Research Article

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Exact solutions for the total variation denoising problem of piecewise constant images in dimension one

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Abstract: A method for obtaining the exact solution for the total variation denoising problem of piecewise constant images in dimension one is presented. The validity of the algorithm relies on some results concerning the behavior of the solution when the parameter λ in front of the fidelity term varies. Albeit some of them are well-known in the community, here they are proved with simple techniques based on qualitative geometrical properties of the solutions.

Keywords: Total variation denoising, signals, exact solutions

MSC 2010: 49K99, 49K21

1 Introduction

When an image is acquired, it comes, unavoidably, with some distortion. Indeed, external conditions, other than defects or limitations of the instruments that are used to obtain them, affect the quality of the acquired data. Thus, in order to perform any task on the image, it is important to be able to recover the *clean* version in the best possible way, i.e., with optimal fidelity. If we denote it by $u : \Omega \rightarrow \mathbb{R}$ and the acquired, corrupted image by $f : \Omega \rightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$ is an open set, it is usually assumed that the two are related via

$$f = Au + n. \quad (1.1)$$

Here A is a bounded linear operator representing, for instance, the blurring effect and n is the instance of the random noise. One of the aims of image reconstruction is deblurring and denoising f in order to recover u (see [8, 24]).

Here we are interested in the denoising problem, i.e., when the operator A is the identity and we have to remove the noise. Problem (1.1) is, in general, ill-posed (in the sense of Hadamard) and thus we need to regularize it (see [1, 55]). A widely used variational technique for this purpose was introduced by Rudin, Osher and Fatemi in [52], where they proposed to recover u via the minimization problem

$$\min_{u \in BV(\Omega), \|u-f\|_{L^2}^2 = \sigma^2} |Du|(\Omega) \quad (1.2)$$

for some fixed $\sigma > 0$, where f is assumed to be in $L^2(\Omega)$ and $|Du|(\Omega)$ denotes the total variation of the function u in Ω . There are some interesting cases though, where the real image is not represented by a function of bounded variation (see [35]). The constrained minimization problem (1.2) has been shown to be equivalent to the following *penalized* minimization problem (known as the total variation denoising model with L^2

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fidelity term):

$$\min_{u \in BV(\Omega)} |Du|(\Omega) + \lambda \|u - f\|_{L^2(\Omega)}^2 \quad (1.3)$$

for some Lagrange multiplier $\lambda > 0$ (see [19]).

Today's literature on the study of problem (1.3) is extensive, and here we limit ourselves to recall that properties of the solutions have been studied, for instance, in [2–4, 9, 10, 16, 17, 22, 25, 29, 31, 36–38, 42, 51, 56, 57], the analysis of variants of (1.3) that use the generalized total variation have been performed in [13, 45, 47, 50], anisotropic models are undertaken in [26, 30, 32, 41], while the effects of considering high-order models have been investigated in [21, 23, 27, 39, 46]. Finally, other variants of (1.2) have been addressed in [6, 7, 33, 34, 44], and algorithmic considerations may be found in [15, 18, 20, 28, 43].

The seeking for an analytical method to find exact solutions to the minimization problem (1.3) (and some variants of it, both for piecewise constant and general initial data f) has been the topic of many studies. For instance, the interested reader can consult [11, 12, 14, 45, 48–50, 53] and the references therein. The starting point of our investigation is the work of Strong and Chan (see [54]), where they consider the minimization problem (1.3) (essentially just in one dimension) where the initial data f is a piecewise constant function with noise. Under certain conditions on the amplitude of the noise, they are able to determine the solution of the minimization problem (1.3) in the case $\lambda \gg 1$, namely when the solution has jumps where f has.

The aim of this paper is to carry on the previous analysis and to determine the solution of the minimization problem (1.3) for f being a piecewise constant function (without noise) for every value of the parameter λ . To allow for more generality in the choice of the fidelity term, we generalize the L^2 norm to an L^p one, with $p \in (1, \infty)$. Some results are also presented in the case $p = 1$. To be precise, we consider the minimization problem

$$\min_{u \in BV(\Omega)} \mathcal{G}(u), \quad (1.4)$$

where $\Omega := (a, b) \subset \mathbb{R}$ and

$$\mathcal{G}(u) := |Du|(\Omega) + \lambda \|u - f\|_{L^p(\Omega)}^p$$

for a given initial piecewise constant data f . Our aim is to provide a method for finding the solution to the minimization problem (1.4) in the case $p > 1$ for all values of $\lambda > 0$.

The novelty of this paper relies on the fact that our approach is aimed at getting the solution u^λ to the minimization problem (1.4) by considering the solution u^λ for large values of λ and then obtaining the ones for small values of λ by decreasing the parameter λ , whereas all the above-mentioned papers used a fixed value of λ . In particular, the main result we obtain is Theorem 5.3 about the behavior of neighboring values of the solution u^λ close to a jump point when the parameter λ is moving. Albeit some of the results could be proved by using the primal-dual optimality condition (see [14, 51]) and the semigroup property of the total variation flow in dimension one (see [53]), here we prefer to employ more elementary techniques to study the problem.

We next explain the main idea behind the strategy we propose. The rigid structure of the initial data forces the solution to be piecewise constant itself, with jump set contained in the one of f (see Corollary 3.2). Moreover, a simple truncation argument shows that the solution takes values between the minimum and the maximum of f . Hence, the minimization problem (1.4) with f of the form

$$f(x) = \sum_{i=1}^k f_i \chi_{(x_{i-1}, x_i)}(x), \quad f_i \in \mathbb{R},$$

is equivalent to the following minimization problem:

$$\min_{v \in Q} G(v), \quad (1.5)$$

where $Q := [\min f, \max f]^k$ and $G : \mathbb{R}^k \rightarrow \mathbb{R}$ is the function defined by

$$G(v) := \sum_{i=2}^k |v_i - v_{i-1}| + \lambda \sum_{i=1}^k L_i |f_i - v_i|^p,$$

with $v = (v_1, \dots, v_k)$ and $L_i := x_i - x_{i-1}$. The function G is convex, but it lacks differentiability on the hyperplanes where $\{v_{i-1} = v_i\}$. Thus, in principle, one should minimize the function G over several compact regions and then compare all the minimum values in order to find the global minimizer.

Our method aims at overcoming this difficulty. We will be able, for each λ , to predict *a priori* – that is, without knowing explicitly u^λ (the minimizer of G corresponding to the parameter λ) – what the relative position of each u_i^λ with respect to u_{i-1}^λ and f_i will be. Knowing that, it is possible to look for the minimizer u^λ only in a specific region of $\mathbb{R}^k$, where the absolute values present in the expression of G can be written explicitly. Hence, u^λ can be found by solving the appropriate Euler–Lagrange equation.

We give a more detailed description of our method: the function $\lambda \mapsto u^\lambda$ is continuous and $u^\lambda \rightarrow f$ as $\lambda \rightarrow \infty$ (see Lemma 5.1). Hence, for $\lambda \gg 1$, we have that u_i^λ is very close to f_i , and this allows us to predict the relative position of u_i^λ with respect to u_{i-1}^λ . Moreover, thanks to the qualitative properties of the solutions we will prove in Lemma 5.2 and Proposition 5.5, we will also be able to tell the relative position of each u_i with respect to f_i . These information allows us to write explicitly the absolute values present in the expression of G , as well as to write explicitly the Euler–Lagrange equation, whose solution will give us the minimizer u^λ . With this reasoning, we find the minimizers for λ large (how large λ has to be will be determined a posteriori).

The idea now is to let λ decrease. Since u^λ is constant for small values of λ (see Lemma 3.6), by continuity of $\lambda \mapsto u^\lambda$ eventually two neighboring values u_i^λ and u_{i-1}^λ will happen to be the same. The main technical result (Theorem 5.3) tells us that the same will be true for all smaller values of λ . As a result we now have to consider the function G restricted to the subspace $\{v_{i-1} = v_i\}$, thus reducing the number of variables. By continuity of $\lambda \mapsto u^\lambda$, it is then possible to predict the relative position of every u_i with respect to u_{i-1} , while the qualitative properties of the solutions will give us the relative position of u_i with respect to f_i . As a consequence, also in this case, we are able to write explicitly the Euler–Lagrange equation.

We observe that the price to pay for applying this method is that, in order to determine the solution of the minimization problem (1.5) for a certain value $\bar{\lambda}$, we first need to know it for all $\lambda > \bar{\lambda}$. This, in the end, boils down to solve some equations (linear if $p = 2$), whose number can be roughly bounded above by $k(k+1)/2$.

