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On the intersection of certain maximal subgroups of a finite group

Adolfo Ballester-Bolinches, James C. Beidleman, Hermann Heineken, Matthew F. Ragland and Jack Schmidt

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Dedicated to Professor Ben Brewster on the occasion of his 70th birthday

Abstract. Let $\Delta(G)$ denote the intersection of all non-normal maximal subgroups of a group G. We introduce the class of T_2-groups which are defined as the groups G for which $G/\Delta(G)$ is a T-group, that is, a group in which normality is a transitive relation. Several results concerning the class T_2 are discussed. In particular, if G is a solvable group, then Sylow permutability is a transitive relation in G if and only if every subgroup H of G is a T_2-group such that the nilpotent residual of H is a Hall subgroup of H.

1 Introduction and statements of results

All groups considered are finite.

It seems that knowing some information about the intersection of certain types of maximal subgroups of a finite group often provides some worthwhile insight into the structure of a finite group. In [11], Gaschütz developed many interesting properties of the Frattini subgroup of a finite group and showed how these properties could be used to find new structural information of such groups. In [11], he introduced a subgroup $\Delta(G)$ similar to the Frattini subgroup and developed some of its properties. Here $\Delta(G)$ is defined as the intersection of all non-normal maximal subgroups of G (and $\Delta(G) = G$ if all maximal subgroups of G are normal, that is, if G is nilpotent). In [11], the following was established:

Theorem 1.1. *Let G be a group. Then*

(i) $\Delta(G)$ *is nilpotent,*

(ii) $\Delta(G)/\Phi(G) = Z(G/\Phi(G))$.

Asaad and Ramadan established several other properties of $\Delta(G)$ and their results can be found in [2]. The second author and Seo [9] used $\Delta(G)$ to determine a number of results about certain nilpotent properties of subnormal subgroups of a group G.

For a group G, we will denote the hypercenter, nilpotent residual, and the Frattini subgroup of G, respectively, by $Z_*(G)$, $\gamma_*(G)$, and $\Phi(G)$. A group G is called a T_0-group provided that $G/\Phi(G)$ is a T-group, that is, a group in which normality is a transitive relation; T_0-groups have been studied in [4, 7, 8, 13, 16]. A group G is called a T_1-group if $G/Z_*(G)$ is a T-group. The second and third authors introduced the concept of a T_1-group and some of the properties of these groups were developed in [7, 8]. One of our purposes in this paper is to introduce a new class of groups which we call the T_2-groups and develop some of the properties of these groups. We call a group G a T_2-group provided that $G/\Delta(G)$ is a T-group. The next two results provide some of the basic properties of T_2-groups.

Theorem A. *Let G be a group and let $N \trianglelefteq G$. Then:*

(i) *If G is a T_2-group, then G/N is a T_2-group.*

(ii) *If $N \leq \Delta(G)$ and G/N is a T_2-group, then G is a T_2-group.*

(iii) *Let $G = H \times K$ for subgroups H and K of G. Then $\Delta(G) = \Delta(H) \times \Delta(K)$.*

(iv) *If G is a solvable T_2-group, then G is supersolvable.*

(v) *G is a T_2-group if and only if $G/\Phi(G)$ is a T_1-group.*

(vi) *If $G/Z_*(G)$ is a T_0-group, then G is a T_2-group.*

In Example 4.5 we will see that the converse of part (vi) of Theorem A is false. If G is a solvable group from Example 4.4 or Example 4.6, then G is a T_2-group which is neither a T_0-group nor a T_1-group.

Theorem B. *Let G be a solvable T_2-group. Then G is a T_0-group if and only if $\gamma_*(G)$ is a Hall subgroup of G.*

Let G be the group in Example 4.1. G is a solvable T_2-group which is also a T_1-group but not a T_0-group. Note that $\gamma_*(G)$ is not a Hall subgroup of G. Also $\Delta(\mathrm{Fit}(G)) = \mathrm{Fit}(G)$ and $\Delta(G) = Z(G)$. This is very different from how the Frattini subgroup of a group behaves. Recall that if H is a group and $X \trianglelefteq H$, then $\Phi(X) \leq \Phi(H)$. Note that every subgroup of G is a T_2-group.

