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Jordan nilpotency in group rings

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Abstract. We investigate the Jordan nilpotency of a group ring RG and, when RG has an involution that is the linear extension of an involution on G, also the Jordan nilpotency of the symmetric elements in RG.

1 The beginning

The *Lie bracket* on an associative ring A is the ring commutator $[x, y] = xy - yx$. It is easy to see that under the Lie bracket, $(A, [\cdot])$ is a Lie ring in the sense that $[x, x] = 0$ for all $x \in A$ and $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. This ring is said to be *Lie nilpotent* of index $n \geq 2$ if $[\dots[[x_1, x_2], x_3], \dots, x_n] = 0$ for all choices of elements $x_1, x_2, \dots, x_n \in A$. If $a \mapsto a^*$ is an involution on A (that is, an antiautomorphism of period 2), then the set $A^- = \{a \in A \mid a^* = -a\}$ of *skew-symmetric* elements of A is closed under the Lie bracket and hence itself a Lie ring. There are obvious questions to ask of a particular associative ring A with involution. When is A Lie nilpotent? When is A^- Lie nilpotent? Does Lie nilpotence of A^- imply Lie nilpotence of A? All these questions have been studied when $A = RG$ is the group ring of a group G over a commutative coefficient ring R with 1 and the involution on A is the linear extension of some involution on G, see [1, 2, 6, 8].

In this paper, we consider analogous questions about *Jordan nilpotence*. The *Jordan product* on an associative ring A, denoted $\circ$, is defined by $x \circ y = xy + yx$. The ring $(A, \circ)$ is Jordan in the sense that it is commutative and satisfies the Jordan identity

$$(x^{\circ 2} \circ y) \circ x = x^{\circ 2} \circ (y \circ x)$$

for all $x, y \in A$. (Here we write $x^{\circ 2} = x \circ x$.) It is *Jordan nilpotent* of index $n \geq 2$ if $(\cdots((x_1 \circ x_2) \circ x_3) \cdots) \circ x_n = 0$ for all $x_1, x_2, \dots, x_n \in A$. If $a \mapsto a^*$ denotes

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an involution on A, then the set $A^+ = \{a \in A \mid a^* = a\}$ of *symmetric* elements is closed under the Jordan product and hence a Jordan ring. In this work, we consider questions related to the Jordan nilpotence of A and of A^+ when $A = RG$ is a group ring and $*$ is the linear extension to RG of an involution on G.

This paper will again focus attention on a class of groups which have become known as *SLC groups* [7], these being groups with a unique nonidentity commutator and the *LC (lack of commutativity) property*, which says roughly that elements do not commute unless the centre is involved: specifically, two elements g and h commute if and only if one of g, h, gh is central. SLC groups made their appearance in the context of nonassociative algebra. They are the building blocks of certain Moufang loops whose loop rings are alternative, but not associative [3], but they have since (unexpectedly?) arisen in many papers involving group rings with involution. It is not hard to see that a group G is SLC if and only if $G/\mathcal{Z}(G) \cong C_2 \times C_2$, see [3, Proposition III.3.6] ($\mathcal{Z}(G)$ denoting the centre of G). If s is the unique nonidentity commutator in an SLC group, then s^{-1} is also a commutator, so $s^2 = 1$. It is also easy to see that s is central.

2 Jordan nilpotence of RG

We begin by introducing a group property whose relevance to this paper will soon be seen.

Definition 2.1. A nonabelian group G has *property* (C) if whenever $a, b, c \in G$ are such that $(a, b) \neq 1$ and $(a, c) \neq 1$, then $(a, b) = (a, c)$.

