begin{aligned} 0 = \int_0^T & \left[(a_i(s) - \xi_i)y_i(s) + [b_i(s) - (a_i(s) - \xi_i)\xi_i]e^{z_i(s)} \right. \\ & \left. - \sum_{j=1}^n c_{ij}(s)f_j(e^{z_j(s)}) + \sum_{j=1}^n d_{ij}(s)f_j(e^{z_j(s-\tau_j(s))}) \right] ds, i \in [n]. \end{aligned} \quad (3.9)$$

Consider the term $\int_0^T \sum_{j=1}^n d_{ij}(s)f_j(e^{z_j(s-\tau_j(s))}) ds$ in (3.9). Using Lemma 2.2, we have

$$\int_0^T \sum_{j=1}^n d_{ij}(s)f_j(e^{z_j(s-\tau_j(s))}) ds = \int_0^T \sum_{j=1}^n \frac{d_{ij}(\mu_j(s))}{1 - \tau'(\mu_j(s))} f_j(e^{z_j(s)}) ds, i \in [n]$$

together with which (3.9), we have

$$0 = \int_0^T \left[(a_i(s) - \xi_i)y_i(s) + [b_i(s) - (a_i(s) - \xi_i)\xi_i]e^{z_i(s)} + \Gamma_i(s)f_j(e^{z_j(s)}) \right] ds, i \in [n]. \quad (3.10)$$

From $\xi_i > 0$ and (3.8), we have

$$y_i(s) > 0, i \in [n]. \quad (3.11)$$

We will prove that the following inequality holds:

$$|z_i(t)| \leq K_i, i \in [n], \forall t \in [0, T], \quad (3.12)$$

where K_i is defined by assumption (H_2) . In fact, if $|z_i(t)| > K_i$, $i \in [n]$, $\forall t \in [0, T]$, then in view of (3.10), (3.11) and condition (3.5) we get

$$0 < \int_0^T \left[(a_i(s) - \xi_i) y_i(s) + [b_i(s) - (a_i(s) - \xi_i) \xi_i] e^{z_i(s)} + \Gamma_i(s) f_j(e^{z_j(s)}) \right] ds, i \in [n]$$

which contradicts (3.10). Hence, (3.12) holds. From integral mean value theorem and (3.8), there is a point $t_1 \in [0, T]$ such that

$$y_i(t_1) = \frac{\int_0^T e^{-z_i(s)} ds}{\xi_i T}, i \in [n]. \quad (3.13)$$

From (3.12) and (3.13), we have

$$|y_i(t_1)| \leq \frac{e^{K_i}}{\xi_i} =: L_i, i \in [n]. \quad (3.14)$$

It follows by (3.7) and (3.14) that

$$\begin{aligned} |y_i(t)| &\leq L_i + \int_0^T |y_i'(s)| ds \\ &\leq T |(a_i(t) - \xi_i)|_\infty |y_i(t)|_\infty + T |(a_i(t) - \xi_i) \xi_i - b_i(t)|_\infty e^{K_i} \\ &\quad + T \sum_{j=1}^n (|c_{ij}(t)|_\infty + d_{ij}(t)|_\infty) M_j + T |I_i(t)|_\infty. \end{aligned} \quad (3.15)$$

Use condition (3.15) and (3.5), there exists a positive constant P_i such that

$$|y_i(t)| \leq P_i, i \in [n]. \quad (3.16)$$

From (3.12) and (3.16), we have

$$\|w\| = \max \left\{ \max_{i \in [n]} K_i, \max_{i \in [n]} P_i \right\} =: M.$$

Let $\Omega_2 = \{w \in \mathcal{X} : \|w\| < M + 1\}$. Then $\forall w \in \Omega_2$, condition (1) of Lemma 2.1 holds. In addition, $\forall w \in \partial\Omega_2 \cap \text{Ker}L$, then $w \in \mathbb{R}^{2n}$ is a constant vector, and there exists at least one $i \in [n]$ such that $|w_{n+i}| = M + 1$ and $|w_{n+j}| \leq M + 1$ for $j \neq i$. We prove that

$$QNw \neq 0 \quad \forall w \in \partial\Omega_2 \cap \text{Ker}L. \quad (3.17)$$

