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# On certain submodules of Weyl modules for $\mathrm{SO}(2n+1,\mathbb{F})$ with $\mathrm{char}(\mathbb{F}) = 2$

Ilaria Cardinali and Antonio Pasini

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**Abstract.** For $k = 1, 2, \ldots, n-1$ let $V_k = V(\lambda_k)$ be the Weyl module for the special orthogonal group $G = \mathrm{SO}(2n+1,\mathbb{F})$ with respect to the $k$-th fundamental dominant weight $\lambda_k$ of the root system of type $B_n$ and put $V_n = V(2\lambda_n)$. It is well known that all of these modules are irreducible when $\mathrm{char}(\mathbb{F}) \neq 2$ while when $\mathrm{char}(\mathbb{F}) = 2$ they admit many proper submodules. In this paper, assuming that $\mathrm{char}(\mathbb{F}) = 2$, we prove that $V_k$ admits a chain of submodules $V_k = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 \supset M_{-1} = 0$ where $M_i \cong V_i$ for $1, \ldots, k-1$ and $M_0$ is the trivial 1-dimensional module. We also show that for $i = 1, 2, \ldots, k$ the quotient $M_i/M_{i-2}$ is isomorphic to the so-called $i$-th Grassmann module for $G$. Resting on this fact we can give a geometric description of $M_{i-1}/M_{i-2}$ as a submodule of the $i$-th Grassmann module. When $\mathbb{F}$ is perfect, $G \cong \mathrm{Sp}(2n,\mathbb{F})$ and $M_i/M_{i-1}$ is isomorphic to the Weyl module for $\mathrm{Sp}(2n,\mathbb{F})$ relative to the $i$-th fundamental dominant weight of the root system of type $C_n$. All irreducible sections of the latter modules are known. Thus, when $\mathbb{F}$ is perfect, all irreducible sections of $V_k$ are known as well.

## 1 Introduction

Let $V := V(2n+1,\mathbb{F})$ be a $(2n+1)$-dimensional vector space over a field $\mathbb{F}$ and let $\eta$ be a non-singular quadratic form of $V$ of Witt index $n$. Let $\Delta$ be the building of type $B_n$ where the elements of type $k = 1, 2, \ldots, n$ are the $k$-dimensional subspaces of $V$ totally singular for $\eta$.


For $1 \leq k \leq n$, let $\mathcal{G}_k$ be the $k$-Grassmannian of $\mathrm{PG}(V)$, that is, $\mathcal{G}_k$ is the point-line geometry where the points are the $k$-dimensional subspaces of $V$ and the lines are the sets $l_{X,Y} = \{Z \mid X \subset Z \subset Y,\ \dim(Z) = k\}$ for a given pair $(X, Y)$ of subspaces of $V$ where $\dim(X) = k-1$ and $\dim(Y) = k+1$. (Note that $X = 0$ when $k = 1$.)

Let $\Delta_k$ be the *$k$-Grassmannian* of $\Delta$. The point-line geometry $\Delta_k$ is the proper subgeometry of $\mathcal{G}_k$ defined as follows. The points of $\Delta_k$ are the $k$-elements of $\Delta$.

When $1 \leq k < n$, the lines of $\Delta_k$ are the lines $l_{X,Y}$ of $\mathcal{G}_k$ with $Y$ totally singular. Note that $\Delta_1$ is the polar space associated to $\eta$.

Let $k = n$. Then the lines of $\Delta_n$ are the sets

$$l_X = \{Z \mid X \subset Z \subset X^\perp,\ \dim(Z) = n,\ Z \text{ totally singular}\} \tag{1.1}$$

where $X$ is a given $(n-1)$-dimensional totally singular subspace of $V$ and $X^\perp$ is the orthogonal of $X$ with respect to (the linearization of) $\eta$. The vector space $X^\perp/X$ is 3-dimensional and $l_X$ is a non-singular conic in the projective plane $\mathrm{PG}(X^\perp/X)$. The geometry $\Delta_n$ is called the *dual polar space* of type $B_n$.

Let $1 \leq k \leq n$ and $W_k := \bigwedge^k V$. Then $\dim(W_k) = \binom{2n+1}{k}$. Let $\iota_k : \mathcal{G}_k \to W_k$ be the embedding of $\mathcal{G}_k$ in $\mathrm{PG}(W_k)$ mapping every point of $\mathcal{G}_k$, that is, every $k$-dimensional subspace $\langle v_1, v_2, \ldots, v_k\rangle$ of $V$, onto the point $\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k\rangle$ of $\mathrm{PG}(W_k)$. Let $\varepsilon_k := \iota_k|_{\Delta_k}$ be the restriction of $\iota_k$ to $\Delta_k$. The map $\varepsilon_k$ is called the *Grassmann embedding* of $\Delta_k$. If $k < n$, then $\varepsilon_k$ maps lines of $\Delta_k$ onto projective lines. If $k = n$, then $\varepsilon_k$ maps lines of $\Delta_k$ onto non-singular conics of $W_k$.

Put $W_k^\circ := \langle \varepsilon_k(\Delta_k)\rangle$. As proved in Cardinali and Pasini [4],

$$\dim(W_k^\circ) = \begin{cases} \binom{2n+1}{k} & \text{if } \mathrm{char}(\mathbb{F}) \neq 2, \\ \binom{2n+1}{k} - \binom{2n+1}{k-2} & \text{if } \mathrm{char}(\mathbb{F}) = 2. \end{cases} \tag{1.2}$$

So, if either $\mathrm{char}(\mathbb{F}) \neq 2$ or $k = 1$, then $W_k^\circ = W_k$ (recall that $\binom{2n+1}{-1} := 0$) while if $\mathrm{char}(\mathbb{F}) = 2$ and $k > 1$, then $W_k^\circ$ is a proper subspace of $W_k$ of codimension $\binom{2n+1}{k-2}$.

Let $G := \mathrm{SO}(2n+1, \mathbb{F})$ be the stabilizer of the form $\eta$ in the special linear group $\mathrm{SL}(V) = \mathrm{SL}(2n+1, \mathbb{F})$. The group $G$ also acts on $W_k$, according to the following rule:

$$g(v_1 \wedge \cdots \wedge v_k) = g(v_1) \wedge \cdots \wedge g(v_k) \quad \text{for } g \in G \text{ and } v_1, \ldots, v_k \in V.$$

Note that $\mathrm{SO}(2n+1, \mathbb{F}) = \mathrm{PSO}(2n+1, \mathbb{F})$, namely $G$ is the adjoint Chevalley group of type $B_n$ defined over $\mathbb{F}$. The universal Chevalley group of type $B_n$ is the spin group $\widetilde{G} = \mathrm{Spin}(2n+1, \mathbb{F})$. If $\mathrm{char}(\mathbb{F}) = 2$, then $\widetilde{G} = G$. If $\mathrm{char}(\mathbb{F}) \neq 2$, then $\widetilde{G} = 2^{\cdot}G$, a non-split central extension of $G$ by a group of order two.

Clearly, $W_k^\circ$ is a $G$-module. It is called the *$k$-Grassmann orthogonal module* for $G$ (also *Grassmann orthogonal module* if $k$ is clear from the context).

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the fundamental dominant weights for the root system of type $B_n$, the nodes of the $B_n$-diagram being numbered in the usual way (see the picture at the beginning of this introduction). For a positive integral combination $\lambda$ of $\lambda_1, \lambda_2, \ldots, \lambda_n$ we denote by $V(\lambda)$ the Weyl module for $\widetilde{G}$ with $\lambda$ as the highest weight. If $\widetilde{G}$ acts unfaithfully on $V(\lambda)$, then it induces $G$ on $V(\lambda)$, namely $V(\lambda)$

is also a $G$-module. For instance, $V(\lambda_k)$ is a $G$-module for every $k < n$ as well as $V(2\lambda_n)$ while if char$(\mathbb{F}) \neq 2$, then $V(\lambda_n)$ is a $\widetilde{G}$-module but not a $G$-module.

We introduce the following notation not to be forced to split our statements into two distinct cases, namely $k < n$ and $k = n$. We put $\lambda_k^\circ = \lambda_k$ for $k < n$ and $\lambda_n^\circ = 2\lambda_n$. We also put $\lambda_0^\circ = 0$ (the null weight). Thus $V(\lambda_0^\circ)$ is the trivial (1-dimensional) $G$-module. Note that

$$\dim(V(\lambda_k^\circ)) = \binom{2n+1}{k} \quad \text{for every } k = 0, 1, \ldots, n, \tag{1.3}$$

as one can see by applying the Weyl dimension formula (see [6, 24.3], for instance). The following is well known (see, e.g., Blok [2]; see also Theorem 2.1 of Section 2.1 of the present paper).

**Theorem 1.1.** *For $k = 1, 2, \ldots, n$ the Grassmann orthogonal module $W_k^\circ$ is a homomorphic image of the Weyl module $V(\lambda_k^\circ)$.*

Therefore, in view of (1.2) and (1.3),

**Corollary 1.2.** *If either* char$(\mathbb{F}) \neq 2$ *or $k = 1$, then $W_k^\circ \cong V(\lambda_k^\circ)$. If* char$(\mathbb{F}) = 2$ *and $k > 1$, then $W_k^\circ$ is a proper homomorphic image of $V(\lambda_k^\circ)$.*

It is worth spending a few more words on the statement of Theorem 1.1. We recall that $V(\lambda_k^\circ)$ is primarily defined as a module for the enveloping algebra of the Lie algebra of $G$. A choice of a $BN$-pair of $G$, equivalently an apartment $\Sigma$ of $\Delta$ and a chamber $C$ of $\Sigma$, is implicit in this way of defining $V(\lambda_k^\circ)$ in such a way that if $V^+$ is the (1-dimensional) weight space of $V(\lambda_k^\circ)$ of weight $\lambda_k^\circ$ and $X_k := \langle e_1, \ldots, e_k \rangle$ is the $k$-element of $C$, then the parabolic subgroup of $G$ stabilizing $X_k$ is the stabilizer of $V^+$ in $G$. The homomorphism $\varphi_k$ from $V(\lambda_k^\circ)$ to $W_k^\circ$ maps $V^+$ onto $\langle e_1 \wedge e_2 \wedge \cdots \wedge e_k \rangle$. This condition uniquely determines $\varphi_k$ modulo scalars. We denote the kernel of $\varphi_k$ by $K_k$. Thus, $W_k^\circ \cong V(\lambda_k^\circ)/K_k$. In view of Corollary 1.2 we have $K_k = 0$ when either char$(\mathbb{F}) \neq 2$ or $k = 1$.

Let char$(\mathbb{F}) = 2$. Then the bilinear form associated to $\eta$ is degenerate with 1-dimensional radical $N_0$. The subspace $N_0$ is called the *nucleus* of the quadratic form $\eta$. It is the unique 1-dimensional subspace of $V$ stabilized by $G$ and it is non-singular.

Suppose firstly that $k > 1$. Given an element $X$ of $\Delta$ of type $k-1$, let St$(X)$ be its upper residue, formed by the elements of $\Delta$ of type $k, k+1, \ldots, n$ that contain $X$. We call St$(X)$ the *star* of $X$. Clearly, St$(X)$ is (the building of) an orthogonal polar space of rank $n-k+1$ defined in $X^\perp/X$. Let $n_X$ be the nucleus of a quadratic form of $X^\perp/X$ associated to the polar space St$(X)$. Then $n_X = N_X/X$ where $N_X = \langle X, N_0 \rangle$. Hence $N_X$ is a point of $\mathcal{G}_k$ and, since $n_X$

belongs to $X^{\perp}/X$, which is spanned by the 1-dimensional subspaces $Y/X$ for $Y$ ranging in the set of points of $\mathrm{St}(X)$, the point $\iota_k(N_X)$ of $\mathrm{PG}(W_k)$ belongs to $\mathrm{PG}(W_k^\circ)$. We define the *$k$-nucleus subspace* $\overline{\mathcal{N}}_k$ of $W_k^\circ$ as the subspace of $W_k^\circ$ spanned by the 1-dimensional subspaces $\iota_k(N_X)$ for $X \in \Delta_{k-1}$ and we denote its preimage in $V(\lambda_k^\circ)$ by the symbol $\mathcal{N}_k$. Thus, $\mathcal{N}_k/K_k \cong \overline{\mathcal{N}}_k$. Note that, since the homomorphism $\varphi_k : V(\lambda_k^\circ) \to W_k^\circ$ is uniquely determined modulo scalars and $\mathcal{N}_k = \varphi_k^{-1}(\overline{\mathcal{N}}_k)$, the submodule $\mathcal{N}_k$ of $V(\lambda_k^\circ)$ is uniquely determined as well.

When $k = 1$, we put $\mathcal{N}_1 = N_0$, regarded as a subspace of $V(\lambda_1^\circ)$ (as we may, since $K_1 = 0$). Clearly, $\mathcal{N}_k$ is stabilized by $G$, whence it is a submodule of $V(\lambda_k^\circ)$. The following is proved in Cardinali and Pasini [4].

**Theorem 1.3.** *Let* $\mathrm{char}(\mathbb{F}) = 2$. *Then*

$$\dim(\mathcal{N}_k) = \dim(V(\lambda_{k-1}^\circ)) = \binom{2n+1}{k-1} \quad \textit{for every } k = 1, 2, \ldots, n,$$

*namely* $\mathcal{N}_k$ *has codimension* $\binom{2n+1}{k} - \binom{2n+1}{k-1} = \binom{2n}{k} - \binom{2n}{k-2}$ *in* $V(\lambda_k^\circ)$.

The following theorem, to be proved in Section 3, is the main result of this paper.

**Theorem 1.4.** *Let* $\mathrm{char}(\mathbb{F}) = 2$. *Then*

$$\mathcal{N}_k \cong V(\lambda_{k-1}^\circ) \quad \textit{for every } k = 1, 2, \ldots, n \quad \textit{(isomorphism of } G\textit{-modules).}$$

*Moreover, if $k > 1$, then the isomorphism from $\mathcal{N}_k$ to $V(\lambda_{k-1}^\circ)$ can be chosen in such a way that it maps $K_k$ onto the submodule $\mathcal{N}_{k-1}$ of $V(\lambda_{k-1}^\circ)$.*

The next corollaries immediately follow from the previous theorem.

**Corollary 1.5.** *Let* $\mathrm{char}(\mathbb{F}) = 2$. *Then $K_k \cong V(\lambda_{k-2}^\circ)$ for every $k = 2, 3, \ldots, n$.*

**Corollary 1.6.** *Let* $\mathrm{char}(\mathbb{F}) = 2$ *and $1 \le k \le n$. Then $V(\lambda_k^\circ)$ admits a chain of submodules $0 \subset M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k = V(\lambda_k^\circ)$ with $M_i \cong V(\lambda_i^\circ)$ for $i = 0, 1, \ldots, k-1$. Moreover, $M_i/M_{i-2} \cong W_i^\circ$ for every $i = 2, 3, \ldots, k$.*

The statements of Theorem 1.3 and Corollary 1.6 can be made sharper when $\mathbb{F}$ is a perfect field of characteristic 2. We shall discuss this case in a few lines, but firstly we recall a few well-known facts regarding symplectic polar Grassmannians and their natural embeddings.