Notice that squares are central in a group with property (C) since if $a^2 b \neq ba^2$, then $ab \neq ba$, so applying property (C) to b, a, a^2, we have $(b, a) = (b, a^2)$ and hence $(a, b) = (a^2, b)$. But this is $a^{-1}b^{-1}ab = a^{-2}b^{-1}a^2 b$, so $b^{-1} = a^{-1}b^{-1}a$ and $ab = ba$, a contradiction. In a group with central squares, any commutator is also central because $a^{-1}b^{-1}ab = a^{-2}(ab^{-1})^2 b^2$ is the product of squares, and any commutator has order at most 2 because $ba = abs$ implies

$$a^2 b = ba^2 = absa = a(bas) = a(abs^2) = a^2 bs^2.$$

Lemma 2.2. *A nonabelian group has property* (C) *if and only if it has a unique nonidentity commutator.*

Proof. One direction is clear. In the other, suppose G is the group in question and $g, h, x, y \in G$ with $(g, h) \neq 1$ and $(x, y) \neq 1$. We wish to show $(g, h) = (x, y)$. If $(g, x) \neq 1$, then property (C) gives $(g, h) = (g, x) = (x, y)$. Similarly, we have the desired result if either g or h fails to commute with either x or y. Suppose then each of g, h commutes with each of x, y and consider the three elements g, hx, y. We have $(g, hx) = (g, h)$ and $(hx, y) = (x, y)$, so property (C) gives $(g, h) = (x, y)$. □

Remark 2.3. We have noted that a group with property (C) has central squares. The lemma says that it has a unique commutator. While these are properties of SLC groups, the two classes of groups are different. Indeed, the group denoted $16\Gamma_2 c_1$ in Hall–Senior notation [5] and $16/9$ in Thomas and Wood [9] has property (C), but it is not SLC.

The next theorem characterizes those groups whose group rings are Jordan nilpotent of index 3.

Theorem 2.4. *Let RG denote the group ring of a group G over a commutative coefficient ring R with 1. Then RG is Jordan nilpotent of index 3 if and only if one of the following statements holds:*

(1) char $R = 4$ *and G is abelian,*

(2) char $R = 2$ *and either G is abelian or G has a unique nonidentity commutator.*

Proof. We first show that RG is Jordan nilpotent in the two situations specified. If char $R = 4$ and G is abelian, then, for any $\alpha, \beta, \gamma \in RG$,

$$(\alpha \circ \beta) \circ \gamma = (2\alpha\beta) \circ \gamma = 4\alpha\beta\gamma = 0.$$

Suppose char $R = 2$. If G is abelian and α, β are in RG, then $\alpha \circ \beta = 0$, so $(\alpha \circ \beta) \circ \gamma = 0$ for any γ. On the other hand, suppose G has a unique nonidentity commutator s, necessarily central and of order 2. To establish Jordan nilpotence of index 3, it suffices to show that $(g \circ h) \circ k = 0$ for any $g, h, k \in G$. This is clear if g and h commute, so suppose $gh \neq hg$. Thus

$$\begin{aligned} (g, h) &= s, \\ g \circ h &= gh + hg = (1 + s)gh, \\ (g \circ h) \circ k &= (1 + s)(gh \circ k). \end{aligned}$$

If gh and k commute, then $gh \circ k = 0$; otherwise, $(gh, k) = s$, so

$$gh \circ k = (gh)k + k(gh) = (1 + s)ghk$$

and

$$(g \circ h) \circ k = (1 + s)^2 ghk = 2(1 + s)ghk = 0.$$

We now attack the converse, assuming that RG is Jordan nilpotent of index 3. Since $(1 \circ 1) \circ 1 = 2 \circ 1 = 4$ must be 0, the characteristic of the ring R is 2 or 4. If char $R = 4$, then, for any $g, h \in G$, one has $(g \circ h) \circ 1 = 0 = 2(gh + hg) = 0$, so $gh = hg$. The group G is abelian.

Suppose char $R = 2$ and G is not abelian. For any $g, h \in G$,

$$0 = (g \circ h) \circ g = ghg + hg^2 + g^2h + ghg,$$

so $g^2h = hg^2$. Thus squares in the group G are central, so commutators are central and, as observed earlier, they have order (1 or) 2 as well. Let $a, b, c \in G$. Then $(a \circ b) \circ c = 0$ implies $abc + bac + cab + cba = 0$, so one has: if $ab \neq ba$, then $abc = cab$ or $abc = cba$.