Finally, we would like to comment on the case $p = 1$. The reason why the strategy described above fails for $p = 1$ is because we cannot use the continuity of the map $\lambda \mapsto u^\lambda$. Indeed, even if for $p = 1$ there is no uniqueness for the solution of the minimization problem (1.5) (see an example in Proposition 4.1), there is always a solution taking only the values that f takes (see Corollary 3.3). But this jumping behavior of the solution prevents us to use continuity arguments, which are at the core of the strategy sketched above. Although it could be possible to obtain a solution of the minimization problem (1.5) in the case $p = 1$ by comparing the value of the functional G over all the vectors $v \in \mathbb{R}^k$ of the form $v_i = f_{\sigma(i)}$, where $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ (see Corollary 3.3), we are currently investigating the possibility to obtain a more efficient analytic method to fulfill the task.

The paper is organized as follows. After a brief recalling of the main properties of one-dimensional functions of bounded variation in Section 2, we devote Section 3 to stating and proving basic results we will need in the sequel concerning the solutions of our minimization problem. In Section 4, we illustrate with a simple case the different behaviors of the solution in the cases $p = 1$ and $p > 1$. Section 5 contains the main technical results needed to justify the strategy to determine the solution of the minimization problem (1.5) we describe. In Section 6, we conclude with an explicit example.

2 Preliminaries

In this section, we review basic definitions of one-dimensional functions of bounded variation. For more details, see [5, 40]. Here we assume $a, b \in \mathbb{R}$ with $a < b$.

Definition 2.1. Let $u : (a, b) \rightarrow \mathbb{R}$. The *pointwise variation* of u in (a, b) is defined by

$$pV(u; a, b) := \sup \left\{ \sum_{i=1}^{n-1} |u(x_{i+1}) - u(x_i)| : a < x_1 < \dots < x_n < b \right\}.$$

Definition 2.2. For $u \in L^1((a, b))$, its *total variation* in (a, b) is given by

$$|Du|((a, b)) := \sup \left\{ \int_a^b \varphi' u dx : \varphi \in C_0^\infty((a, b)), |\varphi| \leq 1 \right\}.$$

If $|Du|((a, b)) < \infty$, we say that u belongs to the space $BV((a, b))$ of functions of bounded variation in (a, b) . In this case, Du is a finite Radon measure on (a, b) .

Definition 2.3. Let $u \in BV((a, b))$. We define the *jump set* of u by

$$J_u := \{x \in (a, b) : |Du|(\{x\}) \neq 0\}.$$

The relation between the total and the pointwise variation is given by the following result. In the following, $\mathcal{L}^1$ will denote the one-dimensional Lebesgue measure on $\mathbb{R}$.

Theorem 2.4. Let $u \in L^1((a, b))$ and define the essential variation of u by

$$eV(u; a, b) := \inf \{pV(v; a, b) : v = u \text{ } \mathcal{L}^1\text{-a.e. in } (a, b)\}. \quad (2.1)$$

The infimum defining $eV(u; a, b)$ in (2.1) is achieved and it coincides with $|Du|((a, b))$.

Theorem 2.4 allows us to single out some well behaving representative of a BV function.

Definition 2.5. Let $u \in BV((a, b))$. Any v with $v = u \text{ } \mathcal{L}^1\text{-a.e. in } (a, b)$ such that

$$pV(v; a, b) = eV(u; a, b) = |Du|((a, b))$$

is called a *good representative* of u .

3 The general structure of the solutions

This section is devoted to stating and proving some basic results concerning the solution of the minimization problem (1.4). Albeit some of these properties may be known, we present here the proofs for the reader's convenience.

We start by stating a well-known result about the jump set of the solution (for a proof see, for instance, [14]).

Theorem 3.1. Let $f \in L^1((a, b))$ and let $u \in BV((a, b))$ be a solution of (1.4). If f is constant in $(c, d) \subset (a, b)$, then u is constant in (c, d) .

In higher dimension, the inclusion $J_u \subset J_f$ has been proved in [16, 56] in the case $p > 1$, while it is known to be not always true if $p = 1$ (see [22, 31]). The above result allows us to study an equivalent finite-dimensional minimization problem in the case in which f is a piecewise constant function.

Corollary 3.2. Let f be a piecewise constant function in (a, b) , i.e.,

$$f(x) = \sum_{i=1}^k f_i \chi_{(x_{i-1}, x_i)}(x), \quad f_i \in \mathbb{R}.$$

Then any solution u of the minimization problem (1.4) is of the form

$$u(x) = \sum_{i=1}^k u_i \chi_{(x_{i-1}, x_i)}(x) \quad (3.1)$$

for some $(u_i)_{i=1}^k \subset \mathbb{R}$, not necessarily distinct from each other.

In particular, a function u of the form (3.1) is a solution of (1.4) if and only if $\bar{u} := (u_1, \dots, u_k) \in \mathbb{R}^k$ is a solution of the minimization problem

$$\min_{v \in \mathbb{R}^k} G(v), \quad (3.2)$$

where $G : \mathbb{R}^k \rightarrow \mathbb{R}$ is the function defined by

$$G(v) := \sum_{i=2}^k |v_i - v_{i-1}| + \lambda \sum_{i=1}^k L_i |f_i - v_i|^p,$$

where $v = (v_1, \dots, v_k)$ and $L_i := x_i - x_{i-1}$.

Thus, we now concentrate on the study of the minimization problem (3.2).

The cases $p = 1$ and $p > 1$ turn out to be quite different. Heuristically, the difference lies in the fact that, in the first case, the two terms of the energy are *of the same order* while, for $p > 1$, the fidelity term is of higher order than the total variation. This leads to very different behavior of the solutions in the two cases.

Because of the strict convexity of the functional G for $p > 1$, the solution of the minimization problem (3.2) is unique, while for $p = 1$ we have lack of uniqueness (see Proposition 4.1). Nevertheless, it is possible to identify a solution with a particular structure.

Corollary 3.3. *For $p = 1$, there exists a solution u of problem (3.2) such that one has $u_i \in \{f_1, \dots, f_k\}$ for every $i = 1, \dots, k$.*

Proof. For any given quadruple of functions

$$\begin{aligned} s_1 : \{2, \dots, k\} &\rightarrow \{0, 1\}, & s_2 : \{1, \dots, k\} &\rightarrow \{0, 1\}, \\ t_1 : \{2, \dots, k\} &\rightarrow \{0, 1\}, & t_2 : \{1, \dots, k\} &\rightarrow \{0, 1\}, \end{aligned}$$

let us consider the set $\mathcal{A}_{s_1, s_2}^{t_1, t_2} \subset \mathbb{R}^k$ such that

$$\begin{aligned} G(u) &= \sum_{i=2}^k (-1)^{s_1(i)} t_1(i) (u_i - u_{i-1}) + \lambda \sum_{i=1}^k (-1)^{s_2(i)} t_2(i) L_i (f_i - u_i) \\ &= v_{\lambda}^{s_1, s_2, t_1, t_2} \cdot u + c_{\lambda}^{s_1, s_2, t_1, t_2} \end{aligned}$$

for all $u \in \mathcal{A}_{s_1, s_2}^{t_1, t_2}$, where

$$c_{\lambda}^{s_1, s_2, t_1, t_2} \in \mathbb{R} \quad \text{and} \quad v_{\lambda}^{s_1, s_2, t_1, t_2} \in \mathbb{R}^k.$$

The result then follows by noticing that G restricted to any $\mathcal{A}_{s_1, s_2}^{t_1, t_2} \subset \mathbb{R}^k$ is always minimized by a vector $u \in \mathbb{R}^k$ with

$$u_i = f_{\sigma(i)}$$

for some function $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ and that

$$\min_{\mathbb{R}^k} G = \min_{s_1, s_2, t_1, t_2} \min_{\mathcal{A}_{s_1, s_2}^{t_1, t_2}} G|_{\mathcal{A}_{s_1, s_2}^{t_1, t_2}},$$

as desired. $\square$

We conjecture that, in the case $p = 1$, non-uniqueness of the solution of the minimization problem (1.4) happens only for a finite number of critical values of λ , where a continuum of solutions is present.

Definition 3.4. We will denote by u^{λ} a solution of the minimization problem (3.2) corresponding to the value λ . This will be *the* solution if $p > 1$, while, for $p = 1$, it will be understood as *a* solution whose structure is those given by the previous result.

Remark 3.5. It is easy to see that $u_i \in [\min f, \max f]$ for every solution u and that, in the case where $p > 1$ and f is not constant, it holds that $u^{\lambda} \in (\min f, \max f)$ for all $\lambda > 0$. In particular, for $p > 1$, the solution u^{λ} can never be equal to the initial data f .

In the rest of this section, we seek to understand the behavior of the solution u^{λ} in the limiting cases for λ , i.e., when $\lambda \ll 1$ and when $\lambda \gg 1$. In the former case the predominant term of the energy is given by the total variation, thus we expect u^{λ} to minimize it. We first treat the case $p > 1$.