Put $\overline{V} := V(2n, \mathbb{F})$. Let $\overline{\Delta}$ be the building of type $C_n$ associated with the symplectic group $\overline{G} := \mathrm{Sp}(2n, \mathbb{F})$ in its natural action on $\overline{V}$ and, for $k = 1, 2, \ldots, n$, let $\overline{\Delta}_k$ be the $k$-Grassmannian of $\overline{\Delta}$. Then $\overline{\Delta}_k$ is a subgeometry of the $k$-Grassmannian $\overline{\mathcal{G}}_k$ of PG$(\overline{V})$. Put $\overline{W}_k := \bigwedge^k \overline{V}$ and let $\bar{\iota}_k : \overline{\mathcal{G}}_k \to$ PG$(\overline{W}_k)$ be the embedding of $\overline{\mathcal{G}}_k$ mapping every $k$-dimensional subspace $\langle v_1, \ldots, v_k\rangle$ of $\overline{V}$ onto the point $\langle v_1 \wedge \cdots \wedge v_k\rangle$ of PG$(\overline{W}_k)$. Let

$$\bar{\varepsilon}_k := \bar{\iota}_k|_{\overline{\Delta}_k}$$

be the restriction of $\bar{\iota}_k$ to $\overline{\Delta}_k$ and let $\overline{W}_k^{\mathrm{sp}}$ be the subspace of $\overline{W}_k$ spanned by $\bar{\varepsilon}_k(\overline{\Delta}_k)$. Then $\bar{\varepsilon}_k$ is a projective embedding of $\overline{\Delta}_k$ in PG$(\overline{W}_k^{\mathrm{sp}})$, called the *natural* or *Grassmann* embedding of $\overline{\Delta}_k$. The subspace $\overline{W}_k^{\mathrm{sp}}$ of $\overline{W}_k$ is stabilized by $\overline{G}$ in its natural action on $\overline{W}_k$. Thus, $\overline{W}_k^{\mathrm{sp}}$ is a $\overline{G}$-module. It is well known that $\overline{W}_k^{\mathrm{sp}}$ has codimension $\binom{2n}{k-2}$ in $\overline{W}_k$ (see, e.g., [3]). Moreover, the subspace $\overline{W}_k^{\mathrm{sp}}$ is isomorphic to the Weyl module for $\overline{G}$ associated to the $k$-th fundamental dominant weight of the root system of type $C_n$. For the sake of completeness we denote by $\overline{W}_0^{\mathrm{sp}}$ the trivial (1-dimensional) $\overline{G}$-module.

Suppose now that $\mathbb{F}$ is a perfect field of characteristic 2. Then we have $G \cong \overline{G}$ and $\Delta \cong \overline{\Delta}$. Accordingly, $\overline{W}_k^{\mathrm{sp}}$ can be regarded as a $G$-module. The following is proved in Cardinali and Pasini [4]:

**Theorem 1.7.** *Let $\mathbb{F}$ be a perfect field of characteristic 2. Then $V(\lambda_k^\circ)/\mathcal{N}_k \cong \overline{W}_k^{\mathrm{sp}}$ and $\mathcal{N}_k/K_k \cong \overline{W}_{k-1}^{\mathrm{sp}}$ for every $k = 1, 2, \ldots, n$.*

Consequently,

**Corollary 1.8.** *Let $\mathbb{F}$ be a perfect field of characteristic 2 and $1 \le k \le n$. Then $V(\lambda_k^\circ)$ admits a chain of submodules*

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k = V(\lambda_k^\circ)$$

*with $M_i \cong V(\lambda_i^\circ)$ and $M_i/M_{i-1} \cong \overline{W}_i^{\mathrm{sp}}$ for $i = 1, 2, \ldots, k$.*

In general the series $0 \subset M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k = V(\lambda_k^\circ)$ is not a composition series. By Corollary 1.8, when $\mathbb{F}$ is perfect, we can produce a composition series of $V(\lambda_k^\circ)$ by inserting a composition series of $\overline{W}_i^{\mathrm{sp}}$ between $M_{i-1}$ and $M_i$, for every $i = 1, 2, \ldots, k$. In other words, when $\mathbb{F}$ is perfect, the irreducible sections of $V(\lambda_k^\circ)$ are the irreducible sections of $\overline{W}_i^{\mathrm{sp}}$ for $i = 1, 2, \ldots, k$. The latter sections are known (Baranov and Suprunenko [1]; also Premet and Suprunenko [8]). Therefore:

**Corollary 1.9.** *Let $\mathbb{F}$ be a perfect field of characteristic 2 and $1 \le k \le n$. Then all irreducible sections of $V(\lambda_k^\circ)$ are known.*

It is likely that $V(\lambda_k^\circ)$ admits a unique series of submodules $M_0, M_1, \ldots, M_k$ satisfying the properties of Corollary 1.6. In Section 4, assuming that $\mathbb{F}$ is perfect, we shall prove that this is indeed the case for $k \le 4$.

## 2 Preliminaries

In this section we recall a number of known facts on Lie algebras, Chevalley groups and their Weyl modules, either assumed as well known in Section 1 or to be exploited in the proof of Theorem 1.4. We firstly recapitulate the construction of Chevalley groups from simple Lie algebras (Section 2.1). We rely on the works of Humphreys [6] and Steinberg [9] for our exposition. We shall give some details, for the convenience of the reader, but not so many. The reader is referred to [6] and [9] for all we shall miss. In the second part of this section (Sections 2.2 and 2.3) we shall give more detailed information on orthogonal and symplectic Lie algebras.

### 2.1 General setting

Let $\mathfrak{L}_{\mathbb{C}} = \mathfrak{H} \oplus (\bigoplus_{\alpha\in\Phi} \mathfrak{X}_\alpha)$ be a simple Lie algebra of rank $n$ over the complex field $\mathbb{C}$, where $\mathfrak{H}$ is the Cartan subalgebra, $\Phi$ is the set of all roots and $\mathfrak{X}_\alpha$ is the 1-dimensional subalgebra of $\mathfrak{L}_{\mathbb{C}}$ corresponding to the root $\alpha$. Suppose we are given a set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots of $\mathfrak{L}_{\mathbb{C}}$ and let $\Phi^+$ and $\Phi^-$ be, respectively, the set of roots that are positive, or negative, with respect to $\Pi$. Let $\{H_1, \ldots, H_n\} \cup \{X_\alpha, Y_\alpha\}_{\alpha\in\Phi^+}$ be a Chevalley basis of $\mathfrak{L}_{\mathbb{C}}$, where $H_i \in \mathfrak{H}$, $X_\alpha \in \mathfrak{X}_\alpha$ and $Y_\alpha \in \mathfrak{X}_{-\alpha}$. Denote by $\mathfrak{A}_{\mathbb{C}}$ the universal enveloping (associative) algebra of $\mathfrak{L}_{\mathbb{C}}$. Given an ordering on the set $\Phi^+$, put

$$\mathcal{A}^- := \{A_t^-\}_{t\in T}, \quad A_t^- := \prod_{\alpha\in\Phi^+} \frac{Y_\alpha^{t_\alpha}}{t_\alpha!},$$

$$\mathcal{A}^0 := \{A_s^0\}_{s\in S}, \quad A_s^0 := \prod_{i=1}^{n} \binom{H_i}{s_i},$$

$$\mathcal{A}^+ := \{A_r^+\}_{r\in T}, \quad A_r^+ := \prod_{\alpha\in\Phi^+} \frac{X_\alpha^{r_\alpha}}{r_\alpha!}$$

where

$$\binom{H_i}{s_i} := \frac{H_i(H_i - I)\ldots(H_i - (s_i-1)I)}{s_i!}$$

($I$ being the identity element of $\mathfrak{A}_{\mathbb{C}}$), the factors $Y_\alpha^{t_\alpha}/t_\alpha!$ and $X_\alpha^{r_\alpha}/r_\alpha!$ in $A_t^-$ and $A_r^+$ are ordered according to the ordering previously chosen on $\Phi^+$, the letters $T$ and $S$ denote the set of mappings from $\Phi^+$ and $\{1, 2, \ldots, n\}$ respectively to $\mathbb{N}$ and, given $t, r \in T$ and $s \in S$, we write $t_\alpha$, $r_\alpha$ and $s_i$ for $t(\alpha)$, $r(\alpha)$ and $s(i)$.

The set $\mathcal{A} := \mathcal{A}^- \cdot \mathcal{A}^0 \cdot \mathcal{A}^+ = \{A_t^- A_s^0 A_r^+\}_{t,r\in T; s\in S}$ is a linear basis of the algebra $\mathfrak{A}_{\mathbb{C}}$ (Humphreys [6, Chapter 26]). We call the elements of $\mathcal{A}^-$, $\mathcal{A}^0$ and $\mathcal{A}^+$ *monomials* of (−)-, (0)- and (+)-*type* respectively and the elements of $\mathcal{A}$ just *monomials*. The identity element $I$ of $\mathfrak{A}_{\mathbb{C}}$ is regarded as a monomial of (−)-type as well as (0)- and (+)-type.

Every positive root $\alpha \in \Phi^+$ is a sum $\alpha = \sum_{i=1}^n k_{\alpha,i}\alpha_i$ where the coefficients $k_{\alpha,i}$ are non-negative integers. For $A_t^+ \in \mathcal{A}^+$ and $i = 1, 2, \ldots, n$ we set

$$d_i(A_t^+) = \sum_{\alpha\in\Phi} t_\alpha k_{\alpha,i}$$

and we call $d_i(A_t^+)$ the *i-degree* of $A_t^+$. The non-negative integer

$$d(A_t^+) := \sum_{i=1}^n d_i(A_t^+)$$

is the *total degree* of $A_t^+$. Similarly, we put

$$d_i(A_t^-) := -\sum_{\alpha\in\Phi} t_\alpha k_{\alpha,i} \quad (i\text{-}degree \text{ of } A_t^-),$$

$$d(A_t^-) := \sum_{i=1}^n d_i(A_t^-) \qquad (total\ degree \text{ of } A_t^-).$$

The *i-degree* (the *total degree*) of a monomial $A_t^- A_s^0 A_r^+$ is

$$d_i(A_t^- A_s^0 A_r^+) := d_i(A_t^-) + d_i(A_r^+) = \sum_{\alpha\in\Phi^+} k_{\alpha,i}(r_\alpha - t_\alpha)$$

(respectively $d(A_t^- A_s^0 A_r^+) := d(A_t^-) + d(A_r^+)$).

Regarding $\mathfrak{A}_{\mathbb{C}}$ as a ring, we denote by $\mathfrak{A}_{\mathbb{Z}}^-$ and $\mathfrak{A}_{\mathbb{Z}}$ the subrings of $\mathfrak{A}_{\mathbb{C}}$ generated by the monomials of type (−) and all monomials respectively. The ring $\mathfrak{A}_{\mathbb{Z}}^-$, regarded as a $\mathbb{Z}$-module, is free over the basis $\mathcal{A}^-$ while $\mathfrak{A}_{\mathbb{Z}}$ is free over $\mathcal{A}$. We also put $\mathfrak{L}_{\mathbb{Z}} := \mathfrak{L}_{\mathbb{C}} \cap \mathfrak{A}_{\mathbb{Z}} = \langle \{H_i\}_{i=1}^n \cup \{X_\alpha\}_{\alpha\in\Phi}\rangle_{\mathbb{Z}}$.

Let $\Lambda$ be the set of weights of the root system $\Phi$ and $\Lambda^+$ the set of dominant weights (relative to the choice of $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ as the set of simple roots). As in Humphreys [6], for two weights $\lambda, \mu \in \Lambda$ we write $\mu \prec \lambda$ if $\lambda - \mu$ is a sum of positive roots.

Given a dominant weight $\lambda \in \Lambda^+$ let $\mathbf{v}^+$ be a vector of highest weight $\lambda$ and $Z_{\mathbb{C}}(\lambda) := \mathfrak{A}_{\mathbb{C}}\mathbf{v}^+$ be the cyclic $\mathfrak{A}_{\mathbb{C}}$-module associated to $\lambda$ (called *Verma module* in Humphreys [7]). We recall that $\mathbf{v}^+$ is uniquely determined modulo scalars by the following conditions: $H\mathbf{v}^+ = \lambda(H)\mathbf{v}^+$ for every $H \in \mathfrak{H}$ and $X_\alpha \mathbf{v}^+ = 0$ for every $\alpha \in \Phi^+$.

We have $Z_{\mathbb{C}}(\lambda) = \bigoplus_{\mu \in \Lambda, \mu \preceq \lambda} \mathbf{V}_\mu$ where $\mathbf{V}_\mu$ is the *weight space* with weight $\mu$ (Humphreys [6, Chapter 20]). More explicitly, $\mu = \lambda - \sum_{i=1}^n m_i \alpha_i$ for non-negative integers $m_1, \ldots, m_n$ and $\mathbf{V}_\mu$ is the direct sum of the 1-dimensional subspaces $\langle A_t^- \mathbf{v}^+ \rangle$ for $A_t^-$ such that $d_i(A_t^-) = -m_i$ for every $i = 1, 2, \ldots, n$. Moreover, $\mathbf{V}_\mu$ is the $\mu$-eigenspace of $\mathfrak{H}$, namely its vectors are the vectors $\mathbf{x} \in Z_{\mathbb{C}}(\lambda)$ such that $H(\mathbf{x}) = \mu(H)\mathbf{x}$ for every $H \in \mathfrak{H}$. In particular, $\mathbf{V}_\lambda = \langle \mathbf{v}^+ \rangle$. Note also that, for every weight $\mu \preceq \lambda$ and every element $X \in \mathcal{A}$ we have

$$X\mathbf{V}_\mu \subseteq \mathbf{V}_{\mu + \sum_{i=1}^n d_i(X)\alpha_i},$$

with the convention that $\mathbf{V}_{\mu + \sum_{i=1}^n d_i(X)\alpha_i} = 0$ if $\lambda \prec \mu + \sum_{i=1}^n d_i(X)\alpha_i$.