Take $a, b, c \in G$ and suppose $ab \neq ba$ and $ac \neq ca$. Write $ab = bas_1$ and $ac = cas_2$ with $s_1 = (a, b) \neq 1$ and $s_2 = (a, c) \neq 1$. Since ab and b do not commute, either $(ab)bc = c(ab)b$ or $(ab)bc = cb(ab)$. The first equation gives $ab^2c = cab^2$, so $ac = ca$, which is false. Thus $ab^2c = cbab$, $acb^2 = cbab$ and $acb = cba$. Thus

$$cba = acb = cabs_2 = cbas_1s_2,$$

so $s_1s_2 = 1$, giving $s_1 = s_2$ (because $s_2^2 = 1$). Thus G has property (C) and the result follows from Lemma 2.2. □

Remark 2.5. In general, if RG is Jordan nilpotent of index $n > 3$, the characteristic of R must be a divisor of 2^{n-1}. If it is 2^{n-1}, our arguments show that G must be abelian. If it is 2^{n-2}, by adding sufficiently many $\circ$ 1s to certain equations, our arguments can be adapted to show that G must again have a unique nonidentity commutator.

3 Jordan nilpotence of $(RG)^+$

Now assume that the group G has an involution $g \mapsto g^*$ and R is a ring of characteristic different from 2. Extending linearly to RG, that is, setting $\alpha^* = \sum \alpha_g g^*$ for $\alpha = \sum \alpha_g g \in RG$, we obtain an involution on RG which we continue to denote $*$. The set

$$(RG)^+ = \{\alpha \in RG \mid \alpha^* = \alpha\}$$

of elements in RG that are symmetric with respect to this involution is a Jordan ring under the Jordan operation $\circ$.

It is convenient here to characterize the elements of $(RG)^+$. Thus we suppose that $\alpha = \sum \alpha_g g \in (RG)^+$. Then $\alpha = \alpha^*$ is the statement $\sum \alpha_g g = \sum \alpha_g g^*$ which implies that for each g in the support of α, there exists an element h in the support of α^* with $\alpha_h h^* = \alpha_g g$. So $\alpha_h = \alpha_g$ and $h = g^*$. It follows that $(RG)^+$

is spanned over R by the set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ where

$$\mathcal{S}_1 = \{g \in G \mid g^* = g\} \quad \text{and} \quad \mathcal{S}_2 = \{g + g^* \mid g \in G,\ g^* \neq g\}.$$

Suppose $(RG)^+$ is Jordan nilpotent of index 3. Thus $(\alpha \circ \beta) \circ \gamma = 0$ for all $\alpha, \beta, \gamma \in \mathcal{S}$. In particular, $(1 \circ 1) \circ 1 = 0$ gives immediately char $R = 4$. Next, taking $\alpha = g$ and $\beta = h$ both in $\mathcal{S}_1$, and $\gamma = 1$, we have $(g \circ h) \circ 1 = 0$, so $2(gh + hg) = 0$ giving $gh = hg$. Thus $\mathcal{S}_1$ is a commutative set, hence closed under multiplication and a subgroup of G.

Suppose $g \in \mathcal{S}_1$ and $h^* \neq h$ (so $h + h^* \in \mathcal{S}_2$). Then $[g \circ (h + h^*)] \circ 1 = 0$ implies $2(gh + gh^* + hg + h^*g) = 0$. This forces some equalities amongst the elements gh, gh^*, hg, h^*g; specifically, since $gh \neq gh^*$, one has either $gh = hg$ or $gh = h^*g$. If the latter, we have $(gh)^* = h^*g^* = h^*g = gh$, so $gh \in \mathcal{S}_1$, implying $h \in \mathcal{S}_1$, a contradiction. Thus $gh = hg$ for all $h \notin \mathcal{S}_1$ (but certainly also for h in the abelian group $\mathcal{S}_1$). This paragraph shows that $\mathcal{S}_1 \subseteq \mathcal{Z}(G)$. In particular, $gg^* \in \mathcal{Z}(G)$ and $gg^* = g^*g$ for any $g \in G$.

To describe the possibilities for a pair of noncommuting elements, we provide a basic lemma.