Lemma 3.6. Fix $p > 1$, positive numbers $(L_i)_{i=1}^k$ and two constants $m < M$. Let

$$\bar{\lambda} := \frac{1}{p(M-m)^{p-1} \max_i L_i}.$$

Then, for any $x_0 < x_1 < \dots < x_k$ with $x_i - x_{i-1} = L_i$ and for any piecewise constant function $f := \sum_{i=1}^k f_i \chi_{(x_{i-1}, x_i)}$ such that $f_i \in [m, M]$ for all $i = 1, \dots, k$, there exists a constant $c \in \mathbb{R}$ such that $u^\lambda \equiv c$ for all $\lambda \in (0, \bar{\lambda}]$.

Proof. Assume that u^λ is not constant and let $i \in \{1, \dots, k\}$ be such that $u_i^\lambda = \min\{u_j^\lambda : j = 1, \dots, k\}$. Let

$$\begin{aligned} r &:= \inf\{j \leq i : u_s = u_i \text{ for all } j \leq s \leq i\}, \\ t &:= \sup\{j \geq i : u_s = u_i \text{ for all } i \leq s \leq j\}. \end{aligned}$$

By hypothesis, either $r > 1$ or $t < k$. Consider, for $\varepsilon > 0$, the vector $u^\varepsilon \in \mathbb{R}^k$ defined by $u_j^\varepsilon := u_j + \varepsilon$ for $j = r, \dots, t$ and $u_j^\varepsilon := u_j^\lambda$ for all the other j 's. Then, recalling that $u_j \in [m, M]$ for all $j = 1, \dots, k$, we have that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{G(u^\varepsilon) - G(u^\lambda)}{\varepsilon} &= a + p\lambda(-1)^{s_i} L_i |u_i - f_i|^{p-1} \\ &\leq a + p\lambda(M-m)^{p-1} \max_{i=1, \dots, k} L_i, \end{aligned} \quad (3.3)$$

where $a \in \{-1, -2\}$ (in particular, $a = -1$ if $r = 1$ or $t = k$ and $a = -2$ otherwise), and $s_i \in \{0, 1\}$. If $\lambda < \bar{\lambda}$, from (3.3) we get that $G(u^\varepsilon) < G(u^\lambda)$. This means that u^λ has to be constant for $\lambda < \bar{\lambda}$. Moreover, it is easy to see that the function G restricted to the set $\{(u_1, \dots, u_k) \in \mathbb{R}^k : u_1 = \dots = u_k\}$ admits a unique minimizer that is independent of λ .

We now have to prove that $u^{\bar{\lambda}}$ is constant. Assume that $u_i^\lambda \equiv c$ for all $\lambda \in (0, \bar{\lambda})$ and all $i = 1, \dots, k$. Let $\bar{c} \in \mathbb{R}^k$ be the vector given by $\bar{c}_i := c$. Then $G_\lambda(c) < G_\lambda(v)$ for all $v \in \mathbb{R}^k$ with $v \neq \bar{c}$ and all $\lambda \in (0, \bar{\lambda})$, where the subscript λ is to underline the dependence of G on λ . By letting $\lambda \nearrow \bar{\lambda}$, we get $G_{\bar{\lambda}}(c) < G_{\bar{\lambda}}(v)$ for all $v \in \mathbb{R}^k$ and thus $u^{\bar{\lambda}} = \bar{c}$. $\square$

In the case $p > 1$, the only thing we can say about the case $\lambda \gg 1$ is that

$$u^\lambda \rightarrow f \quad \text{as } \lambda \rightarrow \infty.$$

Indeed, from

$$\lambda L_i |u_i^\lambda - f_i|^p \leq G(u^\lambda) \leq G(f) < \infty,$$

we conclude that

$$|u_i^\lambda - f_i|^p < \frac{1}{\lambda} \frac{G(f)}{L_i}.$$

We now consider the asymptotic behavior of the solution in the case $p = 1$.

Lemma 3.7. Let $p = 1$ and let $f := \sum_{i=1}^k f_i \chi_{(x_{i-1}, x_i)}$. Set $L_i := x_i - x_{i-1}$ for $i = 1, \dots, k$. Let

$$\bar{\lambda}_1 := \frac{\min_i |f_i - f_{i-1}|}{k(\max_i L_i)(\max f - \min f)}$$

and

$$\bar{\lambda}_2 := \frac{G(f)}{(\min_i L_i)(\min_i |f_i - f_{i-1}|)}.$$

Then, for all $\lambda \in (0, \bar{\lambda}_1]$, every solution u^λ of the minimization problem (3.2) is constant, while for all $\lambda \geq \bar{\lambda}_2$ the solution u^λ is unique and is given by f itself.

Proof. We prove this lemma in two steps.

Step 1: the case $\lambda \ll 1$. Suppose that u^λ is not constant. Recalling that $u_i^\lambda \in \{f_1, \dots, f_k\}$, we have that

$$|Du^\lambda|(\Omega) \geq \min_i |f_i - f_{i-1}|.$$

On the other hand, for any function v such that $v \equiv c \in [\min f, \max f]$ in (x_0, x_k) , it holds that

$$G(v) \leq \lambda k (\max_i L_i) (\max f - \min f).$$

Then, for $\lambda < \bar{\lambda}$, the above estimates show that u^λ must be constant.

Finally, in order to prove that also $u^{\bar{\lambda}}$ is constant, we reason as follows: we know that $u^\lambda = \bar{c}^\lambda$ for $\lambda \in (0, \bar{\lambda})$, for some $\bar{c}^\lambda = (c_\lambda, \dots, c_\lambda) \in \mathbb{R}^k$. Take $\lambda_n \nearrow \bar{\lambda}$. Since $c_{\lambda_n} \in [\min f, \max f]$, up to a not relabeled subsequence, we have that $c_{\lambda_n} \rightarrow c$. We conclude that

$$G_{\bar{\lambda}_1}((c, \dots, c)) \leq G_{\bar{\lambda}}(v)$$

for all $v \in \mathbb{R}^k$.

Step 2: the case $\lambda \gg 1$. Suppose that there exists a sequence $\lambda_j \rightarrow \infty$ for which $u_i^{\lambda_j} \neq f_i$ for all j 's (this is possible since k is finite). By recalling that $u_i^{\lambda_j} \in \{f_1, \dots, f_k\}$, we have, for $\lambda_j > \bar{\lambda}_2$, that

$$G(u^{\lambda_j}) \geq \lambda_j L_i |u_i^{\lambda_j} - f_i| > G(f),$$

contradicting the minimality of u^{λ_j} . $\square$

Remark 3.8. Note that, unlike the case $p > 1$, when $p = 1$ and $\lambda \ll 1$, we cannot conclude that there exists a unique constant $c \in \mathbb{R}$ such that $u^\lambda \equiv c$ for all $\lambda \in (0, \bar{\lambda}_1]$.

4 Explicit solutions in a simple case

Here we study the case where $k = 2$. This analysis, albeit its simplicity, is important to underline some features that distinguish the behavior of the solution of the minimization problem (1.4) in the cases $p = 1$ and $p > 1$.

Proposition 4.1. *Let $f_1 < f_2$. Then the solutions u^λ of the minimization problem (3.2) in the case $p = 1$ are the following:*

- If $L_1 > L_2$, set $\lambda_T^1 := \frac{1}{L_2}$. Then

$$\begin{cases} u_1^\lambda = u_2^\lambda = f_1 & \text{for } \lambda < \lambda_T^1, \\ u_1^\lambda = f_1, u_2^\lambda \in [f_1, f_2] & \text{for } \lambda = \lambda_T^1, \\ u_1^\lambda = f_1, u_2^\lambda = f_2 & \text{for } \lambda > \lambda_T^1. \end{cases}$$

- If $L_1 = L_2$, set $\lambda_T^1 := \frac{1}{L_1}$. Then

$$\begin{cases} u_1^\lambda \in [f_1, f_2], u_2^\lambda \geq u_1 & \text{for } \lambda \leq \lambda_T^1, \\ u_1^\lambda = f_1, u_2^\lambda = f_2 & \text{for } \lambda > \lambda_T^1. \end{cases}$$

- If $L_1 < L_2$, set $\lambda_T^1 := \frac{1}{L_1}$. Then

$$\begin{cases} u_1^\lambda = u_2^\lambda = f_2 & \text{for } \lambda < \lambda_T^1, \\ u_1^\lambda \in [f_1, f_2], u_2^\lambda = f_2 & \text{for } \lambda = \lambda_T^1, \\ u_1^\lambda = f_1, u_2^\lambda = f_2 & \text{for } \lambda > \lambda_T^1. \end{cases}$$

Proof. It is easy to see that we must have $f_1 \leq u_1 \leq u_2 \leq f_2$. Thus, we consider the region

$$\mathcal{T} := \{(u_1, u_2) \in \mathbb{R}^2 : f_1 \leq u_1 \leq u_2 \leq f_2\}, \quad (4.1)$$

and we rewrite the function G in $\mathcal{T}$ as

$$G(u) = [\lambda L_1 - 1]u_1 + [1 - \lambda L_2]u_2 + \lambda[f_2 L_2 - f_1 L_1] = v_\lambda \cdot u + c_\lambda.$$

When minimizing G in $\mathcal{T}$, we can drop the term c_λ . Note that $v_\lambda = 0$ if and only if $L_1 = L_2$ and $\lambda = 1/L_1$. In this case we have $G(u) = f_2 - f_1$ for all u with $u_1 \leq u_2$. Moreover, $v_0 = (-1, 1)$ and $v_\lambda \rightarrow v_\infty$ as $\lambda \rightarrow \infty$, where

$$v_\infty := \left(\frac{L_1}{\sqrt{L_1^2 + L_2^2}}, -\frac{L_2}{\sqrt{L_1^2 + L_2^2}} \right).$$

Figure 1: The case $L_1 < L_2$.