In fact, when $|w_{n+i}| = M + 1$, by (3.7), assumption (H_2) and condition (3.5), we have

$$\int_0^T \left[(a_i(s) - \xi_i) y_i(s) + [b_i(s) - (a_i(s) - \xi_i) \xi_i] e^{z_i(s)} + \Gamma_i(s) f_j(e^{z_j(s)}) \right] ds \neq 0.$$

Hence, (3.17) holds and condition (2) of Lemma 2.1 holds. Let

$$H_i(w_i, \mu) = \mu w_i + (1 - \mu) QNw_i, \mu \in [0, 1], i \in [n] \quad (3.18)$$

and

$$H_{n+i}(w_{n+i}, \mu) = \mu w_{n+i} + (1 - \mu) QNw_{n+i}, \mu \in [0, 1] \quad i \in [n]. \quad (3.19)$$

It is easy to verify that, using assumption (H_2) , (3.18) and (3.19), We have

$$H_i(w_i, \mu) \neq 0 \text{ and } H_{n+i}(w_{n+i}, \mu) \neq 0 \text{ for all } w \in \partial\Omega_2 \cap \text{Ker}L, i \in [n].$$

Based on the property of topological degree and take J to be the identity mapping $I : \text{Im}Q \rightarrow \text{Ker}L$, then

$$\begin{aligned} \deg \{JQN, \Omega_2 \cap \text{Ker}L, 0\} &= \deg \{H(\cdot, 0), \Omega_2 \cap \text{Ker}L, 0\} \\ &= \deg \{H(\cdot, 1), \Omega_2 \cap \text{Ker}L, 0\} \\ &= 1 \neq 0. \end{aligned}$$

So, condition (3) of Lemma 2.1 holds. Therefore, by using Lemma 2.1, we see that the equation $Lx = Nx$ has at least one T -periodic solution w in $\overline{\Omega}_2$. Namely, system (2.1) has at least one positive T -periodic solution. $\square$

Due to assumption (H_3) , the term $f_j(x_j)$, $j \in [n]$ in system (2.4) satisfies Lipschitz condition on $\mathbb{R}$. Thus, by basic results of functional differential equation, we have the following theorem for the unique existence of positive periodic solution to system (2.1).

Theorem 3.2. *Suppose all the conditions of Theorem 3.1 and assumption (H_3) hold. Then system (2.4) has unique T -periodic solution. Namely, system (2.1) has unique positive T -periodic solution.*

4 Asymptotic behaviours of positive periodic solution

Since system (2.3) is equivalent to system (2.1) under the transformation (2.2), then we will consider the asymptotic stability problems of system (2.3).

Definition 4.1. If $w^*(t) = (x_1^*(t), \dots, x_n^*(t), y_1^*(t), \dots, y_n^*(t))^T$ is a positive periodic solution of system (2.3) and $w(t) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_n(t))^T$ is any solution of system (2.3) satisfying

$$\lim_{t \rightarrow +\infty} \sum_{i=1}^n \left[|x_i(t) - x_i^*(t)| + |y_i(t) - y_i^*(t)| \right] = 0.$$

We call $w^*(t)$ is globally asymptotic stable.

Theorem 4.1. *Under the conditions of Theorem 3.2, assume further that*

(i) *there exist input functions $I_i(t)$ such that*

$$\hat{c}_i = 2|(a_i(t) - \xi_i)y_i^*| + 2|(a_i(t) - \xi_i)\xi_i - b_i(t)\|x_i^*| + 2\left(\sum_{j=1}^n (c_{ij}(t) + d_{ij}(t))L_j + 2|I_i(t)|\right) = 0, i \in [n]. \quad (4.1)$$

(ii) *Let $t_i > 0, \kappa_i > 0$, where*

$$t_i = \liminf_{t \rightarrow +\infty} \left[2\xi_i - 1 - [(a_i(t) - \xi_i)\xi_i - b_i(t)]^2 \right], i \in [n]. \quad (4.2)$$

$$\kappa_i = \liminf_{t \rightarrow +\infty} [a_i(t) - \xi_i - 1], i \in [n]. \quad (4.3)$$