The module $Z_{\mathbb{C}}(\lambda)$ admits a unique maximal proper submodule $J_{\mathbb{C}}(\lambda)$. The *Weyl module* relative to $\lambda$ is defined as the quotient $V_{\mathbb{C}}(\lambda) := Z_{\mathbb{C}}(\lambda)/J_{\mathbb{C}}(\lambda)$. The Weyl module $V_{\mathbb{C}}(\lambda)$ is finite dimensional. In particular, $J_{\mathbb{C}}(\lambda)$ contains all weight spaces $\mathbf{V}_\mu$ with $\mu \prec 0$. It follows that for every $\alpha \in \Phi^+$ both elements $X_\alpha$ and $Y_\alpha$ act on $V_{\mathbb{C}}(\lambda)$ (by left multiplication) as nilpotent mappings. Thus, the exponential map

$$e^{X_\alpha t} := \sum_{i=0}^{\infty} \frac{X_\alpha^i t^i}{i!}$$

can be defined on $V_{\mathbb{C}}(\lambda)$ for every $\alpha \in \Phi$ (and every $t \in \mathbb{C}$). This map is (linear and) invertible. If $\mathbf{X}$ denotes the Dynkin type of the root system $\Phi$, the group $G_{\mathbb{C}}(\lambda) = \langle e^{X_\alpha t} \rangle_{\alpha \in \Phi, t \in \mathbb{C}}$ is the complex Chevalley group of type $\mathbf{X}$ associated to the weight $\lambda$.

Before we continue we fix some notation which allows us to immediately recognize if an element is regarded as an element of $V_{\mathbb{C}}(\lambda)$ or $Z_{\mathbb{C}}(\lambda)$. We denote vectors and subspaces of $Z_{\mathbb{C}}(\lambda)$ by bold characters, as we have done so far, keeping normal font letters for the corresponding vectors and subspaces of $V_{\mathbb{C}}(\lambda)$. Thus a letter such as $\mathbf{v}$ stands for a vector of $Z_{\mathbb{C}}(\lambda)$, while $v$ denotes the vector $\pi_{\mathbb{C}}(\mathbf{v})$, where $\pi_{\mathbb{C}}$ is the canonical projection of $Z_{\mathbb{C}}(\lambda)$ onto $V_{\mathbb{C}}(\lambda)$. Similarly, if $\mathbf{V}_\mu$ is a weight space of $Z_{\mathbb{C}}(\lambda)$, then $V_\mu = \pi_{\mathbb{C}}(\mathbf{V}_\mu)$.

We shall now define the analogues $G_{\mathbb{F}}(\lambda)$ and $V_{\mathbb{F}}(\lambda)$ of $G_{\mathbb{C}}(\lambda)$ and $V_{\mathbb{C}}(\lambda)$ over an arbitrary field $\mathbb{F}$. For that goal we must first define suitable $\mathbb{Z}$-analogues $Z_{\mathbb{Z}}(\lambda)$, $J_{\mathbb{Z}}(\lambda)$ and $V_{\mathbb{Z}}(\lambda)$ of $Z_{\mathbb{C}}(\lambda)$, $J_{\mathbb{C}}(\lambda)$ and $V_{\mathbb{C}}(\lambda)$.

The definition of $Z_{\mathbb{Z}}(\lambda)$ is obvious: we put $Z_{\mathbb{Z}}(\lambda) := \mathfrak{A}_{\mathbb{Z}}\mathbf{v}^+$ ($= \mathfrak{A}_{\mathbb{Z}}^-\mathbf{v}^+$ in view of the previous description of $Z_{\mathbb{C}}(\lambda)$ and its weight spaces). In order to define $J_{\mathbb{Z}}(\lambda)$ we must firstly define $V_{\mathbb{Z}}(\lambda)$.

According to Steinberg [9, p. 17], there exists a basis $B$ of $V_{\mathbb{C}}(\lambda)$ such that $\langle B \rangle_{\mathbb{Z}} = \mathfrak{A}_{\mathbb{Z}}^- v^+$. The equality $\langle B \rangle_{\mathbb{Z}} = \mathfrak{A}_{\mathbb{Z}}^- v^+$ implies that we can choose the basis $B$ of $\mathfrak{A}_{\mathbb{Z}}^- v^+$ in such a way that its elements are monomial vectors of the form $A_t^- v^+$, namely $B \subseteq \mathcal{A}^- v^+$. So, we assume that $B$ has been chosen in that way. We put $V_{\mathbb{Z}}(\lambda) := \langle B \rangle_{\mathbb{Z}} = \mathfrak{A}_{\mathbb{Z}}^- v^+$ ($= \mathfrak{A}_{\mathbb{Z}} v^+$, since $\mathfrak{A}_{\mathbb{Z}}^- \mathbf{v}^+ = \mathfrak{A}_{\mathbb{Z}} \mathbf{v}^+$).

We have

$$V_{\mathbb{Z}}(\lambda) = \bigoplus_{\mu \preceq \lambda} V_{\mu,\mathbb{Z}},$$

where $V_{\mu,\mathbb{Z}} = \pi_{\mathbb{C}}(\mathbf{V}_{\mu,\mathbb{Z}})$ and $\mathbf{V}_{\mu,\mathbb{Z}} = \mathbf{V}_\mu \cap Z_{\mathbb{Z}}(\lambda)$. Denoted by $\overline{\mathbf{B}}_\mu$ the monomial basis of $\mathbf{V}_\mu$ contained in $\mathcal{A}^- \mathbf{v}^+$, the set $\overline{\mathbf{B}}_\mu$ contains a subset $\mathbf{B}_\mu$ such that $\pi_{\mathbb{C}}$ induces a bijection from $\mathbf{B}_\mu$ to a basis $B_\mu = \pi_{\mathbb{C}}(\mathbf{B}_\mu)$ of $V_\mu$ contained in $B$. (We warn that $\mathbf{B}_\mu = \emptyset$ is allowed; this is certainly the case when $\mu \prec 0$, since $V_\mu = 0$ in this case.)

We shall now replace the set $\overline{\mathbf{B}}_\mu \setminus \mathbf{B}_\mu$ with an independent set of vectors $\mathbf{B}'_\mu \subset J_{\mathbb{C}}(\lambda) \cap \langle \overline{\mathbf{B}}_\mu \rangle_{\mathbb{Z}}$ (where $\langle \overline{\mathbf{B}}_\mu \rangle_{\mathbb{Z}}$ stands for the set of integral combinations of vectors of $\overline{\mathbf{B}}_\mu$) such that $\langle \mathbf{B}_\mu \cup \mathbf{B}'_\mu \rangle_{\mathbb{Z}} = \langle \overline{\mathbf{B}}_\mu \rangle_{\mathbb{Z}}$ (whence $\mathbf{B}_\mu \cup \mathbf{B}'_\mu$ is a basis of $\mathbf{V}_\mu$). Having done this we will be able to define $J_{\mathbb{Z}}(\lambda)$.

Let $\mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_r$ be the elements of $\overline{\mathbf{B}}_\mu \setminus \mathbf{B}_\mu$, taken in some given order. We construct $\mathbf{B}'_\mu$ recursively as follows. We put

$$\mathbf{B}'_{\mu,0} := \emptyset.$$

Assuming to have already defined $\mathbf{B}'_{\mu,k}$ for $0 \le k < r$, if $\pi_{\mathbb{C}}(\mathbf{b}'_{k+1}) = 0$, then

$$\mathbf{B}'_{\mu,k+1} := \mathbf{B}'_{\mu,k} \cup \{\mathbf{b}'_{k+1}\}.$$

Otherwise the vector $b'_{k+1} := \pi_{\mathbb{C}}(\mathbf{b}_{k+1}) \in V_\mu \cap \mathcal{A}^- v^+ \subset V_{\mu,\mathbb{Z}}$ is an integral combination of the vectors $b_1, b_2, \ldots$ of $B_\mu$, say $b'_{k+1} = \sum_i z_i b_i$. Clearly we have $\mathbf{b}'_{k+1} - \sum_i z_i \mathbf{b}_i \in J_{\mathbb{C}}(\lambda)$. We put

$$\mathbf{B}'_{\mu,k+1} := \mathbf{B}'_{\mu,k} \cup \left\{\mathbf{b}'_{k+1} - \sum_i z_i \mathbf{b}_i\right\}.$$

Finally,

$$\mathbf{B}'_\mu := \mathbf{B}'_{\mu,r}.$$

Put $J_{\mathbb{Z}}(\lambda) := \langle \bigcup_{\mu \preceq \lambda} \mathbf{B}'_\mu \rangle_{\mathbb{Z}}$. Then the set $(\bigcup_{\mu \preceq \lambda} \mathbf{B}_\mu) \cup (\bigcup_{\mu \preceq \lambda} \mathbf{B}'_\mu)$ is a basis of $Z_{\mathbb{Z}}(\lambda)$ and $J_{\mathbb{Z}}(\lambda) = Z_{\mathbb{Z}}(\lambda) \cap J_{\mathbb{C}}(\lambda)$ (as it follows from the definition of $\mathbf{B}_\mu$ and the properties of the sets $\mathbf{B}'_\mu$ defined above). We can now interpret the space $V_{\mathbb{Z}}(\lambda)$ (previously defined as $V_{\mathbb{Z}}(\lambda) := \langle B \rangle_{\mathbb{Z}} = \mathfrak{A}_{\mathbb{Z}} v^+ = \mathfrak{A}_{\mathbb{Z}}^- v^+$) as the quotient $V_{\mathbb{Z}}(\lambda) := Z_{\mathbb{Z}}(\lambda)/J_{\mathbb{Z}}(\lambda)$.

The projection

$$\pi_{\mathbb{C}} : Z_{\mathbb{C}}(\lambda) \to V_{\mathbb{C}}(\lambda)$$

induces a homomorphism $\pi_{\mathbb{Z}}$ from $Z_{\mathbb{Z}}(\lambda)$ to $V(\lambda)_{\mathbb{Z}}$ and $J_{\mathbb{Z}}(\lambda) = \ker(\pi_{\mathbb{Z}})$.

Now let $\mathbb{F}$ be an arbitrary field. Then we put $\mathfrak{A}_{\mathbb{F}} := \mathbb{F} \otimes_{\mathbb{Z}} \mathfrak{A}_{\mathbb{Z}}$, $\mathfrak{A}_{\mathbb{F}}^- := \mathbb{F} \otimes_{\mathbb{Z}} \mathfrak{A}_{\mathbb{Z}}^-$, $Z_{\mathbb{F}}(\lambda) = \mathbb{F} \otimes_{\mathbb{Z}} Z_{\mathbb{Z}}(\lambda)$, $V_{\mathbb{F}}(\lambda) = \mathbb{F} \otimes_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda)$ and $J_{\mathbb{F}}(\lambda) = \mathbb{F} \otimes_{\mathbb{Z}} J_{\mathbb{Z}}(\lambda)$. Since $V_{\mathbb{Z}}(\lambda) = Z_{\mathbb{Z}}(\lambda)/J_{\mathbb{Z}}(\lambda)$, we also have $V_{\mathbb{F}}(\lambda) = Z_{\mathbb{F}}(\lambda)/J_{\mathbb{F}}(\lambda)$. The Chevalley group $G_{\mathbb{F}}(\lambda)$ is defined as a group of linear transformations of $V_{\mathbb{F}}(\lambda)$, by means of the exponentials $e^{X_\alpha t}$ just as in the complex case. The Lie algebra $\mathfrak{L}_{\mathbb{F}} = \mathbb{F} \otimes_{\mathbb{Z}} \mathfrak{L}_{\mathbb{Z}}$ is the Lie algebra of $G_{\mathbb{F}}(\lambda)$.

We finish this subsection with a result on homomorphic images of Weyl modules, already used implicitly in Section 1.

**Theorem 2.1.** *Let $U$ be a finite dimensional $G_{\mathbb{F}}(\lambda)$-module over $\mathbb{F}$, satisfying the following properties.*

(1) *$U$ is also an $\mathfrak{A}_{\mathbb{F}}$-module and the action of $G_{\mathbb{F}}(\lambda)$ on $U$ is induced by the action of $\mathfrak{A}_{\mathbb{F}}$. Explicitly, for every root $\alpha \in \Phi$ the element $X_\alpha$ acts on $U$ as a nilpotent endomorphism and the element $x_\alpha(t) = e^{X_\alpha t}$ of $G_{\mathbb{F}}(\lambda)$ maps $v$ onto $\sum_{n=0}^{\infty} t^n \frac{X_\alpha^n}{n!}(v)$, for every vector $v \in U$.*

(2) *$U$ admits a highest weight vector of weight $\lambda$, namely a vector $\bar{v}^+$ such that $X_\alpha(\bar{v}^+) = 0$ for every root $\alpha \in \Phi^+$ and $H_i(\bar{v}^+) = \lambda(H_i)\bar{v}^+$ for every $i = 1, 2, \ldots, n$.*

*Then there exists a unique homomorphism of $\mathfrak{A}_{\mathbb{F}}$-modules from $V_{\mathbb{F}}(\lambda)$ to $U$ mapping $v^+$ onto $\bar{v}^+$ (whence $g(v^+)$ onto $g(\bar{v}^+)$ for every element $g \in \mathfrak{A}_{\mathbb{F}}$).*

This theorem is implicit in the final result of Humphreys [7, 3.3]. In [7, 3.3] the field $\mathbb{F}$ is assumed to be algebraically closed, but in the hypotheses of Theorem 2.1 that assumption can be dropped. Indeed we can firstly replace $\mathbb{F}$ with its algebraic closure $\overline{\mathbb{F}}$ and apply the theorem of [7, 3.3] to $G_{\overline{\mathbb{F}}}(\lambda)$ acting on $V_{\overline{\mathbb{F}}}(\lambda)$ and $\overline{\mathbb{F}} \otimes_{\mathbb{F}} U$. Next we turn back to $G_{\mathbb{F}}(\lambda)$ acting on $V_{\mathbb{F}}(\lambda)$ and $U$.

### 2.2 A Chevalley basis of $\mathfrak{o}(2n+1, \mathbb{F})$

Given a field $\mathbb{F}$, let $\mathfrak{L}_{\mathbb{F}}^\circ = \mathfrak{o}(2n+1, \mathbb{F}) = \mathfrak{H} \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{X}_\alpha)$ be the Lie algebra of type $B_n$ over $\mathbb{F}$, where $\mathfrak{H}$, $\Phi$ and $\mathfrak{X}_\alpha$ have the same meaning as in Section 2.1. Let $\mathfrak{A}_{\mathbb{F}}^\circ$ be the enveloping algebra of $\mathfrak{L}_{\mathbb{F}}^\circ$, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ a given basis of simple roots of $\Phi$ and $\{X_\alpha, Y_\alpha\}_{\alpha \in \Phi^+} \cup \{H_i\}_{i=1}^n$ a Chevalley basis of $\mathfrak{L}_{\mathbb{F}}^\circ$ where $Y_\alpha := X_{-\alpha}$, as in Section 2.1.