The minimizers of G over the triangle $\mathcal{T}$ are thus simply given by considering the intersection of

$$\partial\mathcal{T} \cap \{(x, y) \in \mathbb{R}^2 : (x, y) \cdot u_\lambda \leq 0\}$$

with a line orthogonal to v_λ . In the case $L_1 < L_2$, the vector $\frac{v_\lambda}{|v_\lambda|}$ spans the two colored regions in Figure 1, for $\lambda \in [0, \infty)$. When in the grey region, namely when $\lambda < 1/L_1$, the minimizer is given by (f_2, f_2) , while for $\lambda > 1/L_1$ the minimizer is given by (f_1, f_2) . Note that the non-uniqueness happens only when the vector v_λ is orthogonal to $\{x = y\} \subset \mathbb{R}^2$. $\square$

In the case $p > 1$ the landscape of the solutions is quite different.

Proposition 4.2. *Let $f_1 < f_2$ and let $p > 1$. Define*

$$\lambda_T^p := \frac{1}{p} \frac{(L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}})^{p-1}}{L_1 L_2 (f_2 - f_1)^{p-1}}.$$

The solution u^λ of the minimization problem (3.2) is the following:

- For $\lambda \leq \lambda_T^p$, we have

$$u_1^\lambda = u_2^\lambda = \frac{L_1^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} f_1 + \frac{L_2^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} f_2, \quad (4.2)$$

- For $\lambda > \lambda_T^p$, we have

$$u_1^\lambda = f_1 + \frac{1}{(p\lambda L_1)^{\frac{1}{p-1}}}, \quad u_2^\lambda = f_2 - \frac{1}{(p\lambda L_2)^{\frac{1}{p-1}}}. \quad (4.3)$$

Proof. Recalling that $f_1 \leq u_1 \leq u_2 \leq f_2$, we just have to consider the region $\mathcal{T}$ defined in (4.1) and to rewrite the function G in that region as

$$G(u_1, u_2) := u_2 - u_1 + \lambda L_1 (u_1 - f_1)^p + \lambda L_2 (f_2 - u_2)^p.$$

The critical point of G is given by

$$u_1 = f_1 + \frac{1}{(p\lambda L_1)^{\frac{1}{p-1}}}, \quad u_2 = f_2 - \frac{1}{(p\lambda L_2)^{\frac{1}{p-1}}},$$

and it belongs to the interior of $\mathcal{T}$, i.e., $u_1^\lambda < u_2^\lambda$, only for $\lambda > \lambda_T^p$. Since G is strictly convex, this critical value turns out to be the global minimizer of G for $\lambda > \lambda_T^p$. In the case $\lambda \leq \lambda_T^p$, the point of minimum has to be on $\partial\mathcal{T}$. Instead of performing all the computations for finding the minimum point in all of the three edges of $\partial\mathcal{T}$ and to compare them, we will use the following argument based on the continuity of the minimizer u^λ with respect to λ (see Lemma 5.1), i.e., we invoke the fact that the function $\lambda \mapsto u^\lambda$ is continuous. Note that for $\lambda \searrow \lambda_T^p$ we have

$$u_\lambda \rightarrow (\bar{u}, \bar{u}),$$

where

$$\bar{u} := \frac{L_1^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} f_1 + \frac{L_2^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} f_2$$

is independent of λ . By using the continuity of the solution, we can conclude that, for $\lambda = \lambda_T^p$, the solution of the minimization problem is given by $(\bar{u}, \bar{u})$. The conclusion for $\lambda < \lambda_T^p$ follows from the result of Theorem 5.3. $\square$

Remark 4.3. We remark a couple of facts:

- (i) We have that $\lambda_T^p \rightarrow \lambda_T^1$ as $p \rightarrow 1^+$ (in each of the cases for the definition of the second one). Indeed, suppose that $L_1 < L_2$. Then

$$\begin{aligned} \lim_{p \rightarrow 1^+} \lambda_T^p &= \lim_{p \rightarrow 1^+} \frac{(L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}})^{p-1}}{L_1 L_2} \\ &= \frac{1}{L_1} \lim_{p \rightarrow 1^+} \left(1 + \left(\frac{L_1}{L_2} \right)^{\frac{1}{p-1}} \right)^{p-1} \\ &= \frac{1}{L_1} \lim_{t \rightarrow 0^+} \exp \left[t \log \left[\left(\frac{L_1}{L_2} \right)^{\frac{1}{t}} + 1 \right] \right] \\ &= \frac{1}{L_2} = \lambda_T^1. \end{aligned}$$

Similar reasonings lead to the claimed result in the other two cases. In particular, note that $\lambda_T^p > \lambda_T^1$.

- (ii) The solutions converge to a solution for $p = 1$, as $p \searrow 1$. Indeed, suppose $\lambda > \lambda_T^1$. Then for p sufficiently close to 1, from the above bullet point, we have that $\lambda > \lambda_T^p$. Thus, the solution of the minimization problem for p is given by (4.3). In this case, it is easy to see that the solution converges to (f_1, f_2) , as $p \searrow 1$. In the case $\lambda < \lambda_T^1$, we can assume as above that p is so close to 1 that the solution of the minimization problem for p is given by (4.2).

If $L_1 > L_2$, then

$$\begin{aligned} \frac{L_1^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} &= \frac{1}{\left(\frac{L_2}{L_1} \right)^{\frac{1}{p-1}} + 1} \rightarrow 1 \quad \text{as } p \rightarrow 1, \\ \frac{L_2^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} &= \frac{1}{\left(\frac{L_1}{L_2} \right)^{\frac{1}{p-1}} + 1} \rightarrow 0 \quad \text{as } p \rightarrow 1. \end{aligned}$$

In the case $L_1 = L_2$, both coefficients are equal to $\frac{1}{2}$.

Finally, in the case $\lambda = \lambda_T^1$, since $\lambda_T^p > \lambda_T^1$, we have that the solution of the minimization problem is given by (4.2). The result follows by arguing as before.

5 The behavior of the solution for $p > 1$

This section contains the main result of this paper, namely Theorem 5.3, that is derived from the qualitative properties of the solutions proved in the following two lemmas and in Proposition 5.5. Although the same result can be deduced by using the semigroup property of the total variation in dimension one [53], we prefer to get it from more elementary observations of qualitative nature on the behavior of the solution u^λ when the parameter λ varies. These observations allow to predict how the solution behaves when the parameter λ varies.

We start by proving the continuity (with respect to the Euclidean topology of $\mathbb{R}^k$) of the solution u^λ with respect to λ .

Lemma 5.1. *Let $p > 1$. Then $\lambda \mapsto u^\lambda$ is continuous and $\lim_{\lambda \rightarrow \infty} u^\lambda = f$.*

Proof. Fix $\bar{\lambda} > 0$ and let $\lambda_n \rightarrow \bar{\lambda}$. Then $G(u^{\lambda_n}) \leq G(v)$ for all $v \in \mathbb{R}^k$, where equality holds if and only if $v = u^{\lambda_n}$. Since

$$|u^{\lambda_n}| \leq \sqrt{k}\|f\|_{\infty},$$

up to a (not relabeled) subsequence, we have that $u^{\lambda_n} \rightarrow \bar{v}$. Using the continuity of G in both v and λ , we have that $G(\bar{v}) \leq G(v)$ for all $v \in \mathbb{R}^k$. By the uniqueness of the solution, we deduce that $\bar{v} = u^{\bar{\lambda}}$, and that $u^{\lambda_n} \rightarrow u^{\bar{\lambda}}$ for all sequences $\lambda_n \rightarrow \bar{\lambda}$.