Then system (2.3) has unique T -periodic solution $w^(t) = (x_1^*(t), \dots, x_n^*(t), y_1^*(t), \dots, y_n^*(t))^T$ which is globally asymptotic stable.*

Proof. Using the results of Theorem 3.2, system (2.3) has unique positive T -periodic solution $w^*(t)$. Suppose $w(t)$ is any solution of system (2.3). Let

$$V_i(t) = (x_i(t) - x_i^*)^2 + (y_i(t) - y_i^*)^2, i \in [n], t \geq 0. \quad (4.4)$$

Derivation of (4.4) along the solution of (2.3) gives

$$\begin{aligned}
V'_i(t) &= -2\xi_i(x_i(t) - x_i^*)^2 + 2(x_i(t) - x_i^*)(y_i(t) - y_i^*) + (2y_i^* - 2\xi_i x_i^*)(x_i(t) - x_i^*) \\
&\quad - 2(a_i(t) - \xi_i)(y_i(t) - y_i^*)^2 - 2(a_i(t) - \xi_i)y_i^*(y_i(t) - y_i^*) \\
&\quad + 2[(a_i(t) - \xi_i)\xi_i - b_i(t)](x_i(t) - x_i^*)(y_i(t) - y_i^*) + 2[(a_i(t) - \xi_i)\xi_i - b_i(t)]x_i^*(y_i(t) - y_i^*) \\
&\quad + 2(y_i(t) - y_i^*) \sum_{j=1}^n c_{ij}(t)f_j(x_j(t)) + 2(y_i(t) - y_i^*) \sum_{j=1}^n d_{ij}(t)f_j(x_j(t - \tau_j(t))) + 2(y_i(t) - y_i^*)I_i(t) \\
&\leq -2\xi_i(x_i(t) - x_i^*)^2 + (x_i(t) - x_i^*)^2 + (y_i(t) - y_i^*)^2 + (2y_i^* - 2\xi_i x_i^*)(x_i(t) - x_i^*) \\
&\quad - 2(a_i(t) - \xi_i)(y_i(t) - y_i^*)^2 + 2|(a_i(t) - \xi_i)y_i^*||y_i(t) - y_i^*| \\
&\quad + [(a_i(t) - \xi_i)\xi_i - b_i(t)]^2(x_i(t) - x_i^*)^2 + (y_i(t) - y_i^*)^2 + 2|(a_i(t) - \xi_i)\xi_i - b_i(t)||x_i^*(y_i(t) - y_i^*)| \\
&\quad + 2(\sum_{j=1}^n (c_{ij}(t) + d_{ij}(t))L_j|y_i(t) - y_i^*| + 2|I_i(t)||y_i(t) - y_i^*| \\
&= -\hat{a}_i(x_i(t) - x_i^*)^2 - \hat{b}_i(y_i(t) - y_i^*)^2 + \hat{c}_i|y_i(t) - y_i^*| + 2(y_i^* - \xi_i x_i^*)(x_i(t) - x_i^*), \tag{4.5}
\end{aligned}$$

where

$$\hat{a}_i = 2\xi_i - 1 - [(a_i(t) - \xi_i)\xi_i - b_i(t)]^2, \hat{b}_i = 2(a_i(t) - \xi_i) - 2.$$

$\hat{c}_i$ is defined by (4.1). Since $w^*(t)$ is a positive periodic solution of system (2.3), we have

$$y_i^* - \xi_i x_i^* = 0, i \in [n]. \tag{4.6}$$

From (4.1), (4.5) and (4.6), we have

$$V'_i(t) \leq -\hat{a}_i(x_i(t) - x_i^*)^2 - \hat{b}_i(y_i(t) - y_i^*)^2, i \in [n]. \tag{4.7}$$