Let G be the group in Example 4.2. Then G is a T_0-group, a T_1-group, and a T_2-group.

The group G in Example 4.3 is a T_2-group and a T_0-group but not a T_1-group. The subgroup H of G is neither a T_2-group nor a T_0-group. Hence the classes of solvable T_2-groups and solvable T_0-groups are not subgroup closed. We note that the class of T_1-groups is subgroup closed by [7, Lemma 1].

Let H and K be subgroups of the group G. The subgroup H is said to permute with K provided that HK is a subgroup of G, and the subgroup H is said to be permutable (S-permutable) if it permutes with every subgroup (Sylow subgroup) of G. Kegel [12] showed that an S-permutable subgroup of G is subnormal.

A group G is said to be a PST-group (PT-group) if H and K are subgroups of G such that H is S-permutable (permutable) in K and K is S-permutable (permutable) in G, then H is S-permutable (permutable) in G; PST-groups and PT-groups have been studied in great detail in [1, 3, 4, 8, 13]. By Kegel's result it follows that a group G is a PST-group (PT-group) if and only if the subnormal subgroups of G are S-permutable (permutable).

The next theorem provides a relationship between solvable PST-groups and T_2-groups.

Theorem C. *Let G be a group. If $G/\Delta(G)$ is a solvable PST-group, then G is a T_2-group.*

By a result of Agrawal [1], we are able to characterize solvable T_2-groups.

Theorem D. *Let G be a solvable group. Then G is a T_2-group if and only if it satisfies:*

(i) *$\gamma_*(G)\Delta(G)/\Delta(G)$ is an abelian Hall subgroup of $G/\Delta(G)$,*

(ii) *G acts by conjugation on $\gamma_*(G)/\Delta(G) \cap \gamma_*(G)$ as a group of power automorphisms.*

For a class of groups, X, let X_0 denote the class of groups G such that $G/\Phi(G)$ is an X-group. Likewise, for a class of groups X, let X_2 denote the class of groups G such that $G/\Delta(G)$ is an X-group. In [13], the fourth author proved that the classes of solvable T_0-groups, solvable PT_0-groups, and solvable PST_0-groups are one and the same. A similar result was established in [8, Theorem B]. We will prove a similar result for solvable T_2-groups.

Theorem E. *The classes of solvable T_2-groups, solvable PT_2-groups, and solvable PST_2-groups are equal.*

Let G be the group in Example 4.3. Then G is a T_0-group and G has a subgroup which is not a T_0-group. We also note that G is not a PST-group. However, the following theorem is established in [4].

Theorem 1.2. *Let G be a group. The following are equivalent:*

(i) *G is a solvable PST-group.*

(ii) *Every subgroup of G is a T_0-group.*

A finite group G is said to satisfy property α if it satisfies the following two conditions:

(α_1) every subgroup of G is a T_2-group,

(α_2) for every subgroup H of G, $\gamma_*(H)$ is a Hall subgroup of H.

The group in Example 4.1 is a T_2-group whose subgroups are T_2-groups; moreover, G is not a T_0-group and its nilpotent residual is not a Hall subgroup.

Remark 1.3. If G satisfies (α_1), then G is supersolvable.

Theorem F. *Let G be a group. The following statements are equivalent:*

(i) *G satisfies conditions (α_1)–(α_2).*

(ii) *Every subgroup of G is a T_0-group.*

(iii) *G is a solvable PST-group.*

Theorem G. *Let G be a group. Then the following statements are equivalent:*

(i) *Every subgroup of G is a T_2-group.*

(ii) *Every subgroup of G is a PT_2-group.*

(iii) *Every subgroup of G is a PST_2-group.*

(iv) *Every subgroup of G is a solvable T_2-group.*

Let p be a prime. A group G satisfies C_p if and only if each subgroup of a Sylow p-subgroup P of G is normal in the normalizer $N_G(P)$. Robinson showed (see [3, Theorem 2.2.2] or [14]) that a group G is a solvable T-group if and only if it is a C_p-group for all primes p.