The roots $\alpha \in \Phi$ are represented by the following vectors of $\mathbb{R}^n$, where $(u_i)_{i=1}^n$ is the natural orthonormal basis of $\mathbb{R}^n$ (see Humphreys [6, 12.1]):

$$(\Pi) \qquad \begin{cases} \alpha_i = u_i - u_{i+1} \text{ for } 1 \le i \le n-1 & \text{(long simple roots)}, \\ \alpha_n = u_n & \text{(short simple root)}, \end{cases}$$

$$(\Phi^+)\qquad \begin{cases} u_i - u_j = \sum_{r=i}^{j-1} \alpha_r & \text{for } 1 \le i < j \le n \quad \text{(long)},\\ u_i = \sum_{r=i}^{n} \alpha_r & \text{for } i = 1, 2, \dots, n \quad \text{(short)},\\ u_i + u_j = \sum_{r=i}^{j-1} \alpha_r + 2\sum_{r=j}^{n} \alpha_r & \text{for } 1 \le i < j \le n \quad \text{(long)}, \end{cases}$$

$$(\Phi^-)\qquad \begin{cases} u_j - u_i = -\sum_{r=i}^{j-1} \alpha_r & \text{for } 1 \le i < j \le n \quad \text{(long)},\\ -u_i = -\sum_{r=i}^{n} \alpha_r & \text{for } i = 1, 2, \dots, n \quad \text{(short)},\\ -u_i - u_j = -\sum_{r=i}^{j-1} \alpha_r - 2\sum_{r=j}^{n} \alpha_r & \text{for } 1 \le i < j \le n \quad \text{(long)}. \end{cases}$$

As in Section 1, let $V = V(2n+1, \mathbb{F})$ and let $\eta$ be a non-singular quadratic form on $V$. Up to rescaling the form $\eta$, we can assume to have chosen an ordered basis $E = (e_1, e_2, \dots, e_{2n+1})$ of $V$ with respect to which

$$\eta(x_1, \dots, x_{2n+1}) = \sum_{i=1}^{n} x_i x_{n+i} + x_{2n+1}^2. \tag{2.1}$$

Regarding $G = \mathrm{SO}(2n+1, \mathbb{F})$ as the subgroup of $\mathrm{SL}(V)$ preserving $\eta$ and $\mathfrak{L}_{\mathbb{F}}^{\circ}$ as the Lie algebra of $G$, the algebra $\mathfrak{L}_{\mathbb{F}}^{\circ}$ acts on $V$ as a subalgebra of $\mathrm{End}(V)$.

We can assume that the chosen Chevalley basis $\{X_\alpha, Y_\alpha\}_{\alpha\in\Phi^+} \cup \{H_i\}_{i=1}^n$ of $\mathfrak{L}_{\mathbb{F}}^{\circ}$ acts as follows on the elements of $E$, where $\delta_{i,j}$ is the Kronecker symbol:

$$\begin{aligned}
X_{u_i-u_j}(e_l) &= \delta_{l,j} e_i - \delta_{l,n+i} e_{n+j} && (1 \le i < j \le n),\\
Y_{u_i-u_j}(e_l) &= \delta_{l,i} e_j - \delta_{l,n+j} e_{n+i} && (1 \le i < j \le n),\\
X_{u_i+u_j}(e_l) &= \delta_{l,n+i} e_j - \delta_{l,n+j} e_i && (1 \le i < j \le n),\\
Y_{u_i+u_j}(e_l) &= -\delta_{l,i} e_{n+j} + \delta_{l,j} e_{n+i} && (1 \le i < j \le n),\\
X_{u_i}(e_l) &= \delta_{l,n+i} e_{2n+1} - 2\delta_{l,2n+1} e_i && (1 \le i \le n),\\
Y_{u_i}(e_l) &= -\delta_{l,i} e_{2n+1} + 2\delta_{l,2n+1} e_{n+i} && (1 \le i \le n),\\
H_i(e_l) &= \delta_{l,i} e_i - \delta_{l,i+1} e_{i+1} - \delta_{l,n+i} e_{n+i}\\
&\qquad + \delta_{l,n+i+1} e_{n+i+1} && (1 \le i < n),\\
H_n(e_l) &= 2\delta_{l,n} e_n - 2\delta_{l,2n} e_{2n}.
\end{aligned}$$

(Recall that $H_i = H_{\alpha_i} = H_{u_i - u_{i+1}}$ if $i < n$ and $H_n = H_{\alpha_n} = H_{u_n}$.) Moreover,

$$H_{u_i-u_j} = \sum_{r=i}^{j-1} H_r = -H_{u_j-u_i} \qquad (1 \le i < j \le n),$$

$$H_{u_i} = 2\sum_{r=i}^{n-1} H_r + H_n = -H_{-u_i} \qquad (1 \le i \le n),$$

$$H_{u_i+u_j} = \sum_{r=1}^{j-1} H_r + 2\sum_{r=j}^{n-1} H_r + H_n = -H_{-u_i-u_j} \quad (1 \le i < j \le n).$$

The elements $X_\alpha$ and $Y_\alpha$ act on $V$ as nilpotent endomorphisms of exponent 2 or 3, according to whether $\alpha$ is long or short. Moreover,

$$\frac{X_{u_i}^2}{2}(e_l) = -\delta_{l,n+i} e_i \quad (1 \le i \le n),$$
$$\frac{Y_{u_i}^2}{2}(e_l) = -\delta_{l,i} e_{n+i} \quad (1 \le i \le n).$$

We now turn to $W_k = \bigwedge^k V$. We firstly fix some notation. We set

$$I := \{1, 2, \dots, 2n+1\} \quad \text{and} \quad E_\wedge := (e_J)_{J \in \binom{I}{k}},$$

where $\binom{I}{k}$ stands for the set of subsets of $I$ of size $k$ and $e_J = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k}$ for every $k$-subset $J = \{j_1, \dots, j_k\}$ of $I$, with the convention that $j_1 < j_2 < \cdots < j_k$. Thus, $E_\wedge$ is the basis of $W_k$ associated to $E$.

The action of $\mathfrak{L}_{\mathbb{F}}^\circ$ on $V$ induces an action of $\mathfrak{L}_{\mathbb{F}}^\circ$ on $W_k$ defined as follows: if $g \in \mathfrak{L}_{\mathbb{F}}^\circ$ and $v = v_1 \wedge v_2 \wedge \cdots \wedge v_k$, then

$$g(v) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge v_{i-1} \wedge g(v_i) \wedge v_{i+1} \wedge \cdots \wedge v_k. \tag{2.2}$$

The action of the elements $X_\alpha^t/t!$ and $Y_\alpha^t/t!$ on $W_k$ can be computed with the help of formula (2.2). We leave these computations for the reader.

Let $\bar{v}^+ := e_1 \wedge e_2 \wedge \cdots \wedge e_k$. Then $\bar{v}^+$ behaves as the maximal vector $v^+$ of $V(\lambda_k^\circ)$ (where $\lambda_k^\circ$ has the meaning stated in Section 1), namely $X_\alpha(\bar{v}^+) = 0$ for any $\alpha \in \Phi^+$ and $H(\bar{v}^+) = \lambda_k^\circ(H)\bar{v}^+$ for every $H \in \mathfrak{H}$. In particular,

$$H_{u_i-u_{i+1}}(\bar{v}^+) = \begin{cases} \bar{v}^+ & \text{if } i = k, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_{u_n}(\bar{v}^+) = \begin{cases} 2\bar{v}^+ & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\lambda_k^\circ = \sum_{i=1}^{k-1} i\alpha_i + k \cdot \sum_{i=k}^{n} \alpha_i \tag{2.3}$$

(see Humphreys [6, 13.2]). Thus,

$$\lambda_k^\circ(H_i) = \langle \lambda_k^\circ, \alpha_i \rangle = \begin{cases} -(i-1) + 2i - (i+1) = 0 & \text{if } i < k < n, \\ -(k-1) + 2k - k = 1 & \text{if } i = k < n, \\ -k + 2k - k = 0 & \text{if } k < i < n, \\ 2k - 2k = 0 & \text{if } k < i = n, \\ -2(n-1) + 2n = 2 & \text{if } k = i = n, \end{cases}$$

which fits with the above values of $H_{u_i - u_{i+1}}(\bar{v}^+)$ and $H_{u_n}(\bar{v}^+)$ (recall that we have $H_i = H_{u_i - u_{i+1}}$ if $i < n$ and $H_n = H_{u_n}$).

Let $W_k^\circ(\mathbb{F}) := \mathfrak{A}_{\mathbb{F}}^\circ(\bar{v}^+)$. Thus $W_k^\circ(\mathbb{F})$ is the module called $W_k^\circ$ in Section 1, but now we keep a record of the field $\mathbb{F}$ in our notation. In view of the above, the homomorphism $\varphi_{k,\mathbb{F}} : V_{\mathbb{F}}(\lambda_k^\circ) \to W_k^\circ(\mathbb{F})$ (Theorem 2.1) can be chosen in such a way that

$$\varphi_{k,\mathbb{F}}(v^+) = \bar{v}^+.$$

This condition uniquely determines $\varphi_{k,\mathbb{F}}$.

As recalled in Section 1, the homomorphism $\varphi_{k,\mathbb{F}}$ is an isomorphism if and only if either $\mathrm{char}(\mathbb{F}) \neq 2$ or $k = 1$. If $\mathrm{char}(\mathbb{F}) = 2$ and $k > 1$, then the kernel $K_k$ of $\varphi_{k.\mathbb{F}}$ has dimension

$$\dim(K_k) = \binom{2n+1}{k-2}.$$

### 2.3 A Chevalley basis of $\mathfrak{sp}(2n, \mathbb{F})$

Let $\mathfrak{L}_{\mathbb{F}}^{\mathrm{sp}} = \mathfrak{sp}(2n, \mathbb{F}) = \mathfrak{C} \oplus (\bigoplus_{\beta \in \overline{\Phi}} \mathfrak{U}_\beta)$ be the Lie algebra of type $C_n$, where $\mathfrak{C}$ is the Cartan subalgebra, $\overline{\Phi}$ is the root system of type $C_n$ and $\mathfrak{U}_\beta$ is the 1-dimensional subalgebra of $\mathfrak{L}_{\mathbb{F}}^{\mathrm{sp}}$ corresponding to the root $\beta$. Let $\mathfrak{A}_{\mathbb{C}}^{\mathrm{sp}}$ denote the enveloping algebra of $\mathfrak{L}_{\mathbb{C}}^{\mathrm{sp}}$. Chosen a basis $\overline{\Pi} = \{\beta_1, \ldots, \beta_n\}$ of simple roots for $\overline{\Phi}$ let $\overline{\Phi}^+$ (respectively $\overline{\Phi}^-$) be the set of roots that are positive (negative) with respect to $\overline{\Pi}$. Let

$$\{C_i\}_{i=1}^n \cup \{U_\beta, V_\beta\}_{\beta \in \overline{\Phi}}$$

be a Chevalley basis of the algebra $\mathfrak{L}_{\mathbb{F}}^{\mathrm{sp}}$, where $C_i \in \mathfrak{C}$ for $i = 1, \ldots, n$, $U_\beta \in \mathfrak{U}_\beta$ and $V_\beta = U_{-\beta} \in \mathfrak{U}_{-\beta}$ for $\beta \in \overline{\Phi}^+$.

The positive roots $\beta \in \overline{\Phi}^+$ are represented by the following vectors of $\mathbb{R}^n$ where $(u_i)_{i=1}^n$ is the natural orthonormal basis of $\mathbb{R}^n$, as in the previous subsection (see Humphreys [6, 12.1]):

$$\begin{aligned}
\beta_i &= u_i - u_{i+1} && \text{for } 1 \leq i \leq n-1 && \text{(short simple)},\\
\beta_n &= 2u_n && && \text{(long simple)},\\
u_i - u_j &= \sum_{r=i}^{j-1} \beta_r && \text{for } 1 \leq i < j \leq n && \text{(short)},\\
2u_i &= 2\sum_{r=i}^{n-1} \beta_r + \beta_n && \text{for } i = 1, 2, \ldots, n && \text{(long)},\\
u_i + u_j &= \sum_{r=i}^{j-1} \beta_r + 2\sum_{r=j}^{n-1} \beta_r + \beta_n && \text{for } 1 \leq i < j \leq n && \text{(short)}.
\end{aligned}$$

The negative roots are the opposite of these. We omit to list them. Note that the short roots of $\overline{\Phi}$ are just the long roots of the root system $\Phi$ of type $B_n$ while the long roots of $\overline{\Phi}$ are the short roots of $\Phi$ multiplied by 2.

With $V = V(2n+1, \mathbb{F})$ and $E = \{e_1, \ldots, e_{2n+1}\}$ as in the previous subsection, let $\overline{V}$ be the hyperplane of $V$ spanned by the subset $\overline{E} = \{e_1, e_2, \ldots, e_{2n}\}$ of $E$. Let $\overline{V}$ be endowed with the non-degenerate alternating form $\varsigma$ represented as follows with respect to the basis $\overline{E}$ of $\overline{V}$:

$$\varsigma((x_1, x_2, \ldots, x_{2n}), (y_1, y_2, \ldots, y_{2n})) = \sum_{i=1}^n x_i y_{n+i} - \sum_{i=1}^n x_{n+i} y_i. \tag{2.4}$$

(Note that if $\mathrm{char}(\mathbb{F}) = 2$, then, according to (2.1), $\varsigma$ is the form induced on $\overline{V}$ by the bilinearization of the quadratic form $\eta$ of $V$.) Regarded $\overline{G} := \mathrm{Sp}(2n, \mathbb{F})$ as the subgroup of $\mathrm{SL}(\overline{V})$ preserving $\varsigma$ and $\mathfrak{L}_{\mathbb{F}}^{\mathrm{sp}}$ as the Lie algebra of $\overline{G}$, the algebra $\mathfrak{L}_{\mathbb{F}}^{\mathrm{sp}}$ acts on $\overline{V}$ as a subalgebra of $\mathrm{End}(\overline{V})$.