By (4.2) and (4.3), for any $\varepsilon > 0$, and $t_i - \varepsilon > 0$ and $\kappa_i - \varepsilon > 0$, there exists a positive constant $\mathbb{T}$ (enough large) such that

$$2\xi_i - 1 - [(a_i(t) - \xi_i)\xi_i - b_i(t)]^2 \geq t_i - \varepsilon \text{ for } t > \mathbb{T}, i \in [n]. \tag{4.8}$$

and

$$2(a_i(t) - \xi_i) - 2 \geq \kappa_i - \varepsilon \text{ for } t > \mathbb{T}, i \in [n] \tag{4.9}$$

From (4.7)–(4.9), we have

$$V'_i(t) \leq -(t_i - \varepsilon)(x_i(t) - x_i^*)^2 - (\kappa_i - \varepsilon)(y_i(t) - y_i^*)^2, \text{ for } t > \mathbb{T}, i \in [n]. \tag{4.10}$$

Take the Lyapunov functional for system (2.3) in the following form:

$$V(t) = \sum_{i=1}^n V_i(t), t \in \mathbb{R}.$$

Derivating it along the solution of system (2.3) which together with (4.10), it follows that

$$V'(t) \leq -\sum_{i=1}^n [(t_i - \varepsilon)(x_i(t) - x_i^*)^2 + (\kappa_i - \varepsilon)(y_i(t) - y_i^*)^2] < 0 \text{ for } t > \mathbb{T}, i \in [n]. \tag{4.11}$$

Integrate both sides of (4.11) from $\mathbb{T}$ to $+\infty$, then

$$V(t) + \int_{\mathbb{T}}^{+\infty} \sum_{i=1}^n [(t_i - \varepsilon)(x_i(t) - x_i^*)^2 + (\kappa_i - \varepsilon)(y_i(t) - y_i^*)^2] \leq V(0).$$

Based on Barbalat's Lemma [25], it follows that

$$\lim_{t \rightarrow +\infty} \sum_{i=1}^n [|x_i(t) - x_i^*| + |y_i(t) - y_i^*|] = 0.$$

The proof of Theorem 4.1 is now finished. $\square$

5 Numerical example

This section presents an example that demonstrate the validity of our theoretical results. Consider the following system:

$$\begin{aligned} \frac{d^2 x_1(t)}{dt^2} &= -a_1(t) \frac{dx_1(t)}{dt} - b_1(t)x_1(t) + \sum_{j=1}^3 c_{1j}(t)f_j(x_j(t)) + \sum_{j=1}^3 d_{1j}(t)f_j(x_j(t - \tau_j(t))) + I_1(t), \\ \frac{d^2 x_2(t)}{dt^2} &= -a_2(t) \frac{dx_2(t)}{dt} - b_2(t)x_2(t) + \sum_{j=1}^3 c_{2j}(t)f_j(x_j(t)) + \sum_{j=1}^3 d_{2j}(t)f_j(x_j(t - \tau_j(t))) + I_2(t), \\ \frac{d^2 x_3(t)}{dt^2} &= -a_3(t) \frac{dx_3(t)}{dt} - b_3(t)x_3(t) + \sum_{j=1}^3 c_{3j}(t)f_j(x_j(t)) + \sum_{j=1}^3 d_{3j}(t)f_j(x_j(t - \tau_j(t))) + I_3(t), \end{aligned} \quad (5.1)$$

where

$$T = 2\pi, a_1(t) = 2.5, a_2(t) = 2.45, a_3(t) = 2.65, b_1(t) = 1.85, b_2(t) = 1.75, b_3(t) = 1.60,$$

$$c_{ij}(t) = d_{ij}(t) = 0.1, \tau_j(t) = \frac{1}{2\pi} \cos t, f_j(u) = \frac{\sin^2 u}{u^2 + 1}, I_1(t) = 0.2895, I_2(t) = 1.5375, I_3(t) = 1.1064.$$