Let $\Delta_p(G)$ be the Sylow p-subgroup of $\Delta(G)$. Then G satisfies $\overline{\overline{C_p}}$ if and only if $G/\Delta_p(G)$ is a C_p-group.

Theorem H. *A group G is a solvable T_2-group if and only if G is a $\overline{\overline{C_p}}$-group for all primes p.*

We now consider how the subgroup $\Delta(G)$ provides some interesting information about certain formations. The following information about formations can be found in [5, 10, 15].

A class of groups $\mathfrak{F}$ is called a formation if it satisfies the following two conditions:

(i) If $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{F}$.

(ii) If G is a group and N and M are normal subgroups such that G/N and G/M belong to $\mathfrak{F}$, then $G/M \cap N \in \mathfrak{F}$.

A formation is said to be saturated provided that if G is a group such that $G/\Phi(G) \in \mathfrak{F}$, then $G \in \mathfrak{F}$. Let P denote the set of primes. Any function f from P to the set of formations is called a formation function. Given a formation function f, we define the class of groups $\mathrm{LF}(f)$ satisfying the following condition: $G \in \mathrm{LF}(f)$ if for all chief factors H/K of G and for primes p dividing $|H/K|$, we have $\mathrm{Aut}_G(H/K) = G/C_G(H/K) \in f(p)$. Then the class $\mathrm{LF}(f)$ is a formation. A class of groups $\mathfrak{F}$ is a local formation if there exists a formation function f such that $\mathfrak{F} = \mathrm{LF}(f)$.

Theorem 1.4 ([5, 10]). *A formation $\mathfrak{F}$ is saturated if and only if $\mathfrak{F}$ is local.*

Let $\mathfrak{F}$ be a saturated formation defined locally by the formation function f. Let G be a group and let H/K be a chief factor of G. If for each prime divisor p of $|H/K|$ we have $G/C_G(H/K) \in f(p)$, then H/K is said to be $\mathfrak{F}$-central. Otherwise H/K is called $\mathfrak{F}$-eccentric. By Theorem 1.4, a group G belongs to $\mathfrak{F}$ if and only if every chief factor of G is $\mathfrak{F}$-central.

Let $\mathfrak{F}$ be a saturated formation containing the class of nilpotent groups. The following theorem is important for our last two results.

Theorem 1.5 ([6]). *Let G be a group and let H be a subnormal subgroup of G containing $\Phi(G)$. If $H/\Phi(G)$ belongs to $\mathfrak{F}$, then H belongs to $\mathfrak{F}$.*

Using Theorem 1.5 we are able to obtain the following two theorems.

Theorem I. *Let G be a group and let H be a subnormal subgroup of G containing $\Delta(G)$. If $H/\Delta(G)$ belongs to $\mathfrak{F}$, then H belongs to $\mathfrak{F}$.*

Theorem J. *Let $\mathfrak{F}$ be a formation containing the class of nilpotent groups. Then $\mathfrak{F}$ is saturated if and only if, for a group G, $G/\Delta(G) \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

2 Preliminary lemmas

Next we present four lemmas which are needed to prove Theorems A–J.

Lemma 2.1 ([11]). *Let $G = H \times K$. Then $\Phi(G) = \Phi(H) \times \Phi(K)$.*

Lemma 2.2 ([7]). *Let R be the nilpotent residual of G and let G be a solvable T_1-group. Then G is a T_0-group if and only if R is a Hall subgroup of G.*

Lemma 2.3 ([13]). *Let G be a solvable T_0-group. Then the nilpotent residual of G is a nilpotent Hall subgroup of G of odd order.*

Lemma 2.4 ([13]). *Let G be a group with nilpotent residual R. Then G is a solvable T_0-group if an only if G/R' is a solvable PST-group and R is nilpotent.*

3 Proofs of the main results

Proof of Theorem A. Let G be a group and let $N \trianglelefteq G$.