We can assume that the chosen Chevalley basis $\{U_\beta, V_\beta\}_{\beta \in \overline{\Phi}^+} \cup \{C_i\}_{i=1}^n$ of $\mathfrak{L}_{\mathbb{F}}^{\mathrm{sp}}$ acts as follows on the elements of $\overline{E}$:

$$\begin{aligned}
U_{u_i - u_j}(e_l) &= \delta_{l,j} e_i - \delta_{l,n+i} e_{n+j} && (1 \leq i < j \leq n),\\
V_{u_i - u_j}(e_l) &= \delta_{l,i} e_j - \delta_{l,n+j} e_{n+i} && (1 \leq i < j \leq n),\\
U_{u_i + u_j}(e_l) &= \delta_{l,n+i} e_j + \delta_{l,n+j} e_i && (1 \leq i < j \leq n),\\
V_{u_i + u_j}(e_l) &= \delta_{l,i} e_{n+j} + \delta_{l,j} e_{n+i} && (1 \leq i < j \leq n),\\
U_{2u_i}(e_l) &= \delta_{l,n+i} e_i && (1 \leq i \leq n),\\
V_{2u_i}(e_l) &= \delta_{l,i} e_{n+i} && (1 \leq i \leq n),
\end{aligned}$$

and

$$C_i(e_l) = \delta_{l,i}e_i - \delta_{l,i+1}e_{i+1} - \delta_{l,n+i}e_{n+i} + \delta_{l,n+i+1}e_{n+i+1} \qquad (1 \leq i < n),$$
$$C_n(e_l) = \delta_{l,n}e_l - \delta_{l,2n}e_{2n}.$$

Note that $U_\beta$ and $V_\beta$ act on $\overline{V}$ as nilpotent endomorphisms of exponent 2 for every $\beta \in \overline{\Phi}^+$. Moreover, the elements

$$X_{u_i-u_j}, Y_{u_i-u_j}, \frac{X_{u_i}^2}{2}, \frac{Y_{u_i}^2}{2}, X_{u_i+u_j}, Y_{u_i+u_j}$$

of $\mathfrak{A}_{\mathbb{F}}^\circ$ stabilize the subspace $\overline{V}$ of $V$ and the following hold for every vector $v \in \overline{V}$

$$\left.\begin{aligned} U_{u_i-u_j}(v) &= X_{u_i-u_j}(v), \quad V_{u_i-u_j}(v) = Y_{u_i-u_j}(v), \\ U_{2u_i}(v) &= -\frac{X_{u_i}^2}{2}(v), \qquad V_{2u_i}(v) = -\frac{Y_{u_i}^2}{2}(v), \end{aligned}\right\} \tag{2.5}$$

$$\left.\begin{aligned} U_{u_i+u_j}(v) &= X_{u_i+u_j}(v) - 2X_{e_i-e_j}\frac{X_{u_j}^2}{2}(v) \\ &= X_{u_i+u_j}(v) + 2U_{e_i-e_j}U_{2u_j}(v), \\ V_{u_i+u_j}(v) &= Y_{u_i+u_j}(v) - 2\frac{Y_{u_j}^2}{2}Y_{e_i-e_j}(v) \\ &= Y_{u_i+u_j}(v) + 2V_{2u_j}V_{e_i-e_j}. \end{aligned}\right\} \tag{2.6}$$

In particular, when char(𝔽) = 2,

$$U_{u_i+u_j}(v) = X_{u_i+u_j}(v) \quad \text{and} \quad V_{u_i+u_j}(v) = Y_{u_i+u_j}(v). \tag{2.7}$$

Moreover,

$$U_{u_i+u_j} = [U_{u_i-u_j}, U_{2u_j}] \quad \text{and} \quad V_{u_i+u_j} = [V_{2u_j}, V_{u_i-u_j}].$$

Consequently,

$$\left.\begin{aligned} U_{u_i+u_j}(v) &= \frac{X_{u_j}^2}{2}X_{u_i-u_j}(v) - X_{u_i-u_j}\frac{X_{u_j}^2}{2}(v), \\ V_{u_i+u_j} &= Y_{u_i-u_j}\frac{Y_{u_j}^2}{2}(v) - \frac{Y_{u_j}^2}{2}Y_{u_i-u_j}(v). \end{aligned}\right\} \tag{2.8}$$

We now turn to $\overline{W}_k = \bigwedge^k \overline{V}$. We take

$$\overline{E}_\wedge := (e_J)_{J\in\binom{\bar{I}}{k}}$$

as a basis for $\overline{W}_k$, where $\bar{I} := \{1, 2, \ldots, 2n\}$. The action of $\mathfrak{L}_{\mathbb{F}}^{\mathrm{sp}}$ on $\overline{V}$ induces an action of $\mathfrak{L}_{\mathbb{F}}^{\mathrm{sp}}$ on $\overline{W}_k$, defined according to the rule (2.2).

Note firstly that $U_\beta$ and $V_\beta$ act on $\overline{W}_k$ as nilpotent endomorphisms of exponent 3 or 2 according to whether $\beta$ is short or long. In view of (2.5) the elements

$$U_{u_i-u_j}, \frac{U^2_{u_i-u_j}}{2}, V_{u_i-u_j}, \frac{V^2_{u_i-u_j}}{2}, U_{2u_j}, V_{2u_j}$$

act on $\overline{W}_k$ as

$$X_{u_i-u_j}, \frac{X^2_{u_i-u_j}}{2}, Y_{u_i-u_j}, \frac{Y^2_{u_i-u_j}}{2}, -\frac{X^2_{u_i}}{2}, -\frac{Y^2_{u_i}}{2}$$

respectively. According to (2.7), denoted by $\langle \overline{E}_\wedge \rangle_{2\mathbb{Z}}$ the set of integral combinations of vectors of $\overline{E}_\wedge$ with even coefficients,

$$\{U_{u_i+u_j}(e_J) - X_{u_i+u_j}(e_J), V_{u_i+u_j}(e_J) - Y_{u_i+u_j}(e_J)\} \subset \langle \overline{E}_\wedge \rangle_{2\mathbb{Z}} \tag{2.9}$$

for every $e_J \in \overline{E}_\wedge$. In particular, if char($\mathbb{F}$) = 2, then $U_{u_i+u_j}$ and $V_{u_i+u_j}$ act on $\overline{W}_k$ in the same way as $X_{u_i+u_j}$ and $Y_{u_i+u_j}$ respectively.

Let $\bar{v}^+ := e_1 \wedge e_2 \wedge \cdots \wedge e_k$ as in the previous subsection and let

$$\overline{W}^{\text{sp}}_k(\mathbb{F}) = \mathfrak{A}^{\text{sp}}_{\mathbb{F}}(\bar{v}^+).$$

Thus, $\overline{W}^{\text{sp}}_k(\mathbb{F})$ is the module called $\overline{W}^{\text{sp}}_k$ in Section 1. By (2.5), (2.6) and (2.7), we see that

$$\overline{W}^{\text{sp}}_k(\mathbb{F}) \subseteq W^\circ_k(\mathbb{F}).$$

Let $\lambda^{\text{sp}}_1, \lambda^{\text{sp}}_2, \ldots, \lambda^{\text{sp}}_n$ be the fundamental dominant weights for the root system of type $C_n$. It is well known that $\overline{W}^{\text{sp}}_k(\mathbb{F})$ is isomorphic to the Weyl module $V_{\mathbb{F}}(\lambda^{\text{sp}}_k)$ and that, if $v^+$ is the highest weight vector chosen in $V_{\mathbb{F}}(\lambda^{\text{sp}}_k)$, we can assume that the isomorphism from $V_{\mathbb{F}}(\lambda^{\text{sp}}_k)$ to $\overline{W}^{\text{sp}}_k$ maps $v^+$ onto $\bar{v}^+$ (see, e.g., Premet and Suprunenko [8]).

## 3 Proof of Theorem 1.4

Throughout this section char($\mathbb{F}$) = 2. If $k = 1$, then $K_k = 0$ and $\mathcal{N}_k$ is 1-dimensional. In fact $\mathcal{N}_1$ is the nucleus of the quadratic form $\eta$ chosen to define $G$. Thus, in this case the first claim of Theorem 1.4 is obvious. The second claim is empty.

Assume $k > 1$. We keep the notation of Sections 2.2 and 2.3. In particular, if $v^+$ is the highest weight vector chosen in $V_{\mathbb{F}}(\lambda^\circ_k)$ and $\bar{v}^+ = e_1 \wedge \cdots \wedge e_k$, we can assume to have chosen the homomorphism $\varphi_{k,\mathbb{F}} : V_{\mathbb{F}}(\lambda^\circ_k) \to W^\circ_k(\mathbb{F})$ in such a way that it maps $v^+$ onto $\bar{v}^+$. We put $K_k := \ker(\varphi_{k,\mathbb{F}})$. Clearly, $K_k$ is an $\mathfrak{A}^\circ_{\mathbb{F}}$-submodule of $V_{\mathbb{F}}(\lambda^\circ_k)$, namely

**Lemma 3.1.** *We have* $\mathfrak{A}^\circ_{\mathbb{F}}(K_k) = K_k$.

Let

$$\begin{aligned} v_1^+ &:= Y_{u_k}(v^+), \\ V_1(\mathbb{F}) &:= \mathfrak{A}^\circ_{\mathbb{F}}(v_1^+), \\ \bar{v}_1^+ &:= \varphi_{k,\mathbb{F}}(v_1^+) = Y_{u_k}(\bar{v}^+) = Y_{u_k}(e_1 \wedge \cdots \wedge e_k) \\ &= -e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{2n+1}. \end{aligned}$$

Our first goal is to prove the following proposition:

**Proposition 3.2.** *We have* $K_k \subseteq V_1(\mathbb{F})$ *and* $V_1(\mathbb{F}) \cong V_{\mathbb{F}}(\lambda^\circ_{k-1})$, *with* $v_1^+ \in V_1(\mathbb{F})$ *corresponding to a highest weight vector of* $V_{\mathbb{F}}(\lambda^\circ_{k-1})$.

### 3.1 Proof of Proposition 3.2

In the sequel we shall often go back and forth between $V_{\mathbb{F}}(\lambda^\circ_k)$ and $V_{\mathbb{C}}(\lambda^\circ_k)$. By a little abuse, we will use the same symbols for vectors of $V_{\mathbb{F}}(\lambda^\circ_k)$ and $V_{\mathbb{C}}(\lambda^\circ_k)$, thus using $v^+$ to denote both the highest weight vector chosen in $V_{\mathbb{F}}(\lambda^\circ_k)$ and the one of $V_{\mathbb{C}}(\lambda^\circ_k)$. Accordingly, the symbol $v_1^+$, which we have introduced to denote the vector $Y_{u_k}(v^+)$ of $V_{\mathbb{F}}(\lambda^\circ_k)$ also denotes the vector $Y_{u_k}(v^+)$ of $V_{\mathbb{C}}(\lambda^\circ_k)$. We do the same in $W_k(\mathbb{C})$. Thus, the symbol $\bar{v}_1^+ = e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{2n+1}$ denotes $\varphi_{k,\mathbb{F}}(v_1^+)$ as well as $\varphi_{k,\mathbb{C}}(v_1^+)$, according to the context. We need one more convention. We put

$$\begin{aligned} 2 \cdot \mathfrak{A}^\circ_{\mathbb{Z}} &:= \{2g \mid g \in \mathfrak{A}^\circ_{\mathbb{Z}}\}, \\ 2 \cdot V_{\mathbb{Z}}(\lambda^\circ_k) &:= \{2v \mid v \in V_{\mathbb{Z}}(\lambda^\circ_k)\} = 2 \cdot \mathfrak{A}^\circ_{\mathbb{Z}}(v^+). \end{aligned}$$

We say that two vectors $v, w \in V_{\mathbb{Z}}(\lambda^\circ_k)$ are *congruent modulo* 2 and we write $v \equiv_2 w$ if $v - w \in 2 \cdot V_{\mathbb{Z}}(\lambda^\circ_k)$. Clearly, the canonical mapping from $V_{\mathbb{Z}}(\lambda^\circ_k)$ to $V_{\mathbb{F}}(\lambda^\circ_k)$ maps $2 \cdot V_{\mathbb{Z}}(\lambda^\circ_k)$ onto 0. Thus, if $v \equiv_2 w$ in $V_{\mathbb{Z}}(\lambda^\circ_k)$, then $v$ and $w$ are equal when regarded as vectors of $V_{\mathbb{F}}(\lambda^\circ_k)$.

**Lemma 3.3.** *The vector* $v_1^+$ *belongs to the weight space* $V_\mu$ *of* $V_{\mathbb{F}}(\lambda^\circ_k)$ *with weight* $\mu = \lambda^\circ_{k-1} = \lambda^\circ_k - \sum_{i=k}^n \alpha_k$.

*Proof.* We have $u_k = \sum_{i=k}^n \alpha_i$. Hence $v_1^+ \in V_\mu$ with $\mu = \lambda^\circ_k - \sum_{i=k}^n \alpha_i$. According to (2.3),

$$\lambda^\circ_k - \sum_{i=k}^{n} \alpha_i = \sum_{i=1}^{k-2} i\alpha_i + (k-1) \sum_{i=k-1}^{n} \alpha_i = \lambda^\circ_{k-1}. \qquad \square$$

**Lemma 3.4.** *We have $X_\alpha(v_1^+) = 0$ for every positive root $\alpha \in \Phi^+$.*

*Proof.* We switch from $V_{\mathbb{F}}(\lambda_k^\circ)$ to $V_{\mathbb{C}}(\lambda_k^\circ)$. Since $V_{\mathbb{C}}(\lambda_k^\circ) \cong W_k^\circ(\mathbb{C})$, we can make our computations inside $W_k^\circ(\mathbb{C})$. It is easy to see that

$$X_{u_i-u_j}(\bar{v}_1^+) = X_{u_i+u_j}(\bar{v}_1^+) = 0$$

for every choice of $1 \le i < j \le n$ while $X_{u_i}(\bar{v}_1^+) = -2e_1 \wedge \cdots \wedge e_{k-1} \wedge e_i$ for every $i = 1, 2, \ldots, n$. Therefore $X_{u_i}(v_1^+) \equiv_2 0$ for every $i = 1, 2, \ldots, n$. It follows that in $V_{\mathbb{F}}(\lambda_k^\circ)$ we have $X_\alpha(v_1^+) = 0$ for every $\alpha \in \Phi^+$. □

**Corollary 3.5.** *The $\mathfrak{A}_{\mathbb{F}}^\circ$-module $V_1(\mathbb{F})$ is a homomorphic image of $V_{\mathbb{F}}(\lambda_{k-1}^\circ)$ and we can choose the homomorphism $\psi_{k-1,\mathbb{F}} : V_{\mathbb{F}}(\lambda_{k-1}^\circ) \to V_1(\mathbb{F})$ in such a way that $\psi_{k-1,\mathbb{F}}$ maps the highest weight vector chosen in $V_{\mathbb{F}}(\lambda_{k-1}^\circ)$ onto $v_1^+$.*

*Proof.* This follows from Theorem 2.1 and Lemmas 3.3 and 3.4. □

Our next step is to prove that $K_k \subseteq V_1(\mathbb{F})$. We firstly compute $Y_{u_i}(v^+)$ and $Y_{u_i}Y_{u_j}(v^+)$ for $i < j$. Recall that $Y_{u_k}(v^+) = v_1^+$ by definition.