To prove the second part of the lemma, we reason as follows. Assume that u^{λ} does not converge to f as $\lambda \rightarrow \infty$. Since $u_i \in [\min f, \max f]$, by compactness, we obtain (up to a not relabeled subsequence) $u^{\lambda} \rightarrow v$ for some $v \neq f$. In particular, there exists an index i such that $|u_i^{\lambda} - f_i| > \varepsilon$ for $\lambda \gg 1$, for some $\varepsilon > 0$. So that

$$+\infty > G(f) \geq G(u^{\lambda}) \geq \lambda |u_i^{\lambda} - f_i| \rightarrow \infty$$

as $\lambda \rightarrow \infty$. This is the desired contradiction. $\square$

We now prove several qualitative properties regarding the behavior of the solution u^{λ} as λ varies. Some of the following results could be stated in a more inclusive way, but since they can be used to deduce qualitative properties of the solutions when no direct analysis can be performed, for clarity of exposition we opt to present each of them separately.

Lemma 5.2. *Let $p > 1$. Then the following properties hold:*

- (i) *Assume that, for $\lambda \in (\lambda_1, \lambda_2)$, there exists a function $\lambda \mapsto \bar{u}^{\lambda}$ such that, for some $r \geq 0$,*

$$\begin{cases} u_i^{\lambda} = u_{i+1}^{\lambda} = \dots = u_{i+r}^{\lambda} = \bar{u}^{\lambda}, \\ u_{i-1}^{\lambda} < \bar{u} < u_{i+r+1}^{\lambda} \quad \text{or} \quad u_{i-1}^{\lambda} > \bar{u} > u_{i+r+1}^{\lambda}; \end{cases}$$

see Figure 2.

Then $\bar{u}^{\lambda}$ is the solution of

$$\min_{c \in (u_{i-1}^{\lambda}, u_{i+r+1}^{\lambda})} \sum_{j=i}^{i+r} L_j |c - f_j|^p.$$

In particular, $\bar{u}^{\lambda}$ is constant in (λ_1, λ_2) .

- (ii) *Assume that, for $\lambda \in (\lambda_1, \lambda_2)$, there exists a function $\lambda \mapsto \bar{u}^{\lambda}$ such that, for some $r \geq 0$,*

$$\begin{cases} u_i^{\lambda} = u_{i+1}^{\lambda} = \dots = u_{i+r}^{\lambda} = \bar{u}^{\lambda}, \\ u_{i-1}^{\lambda}, u_{i+r+1}^{\lambda} < \bar{u}^{\lambda}; \end{cases}$$

see Figure 3.

Then $\lambda \mapsto \bar{u}^{\lambda}$ is increasing.

In particular, in the case $r = 0$, we have

$$u_i^{\lambda} = f_i - \left(\frac{2}{p\lambda L_i} \right)^{\frac{1}{p-1}}.$$

- (iii) *Assume that, for $\lambda \in (\lambda_1, \lambda_2)$, there exists a function $\lambda \mapsto \bar{u}^{\lambda}$ such that, for some $r \geq 0$,*

$$\begin{cases} u_i^{\lambda} = u_{i+1}^{\lambda} = \dots = u_{i+r}^{\lambda} = \bar{u}^{\lambda}, \\ u_{i-1}^{\lambda}, u_{i+r+1}^{\lambda} > \bar{u}^{\lambda}; \end{cases}$$

see Figure 4.

Then $\lambda \mapsto \bar{u}^{\lambda}$ is decreasing.

In particular, in the case $r = 0$, we have

$$u_i^{\lambda} = f_i + \left(\frac{2}{p\lambda L_i} \right)^{\frac{1}{p-1}}.$$

- (iv) *Assume that, for $\lambda \in (\lambda_1, \lambda_2)$, there exists a function $\lambda \mapsto \bar{u}^{\lambda}$ such that, for some $r \geq 0$,*

$$\begin{cases} u_1^{\lambda} = u_2^{\lambda} = \dots = u_r^{\lambda} = \bar{u}^{\lambda}, \\ u_{r+1}^{\lambda} < \bar{u}^{\lambda}; \end{cases}$$

see Figure 5.

Then $\lambda \mapsto \bar{u}^{\lambda}$ is increasing.

In particular, in the case $r = 0$, we have

$$u_i^\lambda = f_1 - \left(\frac{1}{p\lambda L_1} \right)^{\frac{1}{p-1}}.$$

(v) Assume that, for $\lambda \in (\lambda_1, \lambda_2)$, there exists a function $\lambda \mapsto \bar{u}^\lambda$ such that, for some $r \geq 0$,

$$\begin{cases} u_1^\lambda = u_{i+1}^\lambda = \dots = u_r^\lambda = \bar{u}^\lambda, \\ u_{r+1}^\lambda > \bar{u}^\lambda; \end{cases}$$

see Figure 6.

Then $\lambda \mapsto \bar{u}^\lambda$ is decreasing.

In particular, in the case $r = 0$, we have

$$u_i^\lambda = f_1 + \left(\frac{1}{p\lambda L_1} \right)^{\frac{1}{p-1}}.$$

(vi) Assume that, for $\lambda \in (\lambda_1, \lambda_2)$, there exists a function $\lambda \mapsto \bar{u}^\lambda$ such that, for some $r \geq 0$,

$$\begin{cases} u_{k-r}^\lambda = \dots = u_k^\lambda = \bar{u}^\lambda, \\ u_{k-r-1}^\lambda > \bar{u}^\lambda; \end{cases}$$

see Figure 7.

Then $\lambda \mapsto \bar{u}^\lambda$ is decreasing.

In particular, in the case $r = 0$, we have

$$u_k^\lambda = f_k + \left(\frac{1}{p\lambda L_k} \right)^{\frac{1}{p-1}}.$$

(vii) Assume that, for $\lambda \in (\lambda_1, \lambda_2)$, there exists a function $\lambda \mapsto \bar{u}^\lambda$ such that, for some $r \geq 0$,

$$\begin{cases} u_{k-r}^\lambda = \dots = u_k^\lambda = \bar{u}^\lambda, \\ u_{k-r-1}^\lambda < \bar{u}^\lambda; \end{cases}$$

see Figure 8.

Then $\lambda \mapsto \bar{u}^\lambda$ is increasing.

In particular, in the case $r = 0$, we have

$$u_k^\lambda = f_k - \left(\frac{1}{p\lambda L_k} \right)^{\frac{1}{p-1}}.$$

Proof. We start by proving property (i). Suppose that $u_{i-1}^\lambda < \bar{u}^\lambda < u_{i+r+1}^\lambda$. In the other case we argue in a similar way. By hypothesis, the vector u^λ minimizes the function G in the set

$$\{(u_1, \dots, u_k) \in \mathbb{R}^k : u_{i-1} < u_i = \dots = u_{i+r} < u_{i+r+1}\},$$

and in this set the function G can be written as

$$G(u) = \tilde{G}(u_1, \dots, u_{i-1}, u_{i+r+1}, \dots, u_k) + \lambda \sum_{j=i}^{i+r} L_j |\bar{u} - f_j|^p.$$

By keeping $u_1, \dots, u_{i-1}$ and $u_{i+r+1}, \dots, u_k$ fixed, the claim follows by minimizing the above quantity with respect to $\bar{u}$.

Since all the other properties can be proved with an argument whose general lines are similar, we just prove property (ii), leaving the details of the others proofs to the reader.

In the hypothesis of (ii), it holds that u^λ is a minimizer of G in the set

$$\{(u_1, \dots, u_k) \in \mathbb{R}^k : u_{i-1}, u_{i+r+1} < u_i = \dots = u_{i+r}\}.$$

Figure 2: The situation of case (i) of Lemma 5.2.

Figure 3: The situation of case (ii) of Lemma 5.2.

Figure 4: The situation of case (iii) of Lemma 5.2.

Figure 5: The situation of case (iv) of Lemma 5.2.

Figure 6: The situation of case (v) of Lemma 5.2.

Figure 7: The situation of case (vi) of Lemma 5.2.

Figure 8: The situation of case (vii) of Lemma 5.2.

Restricted to this set, the function G can be written as

$$G(u) = \widetilde{G}(u_1, \dots, u_{i-1}, u_{i+r+1}, \dots, u_k) + 2\bar{u} + \lambda \sum_{j=i}^{i+r} L_j |\bar{u} - f_j|^p.$$

So, for $\lambda \in (\lambda_1, \lambda_2)$ and $u_1, \dots, u_{i-1}, u_{i+r+1}, \dots, u_k$ fixed, $\bar{u}^\lambda$ is the minimizer of the strictly convex function

$$H(c) := 2c + \lambda \sum_{j=i}^{i+r} L_j |c - f_j|^p$$

in the set $(\max\{u_i^\lambda, u_{i+r}^\lambda\}, \max f)$.