(i) Assume G is a T_2-group. Then the factor $G/\Delta(G)$ is a T-group. Since $\Delta(G)N/N \leq \Delta(G/N)$ and a homomorphic image of a T-group is a T-group, it follows that $(G/N)/\Delta(G/N)$ is a T-group. Hence G is a T_2-group.

(ii) Assume $N \leq \Delta(G)$ and G/N is a T_2-group. By (i), $G/\Delta(G)$ is a T_2-group. Since $\Delta(G/\Delta(G)) = 1$, G is a T_2-group.

(iii) Let $G = H \times K$. By Lemma 2.1, we obtain $\Phi(G) = \Phi(H) \times \Phi(K)$ and $Z(G/\Phi(G)) = Z(H/\Phi(H)) \times Z(K/\Phi(K))$. It now follows by Theorem 1.1 that $\Delta(G)/\Phi(G) = \Delta(H)/\Phi(H) \times \Delta(K)/\Phi(K)$ so that $\Delta(G) = \Delta(H) \times \Delta(K)$.

(iv) Let G be a solvable T_2-group. Then $G/\Delta(G)$ is a solvable T-group and hence $G/\Delta(G)$ is supersolvable. Consider $(G/\Phi(G))/(\Delta(G)/\Phi(G))$ which is isomorphic to $G/\Delta(G)$. Thus, by induction, $G/\Phi(G)$ is a solvable T-group and hence supersolvable. It follows that G is supersolvable.

(v) Assume G is a T_2-group. Then the factor $G/\Delta(G)$ is a T-group and so $(G/\Phi(G))/(\Delta(G)/\Phi(G))$ is a T-group. But

$$\Delta(G)/\Phi(G) = Z(G/\Phi(G)) = Z_*(G/\Phi(G))$$

so that $G/\Phi(G)$ is a T_1-group. Conversely, assume that $G/\Phi(G)$ is a T_1-group. Then it follows that $G/\Delta(G) \simeq (G/\Phi(G))/Z(G/\Phi(G))$ is a T-group and thus G is a T_2-group.

(vi) Assume that $G/Z_*(G)$ is a T_0-group. Since $Z_*(G) \leq \Delta(G)$, it follows that $G/\Phi(G) \simeq (G/\Phi(G))/Z(G/\Phi(G))$ is a T-group and so G is a T_2-group. □

In Example 4.4 we note that there are T_2-groups which are neither T_0-groups nor T_1-groups.

Proof of Theorem B. Let G be a solvable T_2-group. First, if G is a T_0-group, then by Lemma 2.3, $R = \gamma_*(G)$ is a Hall subgroup of G. So, let us assume that R is a Hall subgroup of G. Since G is a T_2-group, we may assume $\Delta(G) \neq \Phi(G)$. Assume that $\Phi(G) = 1$. Then $\Delta(G) = Z(G)$ by Theorem 1.1 and G is a T_1-group. Hence, by Lemma 2.2, G is a T_0-group. We may assume that $\Phi(G) \neq 1$ and $|G/\Phi(G)| < |G|$. By part (i) of Theorem A, $G/\Phi(G)$ is a solvable T_2-group and $R\Phi(G)/\Phi(G)$ is a Hall subgroup and it is the nilpotent residual of $G/\Phi(G)$. By induction on $|G|$, it follows that $G/\Phi(G)$ is a T_0-group whence G is as well. □

Proof of Theorem C. Let $G/\Delta(G)$ be a solvable *PST*-group. By part (v) of Theorem A, $G/\Delta(G)$ is a T_1-group. Since $Z_*(G/\Delta(G)) \leq \Delta(G/\Delta(G)) = 1$, it follows that $G/\Delta(G)$ is a T_2-group. By part (ii) of Theorem A, G is a solvable T_2-group. □

Proof of Theorem D. Assume that G is a solvable T_2-group. Then $G/\Delta(G)$ is a T-group and by [13, Theorem 3 (i) and (iii)], (i) and (ii) hold.