**Lemma 3.6.** *If $i > k$, then $Y_{u_i}(v^+) = 0$. If $i < k$, then $Y_{u_i}(v^+) = Y_{u_i-u_k}(v_1^+)$.*

*Proof.* We switch to $V_{\mathbb{Z}}(\lambda_k^\circ)$. As $W_k(\mathbb{C}) = W_k^\circ(\mathbb{C}) \cong V_{\mathbb{C}}(\lambda_k^\circ)$, we can replace $v^+$ with $\bar{v}^+ = e_1 \wedge \cdots \wedge e_k$. Computing in $W_k(\mathbb{C})$ we see that if $i > k$, then

$$Y_{u_i}(\bar{v}^+) = 0$$

while if $i < k$, then

$$Y_{u_i}(\bar{v}^+) = Y_{u_i-u_k}Y_{u_k}(\bar{v}^+) = Y_{u_i-u_k}(\bar{v}_1^+).$$

Hence in $V_{\mathbb{Z}}(\lambda_k^\circ)$ we have $Y_{u_i}(v^+) = 0$ if $i > k$ and $Y_{u_i}(v^+) = Y_{u_i-u_k}(v_1^+)$ if $i < k$. Clearly, the same holds in $V_{\mathbb{F}}(\lambda_k^\circ)$. □

**Corollary 3.7.** *We have $Y_{u_i}(v^+) \in V_1(\mathbb{F})$ for every $i = 1, 2, \ldots, n$.*

**Lemma 3.8.** *If $i < j$, then $Y_{u_i}Y_{u_j}(v^+) \in K_k$.*

*Proof.* If $j > k$, then $Y_{u_i}Y_{u_j}(v^+) = 0$ by Lemma 3.6. Let $j = k$. Then

$$Y_{u_i}Y_{u_j}(\bar{v}^+) = Y_{u_i}Y_{u_k}(\bar{v}^+) = Y_{u_i}(\bar{v}_1^+) = 2 \cdot e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{n+i},$$

which is 0 in $W_k(\mathbb{F})$. Therefore $Y_{u_i}Y_{u_k}(v^+) \in K_k$. Finally, let $j < k$. Then

$$Y_{u_i}Y_{u_j}(v^+) = Y_{u_i}Y_{u_j-u_k}Y_{u_k}(v^+)$$

by Lemma 3.6. However $Y_{u_i}$ and $Y_{u_j-u_k}$ commute, since $-u_i - u_j + u_k$ is not a root. Therefore $Y_{u_i}Y_{u_j}(\bar{v}^+) = Y_{u_j-u_k}Y_{u_i}Y_{u_k}(\bar{v}^+) = 0$ since $Y_{u_i}Y_{u_k}(\bar{v}^+) = 0$, as shown above. □

We need to fix a few more conventions. The first convention has to do with the order in which we take the factors $Y_\alpha^r/r!$ of a monomial element of type $(-)$ of $\mathfrak{A}^\circ_\mathbb{F}$ (or $\mathfrak{A}^\circ_\mathbb{Z}$). Recall that we only need these elements to produce $V_\mathbb{F}(\lambda^\circ_k)$ (respectively $V_\mathbb{Z}(\lambda^\circ_k)$) starting from $v^+$. We assume to write each such element $Y$ as a product

$$Y = \overline{Y}Y'$$

where $Y' = Y_{u_{i_1}}Y_{u_{i_2}}\dots Y_{u_{i_r}}$ with $i_1 < i_2 < \dots < i_r$ (and possibly $r = 0$, namely $Y' = 1$) and $\overline{Y}$ is a product of elements as

$$\frac{Y^2_{u_i}}{2},\, Y_{u_i+u_j},\, \frac{Y^2_{u_i+u_j}}{2},\, Y_{u_i-u_j},\, \frac{Y^2_{u_i-u_j}}{2}$$

(and possibly $\overline{Y} = 1$), taken in some given order. Note that there is no need to consider elements as $Y_\alpha^r/r!$ with $r > 2$, since $Y_\alpha$ acts on $V_\mathbb{Z}(\lambda^\circ_k)$ as a nilpotent endomorphisms of exponent 3. (Indeed, as noticed in Section 2.2, this is true for the action of $Y_\alpha$ on $W_k(\mathbb{C})$ and $W_k(\mathbb{C}) \cong V_\mathbb{C}(\lambda^\circ_k)$.)

**Definition.** With $Y = \overline{Y}Y'$ as above, we call $\overline{Y}$ (respectively $Y'$) the *pseudo-symplectic part* (*non-symplectic part*) of $Y$. If $Y' = Y_{u_{i_1}}Y_{u_{i_2}}\dots Y_{u_{i_r}}$, then we put $\delta(Y) := r$ and we call $\delta(Y)$ the *non-symplectic degree* of $Y$. If $\delta(Y) = 0$, namely $Y' = 1$, then we say that $Y$ is *pseudo-symplectic*, otherwise we say that it is *non-symplectic*. The next corollary immediately follows from Corollary 3.7 and Lemma 3.8.

**Corollary 3.9.** *With $Y$ as above, if $\delta(Y) > 0$, then $Y(v^+) \in V_1(\mathbb{F})$. If $\delta(Y) > 1$, then $Y(v^+) \in K_k$.*

When $\delta(Y) = 1$, a sharper statement can be proved. As in Section 2.3, let $\overline{V}$ be the hyperplane of $V$ spanned by $\overline{E} = \{e_1, \dots, e_{2n}\}$ and put $\overline{W}_{k-1} := \bigwedge^{k-1}\overline{V}$. It follows from the information gathered in Sections 2.2 and 2.3 that $\overline{V}$ is stabilized by all monomials of pseudo-symplectic type. Hence these monomials stabilize $\overline{W}_{k-1}$ too.

By Lemma 3.6 we immediately obtain the following:

**Corollary 3.10.** *Let $Y$ be a monomial element of $(-)$-type with $\delta(Y) = 1$. Thus $Y = \overline{Y}Y_{u_i}$, where $\overline{Y}$ is the pseudo-symplectic part of $Y$. Define $\bar{w} \in \overline{W}_{k-1}$ as follows: $\bar{w} = 0$ if $i > k$, $\bar{w} = e_1 \wedge \dots \wedge e_{k-1}$ if $i = k$ and*

$$\bar{w} = Y_{u_i-u_k}(e_1 \wedge \dots \wedge e_{k-1}) = e_1 \wedge \dots \wedge e_{i-1} \wedge e_k \wedge e_{i+1} \wedge \dots \wedge e_{k-1}$$

*if $i < k$. Then $Y(\bar{v}^+) = \overline{Y}(\bar{w}) \wedge e_{2n+1} \in \overline{W}_{k-1} \wedge e_{2n+1}$.*

Suppose now that for some monomials $Y_1, \dots, Y_m$ of $(-)$-type and scalars $t_1, \dots, t_m \in \mathbb{F}$ we have

$$\left(\sum_{i=1}^{m} t_i Y_i\right)(\bar{v}^+) = 0. \tag{3.1}$$

Regarding $\mathbb{F}$ as a vector space over its prime subfield $\mathbb{F}_2$, let $\{\tau_j\}_{j\in J}$ be a basis of $\mathbb{F}$ over $\mathbb{F}_2$. Thus, $t_i = \sum_{j\in J} \tau_j t_{j,i}$ for suitable scalars $t_{j,i} \in \mathbb{F}_2$. (Note that this sum is meaningful even if $J$ is infinite, since in any case all but finitely many of the scalars $t_{j,i}$ are null.) Accordingly, (3.1) is equivalent to a system of equations

$$\left(\sum_{i=1}^{m} t_{j,i} Y_i\right)(\bar{v}^+) = 0 \quad (j = 0, 1, 2, \dots).$$

Thus there is no loss in assuming that in (3.1) the coefficients $t_1, t_2, \dots, t_m$ are taken from $\mathbb{F}_2$.

The next lemma is the main step in the proof of Proposition 3.2. The interplay between the algebras $\mathfrak{A}_{\mathbb{Z}}^{\mathrm{sp}}$ and $\mathfrak{A}_{\mathbb{Z}}^{\circ}$ fully enters the game in the proof of this lemma (see below, claim (3.2)).

**Lemma 3.11.** *Let $g = \sum_{i=1}^{m} t_i Y_i$ be a linear combination of monomials of type $(-)$ with $t_1, t_2, \dots, t_m \in \mathbb{F}_2$. Suppose that $g(\bar{v}^+) = 0$. Then $g(v^+) \in V_1(\mathbb{F})$.*

*Proof.* We can split $g$ as a sum $g = g_0 + g_1 + g_2$ where $g_0$ is a sum of pseudo-symplectic monomials, $g_1$ is a sum of non-symplectic monomials of degree 1 and $g_2$ is a sum of non-symplectic monomials of degree at least 2. We have $(g_1 + g_2)(v^+) \in V_1(\mathbb{F})$ by Lemma 3.6. It remains to prove that $g_0(v^+) \in V_1(\mathbb{F})$. We shall prove more than this. Indeed we shall show that $g_0(v^+) = 0$.

We have $g_2(\bar{v}^+) = 0$ by Lemma 3.8. Moreover, $g_0(\bar{v}^+) \in \overline{W}_k$ (since all monomial elements of pseudo-symplectic type stabilize $\overline{V}$). Let

$$g_1 = t_r Y_r + t_{r+1} Y_{r+1} + \dots + t_s Y_s$$

where $Y_j = \overline{Y}_j Y_{u_{i_j}}$ for $j = r, r+1, \dots, s$, $\overline{Y}_j$ being the pseudo-symplectic part of $Y_j$. Put

$$\widehat{Y}_j = \overline{Y}_j T_j$$

where $T_j = 0$ if $i_j > k$, $T_j = 1$ if $i_j = k$ and $T_j = Y_{u_{i_j} - u_k}$ if $i_j < k$. Let

$$\hat{g}_1 = \sum_{j=r}^{s} t_j \widehat{Y}_j.$$

By Corollary 3.10, we have

$$g_1(\bar{v}^+) = (\hat{g}_1(e_1 \wedge \dots \wedge e_{k-1})) \wedge e_{2n+1}.$$

Therefore

$$g(\bar{v}^+) = g_0(\bar{v}^+) + (\hat{g}_1(e_1 \wedge \cdots \wedge e_{k-1})) \wedge e_{2n+1} = 0.$$

It follows that $g(\bar{v}^+) = 0$ if and only if $g_0(\bar{v}^+) = 0$ and $\hat{g}_1(e_1 \wedge \cdots \wedge e_{k-1}) = 0$.

Thus $g_0(\bar{v}^+) = 0$ (as $g(\bar{v}^+) = 0$ by assumption) and, without loss of generality, we can assume that $g_1 = g_2 = 0$. So, $g_0 = g = t_1Y_1 + t_2Y_2 + \cdots + t_mY_m$ with $\delta(Y) = 0$ for $i = 1, 2, \ldots, m$. The scalars $t_1, t_2, \ldots, t_m$ belong to $\mathbb{F}_2$. So, $g$ can also be regarded as an element of $\mathfrak{A}^\circ_{\mathbb{Z}}$.

In the rest of this proof we will freely go back and forth between $\mathbb{F}$ and $\mathbb{C}$ or $\mathbb{F}$ and $\mathbb{Z}$. In order to avoid any confusion, we slightly modify our notation, writing $v^+_{\mathbb{F}}$ and $\bar{v}^+_{\mathbb{F}}$ instead of $v^+$ and $\bar{v}^+$ if we work in $V_{\mathbb{F}}(\lambda^\circ_k)$ and $W^\circ_k(\mathbb{F})$ and using the symbols $v^+_{\mathbb{C}}$ and $\bar{v}^+_{\mathbb{C}}$ when we deal with $V_{\mathbb{C}}(\lambda^\circ_k)$ or $V_{\mathbb{Z}}(\lambda^\circ)$ and $W^\circ_k(\mathbb{C})$. However we will do so only for the duration of this proof. Afterwards we will revert back to our earlier simpler notation.

Let us denote by $\overline{\mathfrak{A}}^{\mathrm{sp}}_{\mathbb{Z}}$ the subring of $\mathfrak{A}^{\mathrm{sp}}_{\mathbb{Z}}$ generated by the monomials of $(-)$-type. As we have seen in Section 2.3, for every monomial $Y$ of pseudo-symplectic type there is a unique monomial $\sigma(Y) \in \overline{\mathfrak{A}}^{\mathrm{sp}}_{\mathbb{Z}}$ such that

$$Y(\bar{v}^+_{\mathbb{C}}) - \sigma(Y)(\bar{v}^+_{\mathbb{C}}) \in 2 \cdot \mathfrak{A}^\circ_{\mathbb{Z}}(\mathbf{v}^+_{\mathbb{C}}). \tag{3.2}$$

Explicitly,

$$\sigma(Y_{u_i-u_j}) = V_{u_i-u_j}, \quad \sigma\left(\frac{Y^2_{u_i}}{2}\right) = V_{2u_i}, \quad \sigma(Y_{u_i+u_j}) = V_{u_i+u_j}.$$

We extend $\sigma$ to a product $Y_1Y_2\ldots Y_r$ of monomials $Y_1, Y_2, \ldots, Y_r$ as above by setting $\sigma(Y_1Y_2\ldots Y_r) = \sigma(Y_1)\sigma(Y_2)\ldots\sigma(Y_r)$ and finally we extend it to an integral combinations $t_1Y_1 + t_2Y_2 + \cdots + t_mY_m$ of pseudo-symplectic monomials $Y_1, Y_2, \ldots, Y_m$ by setting $\sigma(\sum_{i=1}^m t_iY_i) = \sum_{i=1}^m t_i\sigma(Y_i)$. Thus,

$$\sigma(g) = \sum_{i=1}^m t_i\sigma(Y_i)$$

and

$$g(\bar{v}^+_{\mathbb{C}}) - \sigma(g)(\bar{v}^+_{\mathbb{C}}) \in 2 \cdot \mathfrak{A}^\circ_{\mathbb{Z}}(\mathbf{v}^+_{\mathbb{C}}). \tag{3.3}$$

Turning back to $\mathbb{F}$, since by assumption $g(\bar{v}^+_{\mathbb{F}}) = 0$ in $W^\circ_k(\mathbb{F})$, we also have $\sigma(g)(\bar{v}^+_{\mathbb{F}}) = 0$ in $\overline{W}^{\mathrm{sp}}_k(\mathbb{F})$. However,

$$\overline{W}^{\mathrm{sp}}_k(\mathbb{F}) \cong V_{\mathbb{F}}(\lambda^{\mathrm{sp}}_k).$$

Hence $\sigma(g)(u^+_{\mathbb{F}}) = 0$ in $V_{\mathbb{F}}(\lambda^{\mathrm{sp}}_k)$, where $u^+_{\mathbb{F}}$ stands for the highest weight vector of $V_{\mathbb{F}}(\lambda^{\mathrm{sp}}_k)$ corresponding to $\bar{v}^+_{\mathbb{F}}$ in the isomorphism from $V_{\mathbb{F}}(\lambda^{\mathrm{sp}}_k)$ to $\overline{W}^{\mathrm{sp}}_k(\mathbb{F})$.