Lemma 3.1. *For any elements g and h which do not commute, one of the following holds:*

(1a) $gh = g^*h^*$, $h^*g^* = hg$, $gh^* = g^*h$, $hg^* = h^*g$, $g^{-1}g^* = h^{-1}h^*$ *is a central element of order* 2, *and both g^2 and h^2 are in $\mathcal{S}_1$ and hence central,*

(2a) $gh = hg^*$, $h^*g^* = gh^*$, $g^*h = hg$, $h^*g = g^*h^*$, $g^{-1}g^* = (g,h)$ *and* h^2 *commutes with* g,

(3a) $gh = h^*g$, $h^*g^* = g^*h$, $gh^* = hg$, $hg^* = g^*h^*$, $h^{-1}h^* = (h,g)$ *and* g^2 *commutes with* h.

It will be productive to note that case (3a) is the same as case (2a), but with the roles of g and h interchanged. Thus anything that follows from case (2a) yields an analogous fact in case (3a).

Proof. Since $g^* \neq g$ and $h^* \neq h$, the elements $\alpha = g + g^*$ and $\beta = h + h^*$ are in $\mathcal{S}_2$ and $1 \in \mathcal{S}_1$, so $(\alpha \circ \beta) \circ 1 = 0$; that is,

$$\begin{aligned} 0 &= 2[(g + g^*) \circ (h + h^*)] \\ &= 2(gh + gh^* + g^*h + g^*h^* + hg + hg^* + h^*g + h^*g^*). \end{aligned} \tag{3.1}$$

It follows that there are some equalities amongst the group elements in the support of the right side of this equation. Since $gh \neq hg$ (and $g^* \neq g$ and $h^* \neq h$), there are four cases to examine.

Case 1: $gh = g^*h^*$. Applying the involution, we have also $h^*g^* = hg$, equation (3.1) becomes $2(gh^*+g^*h+hg^*+h^*g) = 0$ and so $gh^* \in \{g^*h, hg^*, h^*g\}$. There are three subcases.

Case 1a. Assume $gh^* = g^*h$ (and so $hg^* = h^*g$). Then

$$g^{-1}g^* = h^*h^{-1} = h^{-1}h^*$$

(because h and h^* commute). Also,

$$h^*h^{-1}h^* = h^*g^{-1}g^* = h^*g^*g^{-1} = hgg^{-1} = h,$$

so $(h^{-1}h^*)^2 = 1$. Since $h^{-1}h^* \neq 1$ (else $h \in \mathcal{S}_1$), the element $g^{-1}g^* = h^{-1}h^*$ has order 2 and $g^2 = (g^2)^* \in \mathcal{S}_1$. Similarly, $h^2 \in \mathcal{S}_1$. In particular, g^2 is central, h^2 is central, and $g^{-1}g^*$ is fixed under $*$ and so central as well. This completes Case 1a.

Case 1b. Assume $gh^* = hg^*$. Thus $gh^* \in \mathcal{S}_1$ and hence central. This cannot be, however; otherwise, $(gh^*)h = h(gh^*)$ with $hh^* = h^*h$ would give $gh = hg$, a contradiction.

Case 1c. Assume $gh^* = h^*g$. Here $gh^*h = h^*gh = h^*g^*h^*$. Since h and h^* commute, we have $ghh^* = h^*g^*h^*$, so $gh = h^*g^* = hg$, again contrary to assumption.

We turn to the other possibilities for gh.

Case 2: $gh = hg^*$. Here, $g^* = h^{-1}gh$, so $g^{-1}g^* = g^{-1}h^{-1}gh = (g, h)$, the commutator of g and h. Also, since $h^*g^* = gh^*$ in this situation, equation (3.1) reduces to $2(g^*h + g^*h^* + hg + h^*g) = 0$. Again there must be some equalities, this time amongst the elements g^*h, g^*h^*, hg, h^*g and, since $g^*h \neq g^*h^*$, there are just two subcases to consider.

Case 2a. Assume $g^*h = hg$ (so also $h^*g = g^*h^*$). Here,

$$g^* = hgh^{-1} = h^{-1}gh,$$

so g and h^2 commute.

Case 2b. Assume $g^*h = h^*g$. Then g^*h is in $\mathcal{S}_1$, hence central, implying commutativity with h^*. This cannot be, however, for $(g^*h)h^* = h^*(g^*h)$ yields $g^*h^*h = h^*g^*h$, $g^*h^* = h^*g^*$ and then $gh = hg$. Case 2b cannot occur.

We turn to Case 3, which is very similar to Case 2.