Conversely, assume (i) and (ii) are satisfied by $G/\Delta(G)$. By [13, Theorem 3 (i) and (iii)], $G/\Delta(G)$ is a solvable *PST*-group and by Theorem C, G is a solvable T_2-group. □

Proof of Theorem E. Let G be a solvable PST_2-group. Then $G/\Delta(G)$ is a solvable *PST*-group and, by Theorem C, G is a solvable T_2-group. Equality of the classes follows easily now as it is clear that $\mathfrak{S} \cap T_2 \subseteq \mathfrak{S} \cap PT_2 \subseteq \mathfrak{S} \cap PST_2$, where $\mathfrak{S}$ is the class solvable groups. □

Proof of Remark 1.3. Let G be a group satisfying condition (α_1). Then every subgroup of G is a T_2-group. Hence, by induction, every subgroup of G is supersolvable. By a well-known result of Huppert, [15, Theorem 10.3.4], G is solvable. Thus, by part (iv) of Theorem A, G is supersolvable. □

Proof of Theorem F. Let G be a group that satisfies condition α. By Remark 1.3, G is supersolvable. Therefore, by Theorem B, every subgroup of G is a T_0-group and so (i) implies (ii).

Now assume (ii) holds. Then G is a solvable *PST*-group by Theorem 1.2 and (iii) holds.

Assume (iii). Then every subgroup of G is also a solvable *PST*-group. Let H be a subgroup of G. Then H is a solvable T_2-group by Theorem C. By Agrawal, [1, Theorem 1], the nilpotent residual of H is a Hall subgroup of H. Thus, G satisfies property α and (i) holds completing the proof. □

Proof of Theorem G. The implications (i) implies (ii), (ii) implies (iii), and (iv) implies (i) are clear.

Let us show (iii) implies (iv). Assume that every subgroup of G is a PST_2-group. Then every proper subgroup is a solvable T_2-group by induction. Then, by part (iv) of Theorem A, we have that every proper subgroup of G is supersolvable. By a result of Huppert (see [15, Theorem 10.3.4]), G is a solvable group. Hence G is a solvable PST_2-group and by Theorem E, G is a solvable T_2-group. □

Proof of Theorem H. Assume G is a $\overline{\overline{C_p}}$-group for all primes. We must show the factor $G/\Delta(G)$ is a solvable T-group, or equivalently, $G/\Delta(G)$ is a C_q-group for all primes q. Assume q does not divide $|\Delta(G)|$. Then G is a C_q-group so that

$G/\Delta(G)$ is a C_q-group. Now assume that q divides $|\Delta(G)|$ and let $\Delta(G)_q$ be the Sylow q-subgroup of $\Delta(G)$. Then the factor $G/\Delta(G)_q$ is a C_q-group and by [8, Lemma 2 (iii)], we have that $(G/\Delta(G)_q)/(\Delta(G)/\Delta(G)_q)$ is a C_q-group. Now $\Delta(G/\Delta_q) = \Delta(G)/\Delta(G)_q$ so that $G/\Delta(G)$ is a C_q-group. Therefore, $G/\Delta(G)$ is a solvable T-group and G is a solvable T_2-group.

Conversely, assume that $G/\Delta(G)$ is a solvable T-group. We want to show that G is a $\overline{\overline{C_q}}$-group for all primes q. Assume q divides $|\Delta(G)|$. Then $G/\Delta(G)_q$ is a solvable T_2-group by part (ii) of Theorem A. If q does not divide $|G/\Delta(G)_q|$, then G is a C_q-group and hence a $\overline{\overline{C_q}}$-group. Now q does not divide $|\Delta(G)/\Delta(G)_q|$ and hence, by induction, $G/\Delta(G)_q$ is a $\overline{\overline{C_q}}$-group. This means $G/\Delta(G)_q$ is a C_q-group and G is a $\overline{\overline{C_q}}$-group.