To study the minimizer of H , we can assume without loss of generality that $f_i < f_{i+1} < \dots < f_{i+r}$. Indeed, we note that the order of the f_j 's does not matter. Moreover, in the case in which $f_p = f_q$ for some $p \neq q$, we can simply collect the two terms in a single one and use $L_p + L_q$ as a corresponding factor in the above summation. We now want to prove that $\lambda \mapsto \bar{u}$ is decreasing. Note that the function H can be written as

$$H(c) = 2c + \lambda \sum_{j=i}^m L_j (c - f_j)^p + \lambda \sum_{j=m+1}^{i+r} L_j (f_j - c)^p$$

if $c \in (f_m, f_{m+1}]$, for some $m \in \{i, \dots, i+r-1\}$, and

$$H(c) = 2c + \lambda \sum_{j=i}^{i+r} L_j (c - f_j)^p$$

if $c \in [f_{i+r}, \max f)$. Consider the function H in the interval (f_m, f_{m+1}) . We have that

$$H'(c) = 2 + p\lambda \left[\sum_{j=i}^m L_j (c - f_j)^{p-1} - \sum_{j=m+1}^{i+r} L_j (f_j - c)^{p-1} \right].$$

Here $H'(c) = 0$ has a solution only if the term in the parenthesis is negative and if so, then let $\lambda \mapsto c^\lambda$ be such a solution. It is easy to see that this function is regular in (f_m, f_{m+1}) . By differentiating the expression $H'_m(c^\lambda)$ with respect to λ , we obtain

$$p \left[\sum_{j=i}^m L_j (c - f_j)^{p-1} - \sum_{j=m+1}^{i+r} L_j (f_j - c)^{p-1} \right] + \lambda \frac{dc^\lambda}{d\lambda} p(p-1) \left[\sum_{j=i}^m L_j (c - f_j)^{p-2} + \sum_{j=m+1}^{i+r} L_j (f_j - c)^{p-2} \right] = 0.$$

Thus, by recalling that the term in the first parenthesis is negative, we get $\frac{dc^\lambda}{d\lambda} < 0$, as desired.

In the case in which the minimizer of the function H is reached at a point $c = f_{m+1}$, we simply consider the function H and we apply the argument above.

Finally, the same reasoning applies when $c \in [f_{i+r}, \max f)$. $\square$

We are now in position to prove the fundamental result we will use to develop our strategy for finding the solution.

Theorem 5.3. *For each $i = 1, \dots, k-1$ there exists $\lambda_i \in (0, \infty)$ such that $u_i^\lambda = u_{i+1}^\lambda$ for $\lambda \leq \lambda_i$, while $u_i^\lambda \neq u_{i+1}^\lambda$ for $\lambda > \lambda_i$.*

Proof. We prove this theorem in two steps.

Step 1. We claim that if $u_i^{\bar{\lambda}} = u_{i+1}^{\bar{\lambda}}$ for some $\bar{\lambda} > 0$, then $u_i^\lambda = u_{i+1}^\lambda$ for all $\lambda \in (0, \bar{\lambda}]$. Indeed, let

$$\bar{\lambda} := \min\{\lambda : u_i^\mu = u_{i+1}^\mu \text{ for all } \mu \in [\lambda, \bar{\lambda}]\},$$

and assume that $\bar{\lambda} > 0$. By continuity of $\lambda \mapsto u^\lambda$, there exists $\varepsilon > 0$ such that $u_i^\lambda \neq u_{i+1}^\lambda$ for $\lambda \in (\bar{\lambda} - \varepsilon, \bar{\lambda})$. Consider the case in which $u_i^\lambda < u_{i+1}^\lambda$ in $(\bar{\lambda} - \varepsilon, \bar{\lambda})$ (the other case can be treated similarly).

If $i = 1$, then property (v) of Lemma 5.2 tells us that $\lambda \mapsto u_i^\lambda$ is decreasing in $(\bar{\lambda} - \varepsilon, \bar{\lambda})$ and thus it is not possible to have $u_i^{\bar{\lambda}} = u_{i+1}^{\bar{\lambda}}$.

If $i > 1$, we can focus, without loss of generality, only on the following two cases: $u_{i-1}^\lambda > u_i^\lambda$ and $u_{i-1}^\lambda < u_i^\lambda$ in $(\bar{\lambda} - \varepsilon, \bar{\lambda})$.

In the first case, we get a contradiction since by property (iii) of Lemma 5.2 the map $\lambda \mapsto u_i^\lambda$ is decreasing in $(\bar{\lambda} - \varepsilon, \bar{\lambda})$ and thus, as above, we cannot have $u_i^\lambda = u_{i+1}^\lambda$.

In the other case, we have $u_{i-1}^\lambda < u_i^\lambda < u_{i+1}^\lambda$ in $(\bar{\lambda} - \varepsilon, \bar{\lambda})$. By using property (i) of Lemma 5.2, we see that this is possible only if $u_i^\lambda = f_i$ for all $\lambda \in (\bar{\lambda} - \varepsilon, \bar{\lambda})$. This yields the desired contradiction.

Step 2. Let us define

$$\lambda_i := \max\{\lambda : u_i^\mu = u_{i+1}^\mu \text{ for all } \mu \leq \lambda\}.$$

Step 1 and the continuity of $\lambda \mapsto u^\lambda$ ensure that λ_i is well defined. Moreover, by Lemma 3.6, we also get that $\lambda_i > 0$ for all $i = i, \dots, k-1$. Finally, the fact that $u^\lambda \rightarrow f$ as $\lambda \rightarrow \infty$ tells us that $\lambda_i < \infty$ for all $i = 1, \dots, k-1$. This concludes the proof. $\square$

Remark 5.4. It is possible to see that the map $\lambda \mapsto u^\lambda$ is smooth (with respect to the Euclidean topology of $\mathbb{R}^k$) for all $\lambda \in (0, \infty) \setminus S$, where $S := \{\lambda_1, \dots, \lambda_{k-1}\} \cup T$, where the λ_i 's are given by Theorem 5.3, and $T := \{\mu_1, \dots, \mu_k\}$ where $\mu_i := \inf\{\lambda : u_i^\lambda = f_i\}$.

Finally, we derive another consequence of Lemma 5.2 that will ensure that the solution is monotone where f is and with the same monotonicity.

Proposition 5.5. Suppose that $f_i < f_{i+1} < \dots < f_{i+r}$. Then the solution u of the minimization problem (3.2) is such that $u_i \leq u_{i+1} \leq \dots \leq u_{i+r}$.

In particular, u has the following structure:

- If $u_i \geq f_{i+r}$, then $u_j = u_i$ for all $j = i, \dots, i+r$.
- If $u_{i+r} \leq f_i$, then $u_j = u_{i+r}$ for all $j = i, \dots, i+r$.
- Otherwise, u is of the form

$$u_j = \begin{cases} u_i & \text{for } j = i, \dots, j_1, \\ f_j & \text{for } j = j_1 + 1, \dots, j_2 - 1, \\ u_{i+r} & \text{for } j = j_2, \dots, k, \end{cases}$$

for some $f_{j_1} \leq u_i < f_{j_1+1}$ and $f_{j_2} \leq u_{i+r} < f_{j_2+1}$, where $j_1 < j_2$.

A similar statement holds in the case $f_i > f_{i+1} > \dots > f_{i+r}$.

Proof. We prove this proposition in two steps.

Step 1. We claim that $u_i \leq u_{i+1} \leq \dots \leq u_{i+r}$.

Suppose that $u_{j-1} > u_j$ for some $j \in \{i+1, \dots, i+r\}$. We have to treat three cases: $u_j < f_j$, $u_j = f_j$ and $u_j > f_j$.

In the first case, we get a contradiction with the minimality of u^λ since it is easy to see that

$$G(u_1^\lambda, \dots, u_{j-1}^\lambda, u_j^\lambda + \varepsilon, u_{j+1}^\lambda, \dots, u_k^\lambda) < G(u^\lambda)$$

for $\varepsilon > 0$ small.

Now, suppose $u_j > f_j$ and that $u_j > u_{j+1}$. Then, for $\varepsilon > 0$ small,

$$G(u_1^\lambda, \dots, u_{j-1}^\lambda, u_j^\lambda - \varepsilon, u_{j+1}^\lambda, \dots, u_k^\lambda) < G(u^\lambda),$$

yielding the desired contradiction.