This implies that $\sigma(g)(u_{\mathbb{C}}^{+}) \in 2 \cdot V_{\mathbb{Z}}(\lambda_k^{\mathrm{sp}})$. Whence

$$\sigma(g)(\bar{v}_{\mathbb{C}}^{+}) \in 2 \cdot \overline{\mathfrak{A}}_{\mathbb{Z}}^{\mathrm{sp}}(\bar{v}_{\mathbb{C}}^{+}).$$

However, for every element $f \in \overline{\mathfrak{A}}_{\mathbb{Z}}^{\mathrm{sp}}$ there is an integral combination $h$ of pseudo-symplectic monomial elements of type $(-)$ of $\mathfrak{A}_{\mathbb{Z}}^{\circ}$ such that $\sigma(h) = f$. We have $h(\bar{v}^{+}) - f(\bar{v}^{+}) \in 2 \cdot \mathfrak{A}_{\mathbb{Z}}^{\circ}(\mathbf{v}_{\mathbb{C}}^{+})$ by (3.2). Therefore

$$2 \cdot \overline{\mathfrak{A}}_{\mathbb{Z}}^{\mathrm{sp}}(\bar{v}_{\mathbb{C}}^{+}) \subseteq 2 \cdot \mathfrak{A}_{\mathbb{Z}}^{\circ}(\bar{v}_{\mathbb{C}}^{+}).$$

It follows that

$$\sigma(g)(\bar{v}_{\mathbb{C}}^{+}) \in 2 \cdot \mathfrak{A}_{\mathbb{Z}}^{\circ}(\mathbf{v}_{\mathbb{C}}^{+}). \tag{3.4}$$

By comparing (3.3) with (3.4) we obtain that

$$g(\bar{v}_{\mathbb{C}}^{+}) \in 2 \cdot \mathfrak{A}_{\mathbb{Z}}^{\circ}(\mathbf{v}_{\mathbb{C}}^{+}). \tag{3.5}$$

Claim (3.5) holds in $W_k^{\circ}(\mathbb{C})$. However $W_k^{\circ}(\mathbb{C}) \cong V_{\mathbb{C}}(\lambda_k^{\circ})$. Therefore

$$g(v_{\mathbb{C}}^{+}) \in 2 \cdot \mathfrak{A}_{\mathbb{Z}}^{\circ}(v_{\mathbb{C}}^{+}) = 2 \cdot V_{\mathbb{Z}}(\lambda_k^{\circ}). \tag{3.6}$$

Namely, we have $g(v_{\mathbb{C}}^{+}) \equiv_2 0$. Claim (3.6) implies that $g(v_{\mathbb{F}}^{+}) = 0$, as we wished to prove. □

**Corollary 3.12.** *We have $K_k \subseteq V_1(\mathbb{F})$.*

*Proof.* This immediately follows from Lemma 3.11. □

Regarding $\mathfrak{A}_{\mathbb{F}}^{\circ}$ as an $\mathbb{F}$-vector space, let $\overline{\mathfrak{A}}_{\mathbb{F}}^{\circ}$ be its subspace spanned by the set of pseudo-symplectic monomials of $(-)$-type. Then $\overline{\mathfrak{A}}_{\mathbb{F}}^{\circ}(\bar{v}^{+})$ is a subspace (but not a submodule) of $W_k^{\circ}(\mathbb{F})$. The following is implicit in the proof of Lemma 3.11.

**Proposition 3.13.** *We have*

$$W_k^{\circ}(\mathbb{F}) = \overline{\mathfrak{A}}_{\mathbb{F}}^{\circ}(\bar{v}^{+}) \oplus \overline{\mathfrak{A}}_{\mathbb{F}}^{\circ}(\bar{v}_1^{+})$$

*(direct sum of vector spaces rather than submodules, since $\overline{\mathfrak{A}}_{\mathbb{F}}^{\circ}(\bar{v}^{+})$ is a subspace but not a submodule of $W_k^{\circ}(\mathbb{F})$). Moreover*

$$\overline{\mathfrak{A}}^{\circ}(\bar{v}^{+}) = \mathfrak{A}_{\mathbb{F}}^{\mathrm{sp}}(\bar{v}^{+}) = \overline{W}_k^{\mathrm{sp}}(\mathbb{F})$$

*(which, as noticed above, is not a submodule of $W_k^{\circ}(\mathbb{F})$) and*

$$\overline{\mathfrak{A}}_{\mathbb{F}}^{\circ}(\bar{v}_1^{+}) = \mathfrak{A}_{\mathbb{F}}^{\circ}(\bar{v}_1^{+})$$

*(which in fact is a submodule of $W_k^{\circ}(\mathbb{F})$).*

**Corollary 3.14.** *We have* $\dim(V_1(\mathbb{F})) = \binom{2n+1}{k-1}$.

*Proof.* Since

$$\dim(\overline{W}^{\mathrm{sp}}_k(\mathbb{F})) = \binom{2n}{k} - \binom{2n}{k-2},$$

Proposition 3.13 implies that $\mathfrak{A}^{\circ}_{\mathbb{F}}\bar{v}^+_1$ has codimension $\binom{2n}{k} - \binom{2n}{k-2}$ in $W^{\circ}_k(\mathbb{F})$. On the other hand, $K_k \subset V_1(\mathbb{F})$ by Corollary 3.12 and $V_1(\mathbb{F})/K_k \cong \mathfrak{A}^{\circ}_{\mathbb{F}}(\bar{v}^+_1)$. Hence $V_1(\mathbb{F})$ has codimension $\binom{2n}{k} - \binom{2n}{k-2}$ in $V_{\mathbb{F}}(\lambda^{\circ}_k)$. We have

$$\binom{2n}{k} - \binom{2n}{k-2} = \binom{2n+1}{k} - \binom{2n+1}{k-1}.$$

Hence $\dim(V_1(\mathbb{F})) = \binom{2n+1}{k-1}$, since $\dim(V_{\mathbb{F}}(\lambda^{\circ}_k)) = \binom{2n+1}{k}$. □

The next corollary finishes the proof of Proposition 3.2

**Corollary 3.15.** *The homomorphism*

$$\psi_{k-1,\mathbb{F}} : V_{\mathbb{F}}(\lambda^{\circ}_{k-1}) \to V_1(\mathbb{F})$$

*of Corollary* 3.5 *is indeed an isomorphism.*

*Proof.* Clear from Corollary 3.14, since $\dim(V_{\mathbb{F}}(\lambda^{\circ}_{k-1})) = \binom{2n+1}{k-1}$. □

**Remark.** We warn that $\overline{\mathfrak{A}}^{\circ}_{\mathbb{F}}$ is not a subalgebra of $\mathfrak{A}^{\circ}_{\mathbb{F}}$. For instance,

$$\left[Y_{u_i-u_j}, \frac{Y^2_{u_j}}{2}\right] = Y_{u_i+u_j} - Y_{u_j}Y_{u_i} = -Y_{u_i+u_j} - Y_{u_i}Y_{u_j}.$$

Each of $Y_{u_i-u_j}$, $\frac{Y^2_{u_j}}{2}$ and $Y_{u_i+u_j}$ belongs to $\overline{\mathfrak{A}}^{\circ}_{\mathbb{F}}$ but the sum $Y_{u_i+u_j} - Y_{u_j}Y_{u_i}$ does not. Therefore

$$\left[Y_{u_i-u_j}, \frac{Y^2_{u_j}}{2}\right] \notin \overline{\mathfrak{A}}^{\circ}_{\mathbb{F}}.$$

In other words, only one of the products $Y_{u_i-u_j}\frac{Y^2_{u_j}}{2}$ and $\frac{Y^2_{u_j}}{2}Y_{u_i-u_j}$ belongs to $\overline{\mathfrak{A}}^{\circ}_{\mathbb{F}}$, depending on the order chosen for the factors occurring in the pseudo-symplectic monomial elements.

On the other hand, $Y_{u_i}Y_{u_j}(\bar{v}^+) = 0$ by Corollary 3.9. Hence

$$\left[Y_{u_i-u_j}, \frac{Y^2_{u_j}}{2}\right](\bar{v}^+) = -Y_{u_i+u_j}(\bar{v}^+).$$

Thus, if for instance

$$Y_{u_i-u_j}\frac{Y_{u_j}^2}{2} \in \overline{\mathfrak{A}}_{\mathbb{F}}^{\circ} \quad \text{(whence } \frac{Y_{u_j}^2}{2}Y_{u_i-u_j} \notin \overline{\mathfrak{A}}_{\mathbb{F}}^{\circ}\text{)},$$

then the space $\overline{\mathfrak{A}}_{\mathbb{F}}^{\circ}(\bar{v}^+)$ also contains the vector $\frac{Y_{u_i}^2}{2}Y_{u_i-u_j}(\bar{v}^+)$, but represented as a sum

$$\frac{Y_{u_i}^2}{2}Y_{u_i-u_j}(\bar{v}^+) = Y_{u_i-u_j}\frac{Y_{u_j}^2}{2}(\bar{v}^+) + Y_{u_i+u_j}(\bar{v}^+).$$

### 3.2 End of the proof of Theorem 1.4

In order to finish the proof of Theorem 1.4 it remains to prove that $V_1(\mathbb{F}) = \mathcal{N}_k$ (notation as in Section 1) and that the isomorphism $\psi_{k-1,\mathbb{F}} : V_{\mathbb{F}}(\lambda_{k-1}^{\circ}) \to V_1(\mathbb{F})$ maps the subspace $\mathcal{N}_{k-1}$ of $V_{\mathbb{F}}(\lambda_{k-1}^{\circ})$ onto $K_k$.

**Proposition 3.16.** *We have* $V_1(\mathbb{F}) = \mathcal{N}_k$.

*Proof.* As $K_k \subseteq V_1(\mathbb{F})$, in order to prove that $V_1(\mathbb{F}) = \mathcal{N}_k$ we only must prove that $V_1(\mathbb{F})/K_k = \mathcal{N}_k/K_k$, namely the homomorphism $\varphi_{k,\mathbb{F}} : V_{\mathbb{F}}(\lambda_k^{\circ}) \to W_k^{\circ}(\mathbb{F})$ maps $V_1(\mathbb{F})$ onto the $k$-nucleus subspace $\overline{\mathcal{N}}_k$ of $W_k^{\circ}(\mathbb{F})$.

We recall that $\overline{\mathcal{N}}_k = \langle \iota_k(N_X)\rangle_{X\in\Delta_{k-1}}$, where $\Delta_{k-1}$ is the set of totally singular $(k-1)$-subspaces of $V$, $N_0$ is the nucleus of the quadric described by $\eta$ and $N_X = \langle X, N_0\rangle$ for $X \in \Delta_{k-1}$.

According to the conventions stated at the beginning of Section 2.2, we have $N_0 = \langle e_{2n+1}\rangle$. Put $X_1 := \langle e_1, e_2, \ldots, e_{k-1}\rangle$. Then we observe that $X_1 \in \Delta_{k-1}$ and $N_{X_1} = \langle e_1, \ldots, e_{k-1}, e_{2n+1}\rangle$. Accordingly,

$$\iota_k(N_{X_1}) = \langle e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{2n+1}\rangle = \langle \bar{v}_1^+\rangle.$$

Therefore $\varphi_{k,\mathbb{F}}(V_1(\mathbb{F}))$ contains $\iota_k(N_{X_1})$. As $\varphi_{k,\mathbb{F}}(\mathbb{F})$ is a $G$-module and $G$ acts transitively on $\Delta_{k-1}$, the module $\varphi_{k,\mathbb{F}}(V_1(\mathbb{F}))$ contains $\overline{\mathcal{N}}_k$.

On the other hand, both subspaces $\overline{\mathcal{N}}_k$ and $\varphi_{k,\mathbb{F}}(V_1(\mathbb{F}))$ have codimension $\binom{2n+1}{k} - \binom{2n+1}{k-1}$ in $W_k^{\circ}(\mathbb{F})$ (see Theorem 1.3 and Corollary 3.14). Hence

$$\varphi_{k,\mathbb{F}}(V_1(\mathbb{F})) = \overline{\mathcal{N}}_k. \qquad \square$$

**Remark.** The argument used in the proof of Proposition 3.16 can be exploited to prove that $\dim(V_1(\mathbb{F})) = \binom{2n+1}{k-1}$ (Corollary 3.14) avoiding Proposition 3.13. Indeed by that argument we obtain that $\varphi_{k,\mathbb{F}}(V_1(\mathbb{F})) \supseteq \overline{\mathcal{N}}_k$. Therefore

$$\dim(V_1(\mathbb{F})) \geq \binom{2n+1}{k-1},$$

since $\overline{\mathcal{N}}_k$ has codimension $\binom{2n+1}{k} - \binom{2n+1}{k-1}$ in $W_k^\circ(\mathbb{F})$. On the other hand,

$$\dim(V_1(\mathbb{F})) \leq \binom{2n+1}{k-1}$$

as $V_1(\mathbb{F})$ is a homomorphic image of $V_{\mathbb{F}}(\lambda_{k-1}^\circ)$. Hence

$$\dim(V_1(\mathbb{F})) = \binom{2n+1}{k-1}.$$

**Proposition 3.17.** *The isomorphism $\psi_{k-1,\mathbb{F}} : V_{\mathbb{F}}(\lambda_{k-1}^\circ) \to V_1(\mathbb{F})$ maps the subspace $\mathcal{N}_{k-1}$ of $V_{\mathbb{F}}(\lambda_{k-1}^\circ)$ onto $K_k$.*

*Proof.* Put $v_2^+ := Y_{k-1}v_1^+$ and $V_2(\mathbb{F}) := \mathfrak{A}_{\mathbb{F}}^\circ(v_2^+)$. We must prove the identity $V_2(\mathbb{F}) = K_k$. We have

$$Y_{k-1}(\bar{v}_1^+) = Y_{k-1}Y_k(\bar{v}^+) = 0$$

(by Corollary 3.9). Therefore $V_2(\mathbb{F}) \subseteq K_k$.