Now assume that q does not divide $|\Delta(G)|$. Then $G/\Delta(G)$ is a C_q-group. Now $\Delta(G)$ is a q'-group and one has $G/\Delta(G) \in C_q$ so that G is a C_q-group by [8, Lemma 2 (ii)]. Thus G is a $\overline{\overline{C_q}}$-group and the proof is complete. □

Proof of Theorem I. Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}$. By [10, Proposition IV, 3.8], there is a unique formation function, say f, defining $\mathfrak{F}$ which is integrated and full.

Let G be a group and H a subnormal subgroup of G such that $H/\Delta(G) \in \mathfrak{F}$. We are to show $H \in \mathfrak{F}$. Note that

$$\Phi(G) \subseteq \Delta(G) \subseteq H$$

and

$$(H/\Phi(G))/(\Delta(G)/\Phi(G)) \simeq H/\Delta(G) \in \mathfrak{F}.$$

Assume $\Phi(G) \neq 1$. By induction on $|G|$, it follows that $H/\Phi(G) \in \mathfrak{F}$. By Theorem 1.5, $H \in \mathfrak{F}$. Hence, we may assume that $\Delta(G) = Z(G)$. Let K/L be a chief factor of H such that $Z(G) \subseteq L \subseteq K \subseteq H$. Then the factor K/L is $\mathfrak{F}$-central since $H/\Delta(G) \in \mathfrak{F}$. Next let Y/X be a p-chief factor of H below $\Delta(G)$. Then Y/X is central in H and so $1 = G/C_H(Y/X) \in f(p)$. This means that each chief factor of H is $\mathfrak{F}$-central and hence $H \in \mathfrak{F}$. This completes the proof. □

Proof of Theorem J. Let $\mathfrak{F}$ be a formation containing $\mathfrak{N}$. Assume first that $\mathfrak{F}$ is saturated and G is a group such that $G/\Delta(G) \in \mathfrak{F}$. By Theorem I it follows that $G \in \mathfrak{F}$.

Conversely, assume that $G \in \mathfrak{F}$ whenever $G/\Delta(G) \in \mathfrak{F}$. We are to show $\mathfrak{F}$ is saturated. Let G be a group such that $G/\Phi(G) \in \mathfrak{F}$. We must show $G \in \mathfrak{F}$. Consider the group $(G/\Phi(G))/(\Delta(G)/\Phi(G)) \simeq G/\Delta(G)$. Now $G/\Phi(G) \in \mathfrak{F}$ and so $G/\Delta(G) \in \mathfrak{F}$ being a homomorphic image of $G/\Phi(G)$. Therefore, $G \in \mathfrak{F}$ and $\mathfrak{F}$ is saturated. □

4 Examples

Example 4.1. Let X be a non-abelian group of order pq where p and q are distinct primes such that q divides $p-1$. Let Y be a group of order p and put $G = X \times Y$. Then $\Phi(G) = 1$, $\Delta(G) = Z(G) = Y$ and $G/\Delta(G) \simeq X$ which is a T-group. Thus G is a T_1-group and a T_2-group. Note G is not a T_0-group. Let W be the Sylow p-subgroup of X and note that $\gamma_*(G) = W$ which is not a Hall subgroup of G. Also, $\Delta(F(G)) = F(G) = W \times Y$, the Fitting subgroup of G. Thus $\Delta(G) = Y \leq \Delta(F(G))$. Recall that $\Phi(H) \leq \Phi(M)$ where H is a normal subgroup of the group M and the Δ-subgroup does not satisfy this property of the Frattini subgroup of a group.

Example 4.2. Let $E = \langle x, y \mid x^3 = y^3 = [x, y] = [[x, y], x] = [[x, y], y] = 1\rangle$ so that E is an extra-special 3-group of order 27 and exponent 3. There is an automorphism σ of E of order 2 given by $x^\sigma = x^{-1}$, $y^\sigma = y^{-1}$ and $[x, y]^\sigma = [x, y]$. Let $G = E \rtimes \langle\sigma\rangle$. Then $Z(G) = Z(E) = \Phi(E) = \Phi(G)$ and G is both a T_1-group and a T_0-group; G is also a T_2-group with $Z(G) = \langle[x, y]\rangle$ and $\Delta(G) = \Phi(G)$.