Case 3: $gh = h^*g$. Here we have also $h^*g^* = g^*h$, and equation (3.1) reduces to $2(gh^* + g^*h^* + hg + hg^*) = 0$. Since $gh^* \neq g^*h^*$, there are again just two possibilities.

Case 3a. Assume $gh^* = hg$ and hence $hg^* = g^*h^*$. Here we have

$$h^* = g^{-1}hg = ghg^{-1},$$

so g^2 commutes with h. Also $h^{-1}h^* = h^{-1}g^{-1}hg = (h, g)$, the commutator of h and g.

Case 3b. Assume $gh^* = hg^*$. Then one has $gh^* \in \mathcal{S}_1$, so this element commutes with h, but $(gh^*)h = h(gh^*)$ implies $ghh^* = hgh^*$ and so $gh = hg$, which is not correct. We conclude that Case 3b cannot occur.

Case 4: $gh = h^*g^*$. This case cannot occur because it says $gh \in \mathcal{S}_1$, giving gh central, contradicting the fact that gh does not commute with h.

This completes the proof of the lemma. □

Not all squares fixed by *

Our results concerning the Jordan nilpotence of $(RG)^+$ depend on whether or not $g^2 \in \mathcal{S}_1$ for all $g \in G$. We consider first the situation where such is not the case.

Theorem 3.2. *Suppose the characteristic of R is different from 2 and $\alpha \mapsto \alpha^*$ is an involution on the group ring RG that extends linearly an involution on G which has the property that not all squares in G are fixed by $*$. Then the Jordan ring $(RG)^+$ of symmetric elements is nilpotent of index 3 if and only if* char $R = 4$ *and G is either abelian or nonabelian with the following properties:*

(a) *any $g \in G$ with $g^* = g$ is central,*

(b) *G has an abelian subgroup A of index 2,*

(c) *there exists $c \notin A$ with $c^* \neq c$, but $(c^2)^* = c^2$,*

(d) *$a^* = c^{-1}ac$ for all $a \in A$.*

We establish the theorem in the straightforward direction first, showing that if char $R = 4$ and either G is abelian or a group satisfying properties (a)–(d), then the ring $(RG)^+$ is Jordan nilpotent of index 3. For this, it suffices to establish $(\alpha \circ \beta) \circ \gamma = 0$ whenever $\alpha, \beta, \gamma \in \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. This is clear if G is abelian.

Suppose that G is a nonabelian group that satisfies properties (a) through (d). Then no element of the form ac can be central (or, hence, fixed by $*$) since ac central would make $G = A \cup A(ac)$ abelian. Thus the elements of $\mathcal{S}$ are elements of $\mathcal{Z}(G)$, the centre of G, or of the form $a + a^*$, $a^* \neq a$, $a \in A$, or of the form $(ac) + (ac)^*$, $a \in A$.

Let $a \in A$. Property (d) of the theorem says $ca^* = ac$ (and, applying the involution, $ac^* = c^*a^*$ too). Properties (a) and (c) imply centrality of c^2, so

$$a^*c = c^{-1}ac^2 = ca$$

(and, applying the involution, $c^*a = a^*c^*$ as well). So we have

$$ca^* = ac, \quad ac^* = c^*a^*, \quad a^*c = ca \quad \text{and} \quad c^*a = a^*c^*.$$

It follows that

$$\begin{aligned} c(a + a^*) &= a^*c + ac = (a + a^*)c, \\ (c + c^*)a &= a^*c + a^*c^* = a^*(c + c^*), \\ (a + a^*)(c + c^*) &= (c + c^*)(a + a^*). \end{aligned}$$

Furthermore, since cc^* is invariant under $*$ and hence central, we have $cc^* = c^*c$, so

$$(c + c^*)^2 = c^2 + 2cc^* + (c^*)^2 = 2c^2 + 2cc^* = 2c(c + c^*).$$

All these ideas will be useful in what comes next.

For instance, to show that $(RG)^+$ is Jordan nilpotent of index 3, it suffices to show that $(\alpha \circ \beta) \circ \gamma = 0$ for α, β, γ of the form

- $z \in A$ central,
- $a + a^*, a \in A$,
- $(ac) + (ac)^* = ac + c^*a^* = ac + ac^* = a(c + c^*)$.