The pre-images $\psi_{k-1,\mathbb{F}}^{-1}(V_2(\mathbb{F}))$ and $\psi_{k-1,\mathbb{F}}^{-1}(v_1^+)$ of $V_2(\mathbb{F})$ and $v_1^+$ are the analogues of $V_1(\mathbb{F})$ and $v_0^+$ respectively, but in $V_{\mathbb{F}}(\lambda_{k-1}^\circ)$ instead of $V_{\mathbb{F}}(\lambda_k^\circ)$. So, we can apply Proposition 3.2 to them, obtaining that

$$\psi_{k-1,\mathbb{F}}^{-1}(V_2(\mathbb{F})) \cong V_{\mathbb{F}}(\lambda_{k-2}^\circ),$$

whence $V_2(\mathbb{F}) \cong V_{\mathbb{F}}(\lambda_{k-2}^\circ)$. It follows that

$$\dim(V_2(\mathbb{F})) = \binom{2n+1}{k-2}.$$

However we also have

$$\dim(K_k) = \binom{2n+1}{k-2}$$

(see Section 1, (1.2)). Hence $V_2(\mathbb{F}) = K_k$, since $V_2(\mathbb{F}) \subseteq K_k$. □

The proof of Theorem 1.4 is complete.

## 4 Uniqueness conjectures

So far we have proved that $V_{\mathbb{F}}(\lambda_k^\circ)$ admits a series of submodules

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k = V_{\mathbb{F}}(\lambda_k^\circ)$$

where

$$M_i = Y_{u_{i+1}}Y_{u_{i+2}} \ldots Y_{u_{k-1}}Y_{u_k}(v^+) \cong V_{\mathbb{F}}(\lambda_i^\circ) \quad (i = 0, 1, 2, \ldots, k-1). \quad (4.1)$$

Moreover $M_i$ corresponds to the submodule $\mathcal{N}_{i+1}$ of $V_{\mathbb{F}}(\lambda^\circ_{i+1})$ as well as to the kernel $K_{i+2}$ of the homomorphism $\varphi_{i+2,\mathbb{F}} : V_{\mathbb{F}}(\lambda^\circ_{i+2}) \to W^\circ_{i+2}(\mathbb{F})$ (if $i < k-1$). Avoiding any mention of the elements $Y_{u_{i+1}} \dots Y_{u_k}(v^+)$ we can rephrase the above conditions as follows:

$$M_i \cong V_{\mathbb{F}}(\lambda^\circ_i) \quad (i = 0, 1, 2, \dots, k), \tag{4.2}$$

$$M_i/M_{i-2} \cong W^\circ_i(\mathbb{F}) \quad (i = 1, 2, \dots, k). \tag{4.3}$$

Recall that $V_{\mathbb{F}}(\lambda^\circ_0)$, to be considered in (4.2) for $i = 0$, is 1-dimensional, by convention. We allow $i = 1$ in (4.3). This forces us to define $M_{-1}$ too. Put $M_{-1} := 0$.

When $\mathbb{F}$ is perfect, we also know that

$$M_i/M_{i-1} \cong \overline{W}^{\mathrm{sp}}_i(\mathbb{F}) \tag{4.4}$$

(Theorem 1.7 and Corollary 1.8). The following conjecture is quite natural.

**Conjecture 4.1.** The series of submodules defined as in (4.1) is the unique series of $G$-submodules of $V_{\mathbb{F}}(\lambda^\circ_k)$ satisfying conditions (4.2) and (4.3).

If the $G$-module $V_{\mathbb{F}}(\lambda^\circ_k)$ were rigid, namely all of its automorphisms as a $G$-module fix $\langle v^+\rangle$, then Conjecture 4.1 would be not so difficult to prove, but we do not know if $V_{\mathbb{F}}(\lambda^\circ_k)$ is rigid (although it is certainly rigid as an $\mathfrak{A}^\circ_{\mathbb{F}}$-module). Recall that all irreducible Weyl modules are rigid (Humphreys [7, 2.2]), but when $\mathrm{char}(\mathbb{F}) = 2$, as we are assuming here, $V_{\mathbb{F}}(\lambda^\circ_k)$ is not irreducible.

When $\mathbb{F}$ is perfect, we can consider the following variation of Conjecture 4.1.

**Conjecture 4.2.** Let $\mathbb{F}$ be a perfect field. Then the series of submodules defined as in (4.1) is the unique series of $G$-submodules of $V_{\mathbb{F}}(\lambda^\circ_k)$ satisfying conditions (4.2) and (4.4).

We shall prove Conjecture 4.2 for $k \le 4$. We will rely on the information on the lattice of submodules of $\overline{W}^{\mathrm{sp}}_k(\mathbb{F})$ provided in Baranov and Suprunenko [1] (see also Premet and Suprunenko [8]). As the reader will see, in principle our method can be applied to check Conjecture 4.2 for every particular value of $k$, but apparently there is no way to make a general argument out of it.

**Theorem 4.3.** *Conjecture* 4.2 *holds true when* $k \le 4$.

*Proof.* Let $\mathbb{F}$ be a perfect field of characteristic 2. Let $(M_i)^k_{i=0}$ be the series defined as in (4.1) and let $(M'_i)^k_{i=0}$ be another series satisfying conditions (4.2) and (4.4). We must prove that if $k \le 4$, then

$$(M_i)^k_{i=0} = (M'_i)^k_{i=0}. \tag{4.5}$$

When $k = 1$, this equality is trivially true. Thus, we may assume that $2 \le k \le 4$.

Let $k = 2$ and suppose by contradiction that $M_1' \neq M_1$. Then $M_1 \cap M_1'$ equals either $M_0$ or 0. Hence $(M_1 + M_1')/M_1$ occurs in $M_2/M_1 \cong \overline{W}_2^{\mathrm{sp}}(\mathbb{F})$ as a submodule isomorphic to either $\overline{W}_1^{\mathrm{sp}}(\mathbb{F})$ or $W_1^{\circ}(\mathbb{F})$. But according to [1] and [8] no such submodules occur in $\overline{W}_2^{\mathrm{sp}}(\mathbb{F})$.

Therefore $M_1' = M_1$. As $M_1' = M_1 \cong V_{\mathbb{F}}(\lambda_1^{\circ})$ admits a unique 1-dimensional submodule, we also have $M_0' = M_0$. Hence equality (4.5) holds true when $k = 2$.

Let $k = 3$. According to [1] and [8], if $n$ is odd, the module $M_2/M_1 \cong \overline{W}_2^{\mathrm{sp}}(\mathbb{F})$ is irreducible, while if $n$ is even, it admits a unique proper submodule $\overline{M}_{2,1}$ of dimension 1. In this case let $M_{2,1} \supset M_1$ be such that $M_{2,1}/M_1 = \overline{M}_{2,1}$. As for $M_1 \cong W_1^{\circ}(\mathbb{F})$, we know that $M_0$ is its unique proper submodule. The module $W_2^{\circ}(\mathbb{F})$ does not admit any 1-dimensional submodule, as one can check by direct computations. Hence $M_{2,1}/M_0$ does not split as a direct sum of $M_1/M_0$ and a 1-dimensional module.

It follows that if $M_2' \neq M_2$, the following are the only possibilities for $M_2 \cap M_2'$.

| $M_2 \cap M_2'$ | $\dim(M_2 \cap M_2')$ | $\dim((M_2 + M_2')/M_2)$ |
|---|---|---|
| 0 | 0 | $\binom{2n+1}{2}$ |
| $M_0$ | 1 | $\binom{2n+1}{2} - 1$ |
| $M_1$ | $2n+1$ | $\binom{2n+1}{2} - 2n - 1$ |
| $M_{2,1}$ | $2n+2$ | $\binom{2n+1}{2} - 2n - 2$ (only if $n$ is even) |

On the other hand, according to [1] and [8], if $n$ is even, the module $\overline{W}_3^{\mathrm{sp}}(\mathbb{F})$ is irreducible, while if $n$ is odd, it admits a unique proper submodule, isomorphic to $\overline{W}_1^{\mathrm{sp}}(\mathbb{F})$. It follows that $\binom{2n}{3} - 2n$ and $2n$ are the only possible values for $\dim((M_2 + M_2')/M_2)$. Neither of the numbers listed in the second column of the above table is equal to any of these two numbers, except for $\binom{2n+1}{2} - 2n - 1$, which is equal to $\binom{2n}{3} - 2n$ for $n = 3$.

So, we are left with the case of $n = 3$,

$$M_2' \cap M_2 = M_1 \quad \text{and} \quad M_2' + M_2 = M_3 = V_{\mathbb{F}}(\lambda_3^{\circ}).$$

However $W_3^{\circ}(\mathbb{F}) = V_{\mathbb{F}}(\lambda_3^{\circ})/K_3 = M_3/M_1$. Whence $W_3^{\circ}(\mathbb{F})$ now splits as the direct sum of two copies $M_2/M_1$ and $M_2'/M_1$ of $\overline{W}_2^{\mathrm{sp}}(\mathbb{F})$. (Note that, in view of the analysis made in the case $k = 2$, $M_1 = M_1'$ is the unique submodule of $M_2'$ of dimension $2n + 1 = 7$.) Accordingly, the stabilizer in $G$ of the 1-dimensional subspace spanned by $\bar{v}_1^+ = Y_{u_3}\bar{v}^+ \in M_2/M_1$ in $G$ also stabilizes a 1-dimensional subspace of $M_2'/M_1$. It is not difficult to see by direct computations that this is not the case.

It follows that $M_2' = M_2$. The equalities $M_1' = M_1$ and $M_0' = M_0$ now follow by applying to $M_2 = V_{\mathbb{F}}(\lambda_2^\circ)$ the analysis done in case $k = 2$. So, (4.5) also holds when $k = 3$.

Finally, let $k = 4$. As seen in the previous cases, if $n$ is odd, then $M_0$ and $M_1$ are the unique non-zero proper submodules of $M_2$, while if $n$ is even, then $M_2$ also admits a submodule $M_{2,1} \supset M_1$ with $\dim(M_{2,1}/M_1) = 1$. We have also seen that $M_3/M_2$ is irreducible if $n$ is even while if $n$ is odd, then $M_3/M_2$ admits a unique proper submodule $\overline{M}_{3,1}$, isomorphic to $\overline{W}_1^{\mathrm{sp}}(\mathbb{F})$. Let $M_{3,1} \supset M_2$ be such that $M_{3,1}/M_2 = \overline{M}_{3,1}$. By considering the action of $G$ on $M_3/M_1 = W_3^\circ(\mathbb{F})$ it is not difficult to see that $M_{3,1}/M_1$ cannot split as a direct sum of $G$-submodules:

$$M_{3,1}/M_1 \neq M_2/M_1 \oplus \overline{M}_{3,1}.$$

Therefore the following are the unique non-zero proper $G$-submodules of $M_3$.

| $n$ even | | $n$ odd | |
|---|---|---|---|
| submodule | dimension | submodule | dimension |
| $M_0$ | $1$ | $M_0$ | $1$ |
| $M_1$ | $2n+1$ | $M_1$ | $2n+1$ |
| $M_{2,1}$ | $2n+2$ | $M_2$ | $\binom{2n+1}{2}$ |
| $M_2$ | $\binom{2n+1}{2}$ | $M_{3,1}$ | $\binom{2n+1}{2} + 2n$ |
| $M_3$ | $\binom{2n+1}{3}$ | $M_3$ | $\binom{2n+1}{3}$ |

Accordingly, the following are the only possibilities for $M_3 \cap M_3'$ if $M_3' \neq M_3$.

| $M_3 \cap M_3'$ | $\dim((M_3 + M_3')/M_3)$ | |
|---|---|---|
| $0$ | $\binom{2n+1}{3}$ | |
| $M_0$ | $\binom{2n+1}{3} - 1$ | |
| $M_1$ | $\binom{2n+1}{3} - 2n - 1$ | |
| $M_{2,1}$ | $\binom{2n+1}{3} - 2n - 2$ | (only if $n$ is even) |
| $M_2$ | $\binom{2n+1}{3} - \binom{2n+1}{2}$ | |
| $M_{3,1}$ | $\binom{2n+1}{3} - \binom{2n+1}{2} - 2n$ | (only if $n$ is odd) |

According to [1] and [8], if $n \equiv 3 \pmod 4$, then the module $\overline{W}_4^{\mathrm{sp}}(\mathbb{F})$ is irreducible, while if $n \equiv 1 \pmod 4$, then it admits a unique non-zero proper submodule, which is 1-dimensional. If $n \equiv 0 \pmod 4$, then $\overline{W}_4^{\mathrm{sp}}(\mathbb{F})$ admits a unique non-zero proper submodule, which has dimension equal to $\binom{2n}{2} - 2$ and is isomorphic to the unique irreducible quotient of $\overline{W}_2^{\mathrm{sp}}(\mathbb{F})$. Finally, if $n \equiv 2 \pmod 4$, then $\overline{W}_4^{\mathrm{sp}}(\mathbb{F})$ admits just two proper submodules, namely either $\overline{W}_2^{\mathrm{sp}}(\mathbb{F})$ and its unique 1-dimensional submodule or the dual of $\overline{W}_2^{\mathrm{sp}}(\mathbb{F})$ and its unique submodule of codimension 1. Therefore the following are the only possibilities for $\dim((M_3 + M_3')/M_3)$:

$$\begin{aligned} &1 \quad (\text{only if } n \equiv 1 \pmod 4 \text{ or } n \equiv 2 \pmod 4), \\ &\tbinom{2n}{2} - 2 \quad (\text{only if } n \text{ is even}), \\ &\tbinom{2n}{2} - 1 \quad (\text{only if } n \equiv 2 \pmod 4), \\ &\tbinom{2n}{4} - \tbinom{2n}{2}. \end{aligned}$$

Comparing the four values with the feasible dimensions previously listed for $(M_3 + M_3')/M_3$, we see that $M_3 \cap M_3' = 0$ with $n = 5$ is the unique possibility. In this case

$$(M_3 + M_3')/M_3 = M_4/M_3.$$

This implies that $M_4/M_2 \cong W_4^{\circ}(\mathbb{F})$ splits as the direct sum of two $G$-submodules, one of which is

$$M_3/M_2 \cong \overline{W}_3^{\mathrm{sp}}(\mathbb{F})$$

and the other one is

$$(M_3' + M_2)/M_2 \cong M_3' \cong V_{\mathbb{F}}(\lambda_3^{\circ}).$$

By an argument similar to the one used in a similar situation in case $k = 3$, one can see that $W_4^{\circ}(\mathbb{F})$ does not admit such a splitting.

Therefore $M_3' = M_3$. We can now apply to $M_3 \cong V_{\mathbb{F}}(\lambda_3^{\circ})$ the analysis made in case $k = 3$, obtaining that $M_2' = M_2$, $M_1' = M_1$ and $M_0' = M_0$. Equality (4.5) is proved for $k = 4$ too. □

**Remark.** An analysis as above has also been done in [5], but only for $k \leq n \leq 3$.

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**Author information**

Ilaria Cardinali, Department of Information Engineering and Mathematics,
University of Siena, Via Roma 56, 53100 Siena, Italy.
E-mail: ilaria.cardinali@unisi.it

Antonio Pasini, Department of Information Engineering and Mathematics,
University of Siena, Via Roma 56, 53100 Siena, Italy.
E-mail: antonio.pasini@unisi.it