Example 4.3. Let $P = \langle x, y \mid x^5 = y^5 = [x, y]^5 = 1\rangle$ be an extra-special group of order 125 and exponent 5. Let $z = [x, y]$ and note that $Z(P) = \Phi(P) = \langle z\rangle$. Note that P has an automorphism a of order 4 given by $x^a = x^2$, $y^a = y^2$, and $z^a = z^4 = z^{-1}$. Put $G = P \rtimes \langle a\rangle$. Now $Z(G) = 1$, $\Delta(G) = \langle z\rangle$, and $G/\Delta(G)$ is a T-group. Thus G is a T_2-group and note it is also a T_0-group. Let $H = \langle y, z, a\rangle$ and notice that $\Phi(H) = \Delta(H) = 1$. Further observe that H is not a T-group since $\gamma_*(H) = \langle y, z\rangle$ and a does not act as a power automorphism on the abelian subgroup $\gamma_*(H)$. This means that H is neither a T_2-group nor a T_0-group. Therefore, the class of solvable T_2-groups is not subgroup-closed. Likewise, the class of solvable T_0-groups is not subgroup-closed.

Example 4.4. Let p be an odd prime and let $C = \langle x\rangle$ be a cyclic group of order p^2. Let $S = \langle y, z \mid y^p = z^2 = 1, y^z = y^{-1}\rangle$ be a dihedral group of order $2p$. Let S act on C as follows: $x^z = x^{-1}$ and $x^y = x$. Put $H = C \rtimes S$. Note that $\Phi(H) = \langle x^p\rangle$ and $Z(H) = 1$. Further observe that H is a solvable T_0-group and a T_2-group but not a T_1-group.

Let $G = H \times C_p$ where C_p is a cyclic group of order p. Note that $\Phi(G) = \langle x^p\rangle$ and $Z(G) = C_p$ is the hypercenter of G. G is neither a T_0-group nor a T_1-group. Now $\Delta(G)/\Phi(G) = Z(G/\Phi(G)) \simeq C_p$ and so G is a T_2-group.

Example 4.5. Let $C = \langle x\rangle$ be a cyclic group of order 9 and $D = \langle y, z\rangle = \langle y\rangle \times \langle z\rangle$ be a cyclic group of order 6 with $|y| = 3$ and $|z| = 2$ where D is viewed as the automorphism group of C. Put $G = C \rtimes D$. Then $\Phi(G) = \Phi(C) = \langle x^3\rangle$,

$Z_*(G) = 1$ and $G/\Phi(G)$ is not a T-group. Note that $\Delta(G) = \langle x^3 \rangle \times \langle y \rangle$ and $G/\Delta(G)$ is a T-group. Hence G is a T_2-group but not a T_0-group.

Example 4.6. Let $G = P \rtimes \langle a \rangle$ be the group in Example 4.3 and let $X = G \times \langle t \rangle$ where $\langle t \rangle$ is a cyclic group of order 5. Then X is a solvable T_2-group which is neither a T_0-group nor a T_1-group.

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Author information

Adolfo Ballester-Bolinches, Departament d'Àlgebra, Universitat de València,
Dr. Moliner, 50, 46100 Burjassot, València, Spain.
E-mail: Adolfo.Ballester@uv.es

James C. Beidleman, Department of Mathematics, University of Kentucky,
Lexington, KY 40605-0027, USA.
E-mail: clark@ms.uky.edu

Hermann Heineken, Institut für Mathematik, Universität Würzburg,
Am Hubland, 97074 Würzburg, Germany.
E-mail: heineken@mathematik.uni-wuerzburg.de

Matthew F. Ragland, Department of Mathematics, Auburn University at Montgomery,
P.O. Box 244023, Montgomery, AL 36124-4023, USA.
E-mail: mragland@aum.edu

Jack Schmidt, Department of Mathematics, University of Kentucky,
Lexington, KY 40605-0027, USA.
E-mail: jack.schmidt@uky.edu