Finally, we can treat all the remaining cases (namely when $u_j = f_j$ or the case where $u_j > f_j$ and $u_{j+1} > u_j$) simultaneously as follows: let us denote by $j_m \in \{i, \dots, j\}$ the minimum index r such that $u_r > u_{r+1}$. In both cases we have $u_{j_m} > f_{j_m}$, and thus

$$G(u_1^\lambda, \dots, u_{j_m-1}^\lambda, u_{j_m}^\lambda - \varepsilon, u_{j_m+1}^\lambda, \dots, u_k^\lambda) < G(u^\lambda)$$

for $\varepsilon > 0$ small.

Step 2. Using Step 1, we have that

$$\sum_{j=i+1}^{i+r} |u_j^\lambda - u_{j-1}^\lambda| = u_{i+r}^\lambda - u_i^\lambda.$$

Since this value is invariant under modification of u_j^λ for $j = i + 1, \dots, i + r - 1$, if we keep u_i and u_{i+r} fixed, the minimality of u^λ implies that

$$\sum_{j=i}^{i+r} |u_j - f_j|^p = \min_{\mathcal{A}} \sum_{j=i}^{i+r} |v_j - f_j|^p,$$

where

$$\mathcal{A} := \{(v_{i+1}, \dots, v_{i+r-1}) \in \mathbb{R}^{i+r-2} : u_i \leq v_{i+1} \leq \dots \leq v_{i+r-1} \leq u_{i+r}\}.$$

This proves the second part of the statement of the proposition. $\square$

6 A method for finding the solution

In this section, we describe the method we propose in order to identify the solution of the minimization problem (3.2). The general idea is, for every $\lambda > 0$, to be able to tell *a priori* the relative position of each u_i^λ with respect to u_{i-1}^λ and f_i . Knowing that allows us to

(i) know if the minimization of G has to take place in some subspace

$$\{v_{i_1-1} = v_{i_1}\} \cap \dots \cap \{v_{i_r-1} = v_{i_r}\},$$

and hence if we have to reduce the number of variables G depends on;

(ii) write explicitly the absolute values present in the expression of G .

If we are able to do that, we can reduce the problem of minimizing the functional G to the problem of minimizing a strictly convex functional of class C^1 , and thus the minimizer can be found by solving the appropriate Euler–Lagrange equation.

Let $f := \sum_{i=1}^k f_i \chi_{(x_{i-1}, x_i]}$ and set $L_i := x_i - x_{i-1}$ for $i = 1, \dots, k$. Fix $p > 1$ and $\bar{\lambda} > 0$.

Step 0: initialization. For every $i \in \{2, \dots, k\}$ set

$$\bar{r}_i := \text{sgn}(f_i - f_{i-1}),$$

$$\bar{t}_i := 0,$$

$$T := \sum_{i=2}^k t_i.$$

Finally, set

$$\bar{s}_1 := \text{sgn}(f_2 - f_1),$$

$$\bar{s}_k := \text{sgn}(f_k - f_{k-1})$$

and, for every $i \in \{2, \dots, k-1\}$,

$$\bar{s}_i := \frac{\text{sgn}(f_i - f_{i-1}) + \text{sgn}(f_i - f_{i+1})}{2}.$$

Step 1: Solving the Euler–Lagrange equations. Consider the functional $\tilde{G} : V \rightarrow [0, \infty)$ defined by

$$\tilde{G}(v) := \sum_{i=2}^k \bar{r}_i (v_i - v_{i-1}) + \lambda \sum_{i=1}^k L_i \bar{s}_i (f_i - v_i)^p,$$

where

$$V := \{v = (v_1, \dots, v_k) \in \mathbb{R}^k : v_i = v_{i-1} \text{ if } \bar{t}_i = 1\}.$$

Find a solution u_i^λ of the i -th Euler–Lagrange equation of G . Note that in the case $p = 2$ this is a set of $k - T$ linear equations. In the case $\bar{s}_i = 0$, set $u_i^\lambda := f_i$. It can happen that $\text{sgn}(f_i - u_i^\lambda)$ changes when varying λ . In that case we have to change $\bar{s}_i$.

Step 2: Critical threshold. Find $\bar{\lambda}$ as the greatest value of λ for which there exists $i \in \{2, \dots, k\}$ such that $u_i^\lambda = u_{i-1}^\lambda$.

Step 3: Determination of the new functional. For every $i = 2, \dots, k$ set

$$\begin{aligned}\bar{r}_i &:= \operatorname{sgn}(u_{i+1}^{\bar{\lambda}} - u_i^{\bar{\lambda}}), \\ \bar{t}_i &:= \begin{cases} 1 & \text{if } u_i^{\bar{\lambda}} = u_{i-1}^{\bar{\lambda}}, \\ 0 & \text{otherwise,} \end{cases} \\ T &:= \sum_{i=2}^k t_i, \\ \bar{s}_i &:= \operatorname{sgn}(f_i - u_i^{\bar{\lambda}}).\end{aligned}$$

Step 4: Cycle. Repeat Steps 1, 2 and 3 until $T = k$ or $\lambda \leq \bar{\lambda}$.

The algorithm terminates after, at most, k iterations. Since at every step we have to solve $k - T$ equations (linear in the case $p = 2$), the complexity of the algorithm is $O(k^2)$.

Example. We illustrate the above strategy with a concrete example.

Let $p = 2$, $k = 6$, $L_1 = L_3 = L_5 = 1$ and $L_2 = L_4 = L_6 = 2$. Consider the initial data f given by

$$f_1 = 2, \quad f_2 = 1, \quad f_3 = 3, \quad f_4 = 5, \quad f_5 = 6, \quad f_6 = 4;$$

see Figure 9.

Let us apply the algorithm.

Step 0. We have

$$\bar{r}_2 = -1, \quad \bar{r}_3 = 1, \quad \bar{r}_4 = 1, \quad \bar{r}_5 = 1, \quad \bar{r}_6 = -1$$

and

$$\bar{s}_1 = 1, \quad \bar{s}_2 = -1, \quad \bar{s}_3 = 0, \quad \bar{s}_4 = 0, \quad \bar{s}_5 = 1, \quad \bar{s}_6 = -1.$$

Moreover, $\bar{t}_i = 0$ for every $i = 2, \dots, k$ and $T = 0$.

Iteration 1, step 1. We have to consider the functional

$$\widetilde{G}(v_1, v_2, v_3, v_4, v_5, v_6) := v_1 - 2v_2 + 2v_5 - v_6 + \lambda[(2 - v_1)^2 + 2(1 - v_2)^2 + |v_2 - 3|^2(6 - v_5)^2 + 2(v_6 - 4)^2].$$

The solution u^λ of the Euler–Lagrange equation of $\widetilde{G}$ is given by

$$(u_1^\lambda, u_2^\lambda, u_3^\lambda, u_4^\lambda, u_5^\lambda, u_6^\lambda) = \left(2 - \frac{1}{2\lambda}, 1 + \frac{1}{2\lambda}, 3, 5, 6 - \frac{1}{\lambda}, 4 + \frac{1}{4\lambda}\right).$$

Iteration 1, step 2. For $\bar{\lambda} = 1$, we get $u_1^{\bar{\lambda}} = u_2^{\bar{\lambda}}$ and $u_4^{\bar{\lambda}} = u_5^{\bar{\lambda}}$.

Iteration 1, step 3. We have

$$(u_1^{\bar{\lambda}}, u_2^{\bar{\lambda}}, u_3^{\bar{\lambda}}, u_4^{\bar{\lambda}}, u_5^{\bar{\lambda}}, u_6^{\bar{\lambda}}) = \left(\frac{3}{2}, \frac{3}{2}, 3, 5, 5, \frac{17}{4}\right).$$

Thus,

$$\bar{r}_2 = 0, \quad \bar{r}_3 = 1, \quad \bar{r}_4 = 1, \quad \bar{r}_5 = 1, \quad \bar{r}_6 = -1$$

and

$$\bar{s}_1 = 1, \quad \bar{s}_2 = -1, \quad \bar{s}_3 = 0, \quad \bar{s}_4 = -0, \quad \bar{s}_5 = 1, \quad \bar{s}_6 = -1.$$

Moreover,

$$\bar{t}_2 = 1, \quad \bar{t}_3 = 0, \quad \bar{t}_4 = 0, \quad \bar{t}_5 = 1, \quad \bar{t}_6 = 0$$

and $T = 2$; see Figure 10.

Iteration 2, step 1. We have to consider the functional

$$\begin{aligned}\widetilde{G}(v_1, v_2, v_3, v_4) \\ := 2v_3 - v_1 - v_4 + \lambda[(2 - v_1)^2 + 2(v_1 - 1)^2 + |v_2 - 3|^2 + 2(5 - v_3)^2 + (6 - v_3)^2 + 2(v_4 - 4)^2].\end{aligned}$$

Figure 9: The initial data f .