Solving system (5.1) by Matlab toolbox, we obtain that system (5.1) has unique positive solution $x^* = (5.7930, 8.5512, 4.2663)^\top$. Let $\xi_i = 1, i = 1, 2, 3$. By (2.3), we have $y^* = (5.7930, 8.5512, 4.2663)^\top$. For the above parameters, it easily check condition (4.1) holds for $i = 1, 2, 3$. Next, we check conditions (4.2) and (4.3) hold:

$$\begin{aligned} \iota_1 &= \liminf_{t \rightarrow +\infty} [2\xi_1 - 1 - [(a_1(t) - \xi_1)\xi_1 - b(t)^2]] = 0.8775 > 0, \\ \iota_2 &= \liminf_{t \rightarrow +\infty} [2\xi_2 - 1 - [(a_2(t) - \xi_2)\xi_2 - b_2(t)^2]] = 0.81 > 0 \\ \iota_3 &= \liminf_{t \rightarrow +\infty} [2\xi_3 - 1 - [(a_3(t) - \xi_3)\xi_3 - b_3(t)^2]] = 0.9775 > 0, \\ \kappa_1 &= \liminf_{t \rightarrow +\infty} [a_1(t) - \xi_1 - 1] = 0.5 > 0, \\ \kappa_2 &= \liminf_{t \rightarrow +\infty} [a_2(t) - \xi_2 - 1] = 0.45 > 0, \\ \kappa_3 &= \liminf_{t \rightarrow +\infty} [a_3(t) - \xi_3 - 1] = 0.65 > 0. \end{aligned}$$

Thus, all assumptions of Theorem 4.1 hold and the periodic solution of (5.1) is globally asymptotic stable. The corresponding numerical simulations are presented in Figures 1–4 with random initial conditions. Specifically, Figures 1–3 describe 40 sample paths of the states of $x_1(t)$, $x_2(t)$ and $x_3(t)$ for system (5.1), respectively. We find that all state orbits of system (5.1) converge to equilibrium x^* . Furthermore, Figure 4 describes 40 sample paths of the corresponding phase diagram of system (5.1). It is easy to see that all solutions of system (5.1) also converge to equilibrium x^* .

Remark 5.1. For all we know, the positive periodic solutions problems of INNs with delays are considered in the present paper for the first time. Using coincidence degree theory and constructing proper Lyapunov functional, we get some brand new results on the existence, uniqueness, and asymptotic stability of positive periodic solution of positive INNs. We can confirm the truth of the proposed methods, for example, in [12, 26–28] cannot be generalized to the problems studied in this article. Besides that, for system (5.1) in numerical example 1, the results of Theorems 2 and 3 in [12], Theorems 1 in [4] are not applicable for the positive periodic solution problems of system (5.1). It is important to point out that globally exponentially stable results of positive

Figure 1: The positive periodic solution $x_1(t)$ of system (5.1).

Figure 2: The positive periodic solution $x_2(t)$ of system (5.1).

Figure 3: The positive periodic solution $x_3(t)$ of system (5.1).

Figure 4: The phase diagram of system (5.1).

equilibrium in Lagrange sense were obtained by Theorem 3 in [12], and in this paper we only obtain some sufficient conditions for asymptotic stability of positive periodic solution of positive INNs. The main reason is that constructing proper Lyapunov functional is very difficult in periodic function space. We hope to study the globally exponentially stability of the positive periodic solution of system (2.1) in future research.

6 Conclusions and discussions

In this paper we study the problems of positive periodic solutions for inertial neural networks with multiple variable delays. First, by applying Mawhin's continuous theorem to the system, we get some sufficient conditions for guaranteeing the existence and uniqueness of positive periodic solutions. Then, on the basis of positivity results, by constructing proper Lyapunov functional, we obtain the asymptotic stability of positive periodic solutions of system (2.1). A numerical example verifies the correctness of the obtained results.

It should be pointed out that only asymptotic stability results are obtained in this paper. Because it is difficult to construct proper Lyapunov functional in the periodic function space, the exponential stability is not obtained in this paper. We hope that researchers will continue this research in the future. In addition, for neural network with positivity constraints, there are still many problems unsolved, such as the state estimation problem, the synchronization problem, the pulse-perturb problem, etc. The above questions will be our focus.

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