Figure 10: The behavior of the solution for $\lambda \gg 1$ as λ decreases.

The solution u^λ of the Euler–Lagrange equation of $\widetilde{G}$ is given by

$$(u_1^\lambda, u_2^\lambda, u_3^\lambda, u_4^\lambda, u_5^\lambda, u_6^\lambda) = \left(\frac{4}{3} + \frac{1}{6\lambda}, \frac{4}{3} + \frac{1}{6\lambda}, 3, \frac{16}{3} - \frac{1}{3\lambda}, \frac{16}{3} - \frac{1}{3\lambda}, 4 + \frac{1}{4\lambda} \right).$$

Iteration 2, step 2. For $\bar{\lambda} = \frac{7}{16}$, we get $u_6^{\bar{\lambda}} = u_5^{\bar{\lambda}}$.

Iteration 2, step 3. We have

$$(u_1^{\bar{\lambda}}, u_2^{\bar{\lambda}}, u_3^{\bar{\lambda}}, u_4^{\bar{\lambda}}, u_5^{\bar{\lambda}}, u_6^{\bar{\lambda}}) = \left(\frac{12}{7}, \frac{12}{7}, 3, \frac{32}{7}, \frac{32}{7}, \frac{32}{7} \right).$$

Thus,

$$\bar{r}_2 = 0, \quad \bar{r}_3 = 1, \quad \bar{r}_4 = 1, \quad \bar{r}_5 = 0, \quad \bar{r}_6 = 0$$

and

$$\bar{s}_1 = 1, \quad \bar{s}_2 = -1, \quad \bar{s}_3 = 0, \quad \bar{s}_4 = 1, \quad \bar{s}_5 = 1, \quad \bar{s}_6 = -1.$$

Moreover,

$$\bar{t}_2 = 1, \quad \bar{t}_3 = 0, \quad \bar{t}_4 = 0, \quad \bar{t}_5 = 1, \quad \bar{t}_6 = 1$$

and $T = 3$; see Figure 11.

Iteration 3, step 1. We have to consider the functional

$$\widetilde{G}(v_1, v_2, v_3) := v_3 - v_1 + \lambda[(2 - v_1)^2 + 2(v_1 - 1)^2 + |v_2 - 3|^2 + 2(5 - v_3)^2 + (6 - v_3)^2 + 2(v_3 - 4)^2].$$

The solution u^λ of the Euler–Lagrange equation of $\widetilde{G}$ is given by

$$(u_1^\lambda, u_2^\lambda, u_3^\lambda, u_4^\lambda, u_5^\lambda, u_6^\lambda) = \left(\frac{4}{3} + \frac{1}{6\lambda}, \frac{4}{3} + \frac{1}{6\lambda}, 3, \frac{16}{3} - \frac{1}{3\lambda}, \frac{16}{3} - \frac{1}{3\lambda}, \frac{16}{3} - \frac{1}{3\lambda} \right).$$

Note that for $\lambda = \frac{1}{4}$ we have $u_1^\lambda = f_1$. Thus, for $\lambda < \frac{1}{4}$, we have to consider the functional

$$G(v_1, v_2, v_3) := v_3 - v_1 + \lambda[(v_1 - 2)^2 + 2(v_1 - 1)^2 + |v_2 - 3|^2 + 2(5 - v_3)^2 + (6 - v_3)^2 + 2(v_3 - 4)^2].$$

Hence, the solutions of the Euler–Lagrange equations remain equal to the previous ones.

Iteration 3, step 2. For $\bar{\lambda} = \frac{1}{7}$, we get $u_2^{\bar{\lambda}} = u_3^{\bar{\lambda}}$.

Figure 11: The behavior of the solution for $\lambda \in (\frac{7}{16}, 1]$ as λ decreases.

Figure 12: The behavior of the solution for $\lambda \in (\frac{1}{7}, \frac{7}{16}]$ as λ decreases.

Iteration 3, step 3. We have

$$(u_1^{\bar{\lambda}}, u_2^{\bar{\lambda}}, u_3^{\bar{\lambda}}, u_4^{\bar{\lambda}}, u_5^{\bar{\lambda}}, u_6^{\bar{\lambda}}) = \left(\frac{15}{6}, \frac{15}{6}, 3, 3, 3, 3\right).$$

Thus,

$$\bar{r}_2 = 0, \quad \bar{r}_3 = 0, \quad \bar{r}_4 = 1, \quad \bar{r}_5 = 0, \quad \bar{r}_6 = 0$$

and

$$\bar{s}_1 = -1, \quad \bar{s}_2 = -1, \quad \bar{s}_3 = -1, \quad \bar{s}_4 = 1, \quad \bar{s}_5 = 1, \quad \bar{s}_6 = -1.$$

Moreover,

$$\bar{t}_2 = 1, \quad \bar{t}_3 = 1, \quad \bar{t}_4 = 0, \quad \bar{t}_5 = 1, \quad \bar{t}_6 = 1$$

and $T = 4$; see Figure 12.

Iteration 4, step 1. We have to consider the functional

$$G(v_1, v_2) := v_2 - v_1 + \lambda[(2 - v_1)^2 + 2(v_1 - 1)^2 + |v_2 - 3|^2 + 2(5 - v_2)^2 + (6 - v_2)^2 + 2(v_2 - 4)^2].$$

The solution u^λ of the Euler–Lagrange equation of $\tilde{G}$ is given by

$$(u_1^\lambda, u_2^\lambda, u_3^\lambda, u_4^\lambda, u_5^\lambda, u_6^\lambda) = \left(\frac{7}{4} + \frac{1}{8\lambda}, \frac{7}{4} + \frac{1}{8\lambda}, \frac{7}{4} + \frac{1}{8\lambda}, \frac{24}{5} - \frac{1}{10\lambda}, \frac{24}{5} - \frac{1}{10\lambda}, \frac{24}{5} - \frac{1}{10\lambda}\right).$$

Iteration 4, step 2. For $\tilde{\lambda} = \frac{9}{122}$, we get that all the components are equal to each other.

Iteration 4, step 3. We have

$$(u_1^{\tilde{\lambda}}, u_2^{\tilde{\lambda}}, u_3^{\tilde{\lambda}}, u_4^{\tilde{\lambda}}, u_5^{\tilde{\lambda}}, u_6^{\tilde{\lambda}}) = \left(\frac{31}{9}, \frac{31}{9}, \frac{31}{9}, \frac{31}{9}, \frac{31}{9}, \frac{31}{9}\right).$$

Thus,

$$\bar{r}_2 = 0, \quad \bar{r}_3 = 0, \quad \bar{r}_4 = 0, \quad \bar{r}_5 = 0, \quad \bar{r}_6 = 0$$

and

$$\bar{s}_1 = -1, \quad \bar{s}_2 = -1, \quad \bar{s}_3 = -1, \quad \bar{s}_4 = 1, \quad \bar{s}_5 = 1, \quad \bar{s}_6 = 1.$$

Moreover,

$$\bar{t}_2 = 1, \quad \bar{t}_3 = 1, \quad \bar{t}_4 = 1, \quad \bar{t}_5 = 1, \quad \bar{t}_6 = 1$$

and $T = 5$; see Figure 13.

Figure 13: The behavior of the solution for $\lambda \in (\frac{9}{122}, \frac{1}{10}]$ as λ decreases.

Figure 14: The behavior of the solution for $\lambda < \frac{9}{122}$.

Iteration 5. Finally, for $\lambda \leq \frac{9}{122}$ we have that the solution is given by

$$u_1^\lambda = u_2^\lambda = u_3^\lambda = u_4^\lambda = u_5^\lambda = u_6^\lambda := \frac{31}{9};$$

see Figure 14.

Remark 6.1. The previous example allows us to draw some conclusions on properties of the solution u^λ :

- (i) It is *not* true that if $u_i^\lambda = f_i$, then $u_i^\lambda = f_i$ for all $\lambda \geq \bar{\lambda}$.
- (ii) The function $\lambda \mapsto u_i^\lambda$ is *not* monotone in general. Nevertheless, a change in the monotonicity can happen only if $\lambda = \lambda_i$ or $\lambda = \lambda_{i-1}$.

Remark 6.2. Let us denote by $u^{\lambda,p}$ the solution of problem (3.2) corresponding to p and λ . Although we know that, for every λ fixed, $u^{\lambda,p} \rightarrow v$ as $p \searrow 1$, where v is a solution of problem (3.2) corresponding to λ and $p = 1$, we cannot apply directly our method to find v since analytic computations are difficult to perform in the case $p \in (1, 2)$. Nevertheless, a finer analysis of the behavior of the solution $u^{\lambda,p}$ for $p \in (1, 2)$ is currently under investigation.

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