We now show that $(RG)^+$ is indeed Jordan nilpotent of index 3.

If two of α, β, γ are central, then $(\alpha \circ \beta) \circ \gamma = 4\alpha\beta\gamma = 0$. If just one of α, β, γ is central, there is no loss of generality in assuming that such element is $\alpha = z \in A$. There are four possibilities for β and γ.

(i) If $\beta = a_1 + a_1^*$ and $\gamma = a_2 + a_2^*$, then

$$(\alpha \circ \beta) \circ \gamma = [2z(a_1 + a_1^*)] \circ (a_2 + a_2^*) = 0$$

because $z(a_1 + a_1^*)$ and $a_2 + a_2^*$ commute (and char $R = 4$).

(ii) If $\beta = a_1 + a_1^*$ and $\gamma = a_2(c + c^*)$, then

$$\begin{aligned} (\alpha \circ \beta) \circ \gamma &= [2z(a_1 + a_1^*)] \circ [a_2(c + c^*)] \\ &= 2z[(a_1 + a_1^*)a_2(c + c^*) + a_2(c + c^*)(a_1 + a_1^*)] = 0 \end{aligned}$$

because $a_1 + a_1^*$ commutes with both a_2 and $c + c^*$. (The elements a_2 and $a_1 + a_1^*$ are each in the abelian subgroup A.)

(iii) If $\beta = a_1(c + c^*)$ and $\gamma = a_2 + a_2^*$, then

$$(\alpha \circ \beta) \circ \gamma = [2za_1(c + c^*)] \circ (a_2 + a_2^*)$$
$$= 2z[a_1(c + c^*)(a_2 + a_2^*) + (a_2 + a_2^*)a_1(c + c^*)] = 0$$

because $a_2 + a_2^*$ commutes with both $c + c^*$ and a_1.

(iv) If $\beta = a_1(c + c^*)$ and $\gamma = a_2(c + c^*)$, then

$$(\alpha \circ \beta) \circ \gamma = [2za_1(c + c^*)] \circ [a_2(c + c^*)]$$
$$= 2z[a_1(c + c^*)a_2(c + c^*) + a_2(c + c^*)a_1(c + c^*)]$$
$$= 2z[a_1a_2^*(c + c^*)^2 + a_2a_1^*(c + c^*)^2]$$

which is 0 because $(c + c^*)^2 = 2c(c + c^*)$.

It remains to show that $(\alpha \circ \beta) \circ \gamma = 0$ in the eight cases that arise with none of α, β, γ central, that is, where each has the form $a + a^*$ or $a(c + c^*)$ with $a \in A$.

(1) If $\alpha = a_1 + a_1^*$, $\beta = a_2 + a_2^*$, $\gamma = a_3 + a_3^*$, then $(\alpha \circ \beta) \circ \gamma = 4\alpha\beta\gamma = 0$ because α, β, γ are in the abelian subgroup A.

(2) If $\alpha = a_1 + a_1^*$, $\beta = a_2 + a_2^*$, $\gamma = a_3(c + c^*)$, then

$$(\alpha \circ \beta) \circ \gamma = [2(a_1 + a_1^*)(a_2 + a_2^*)] \circ [a_3(c + c^*)]$$
$$= 2[(a_1 + a_1^*)(a_2 + a_2^*)a_3(c + c^*)$$
$$+ a_3(c + c^*)(a_1 + a_1^*)(a_2 + a_2^*)] = 0$$

because $a + a^*$ and $c + c^*$ commute for any $a \in A$.

(3) If $\alpha = a_1 + a_1^*$, $\beta = a_2(c + c^*)$, $\gamma = a_3 + a_3^*$, then

$$\alpha \circ \beta = (a_1 + a_1^*)a_2(c + c^*) + a_2(c + c^*)(a_1 + a_1^*)$$
$$= 2a_2(a_1 + a_1^*)(c + c^*)$$

because $a_1 + a_1^*$ commutes with a_2, and $c + c^*$ commutes with $a_1 + a_1^*$. So

$$(\alpha \circ \beta) \circ \gamma = [2a_2(a_1 + a_1^*)(c + c^*)] \circ (a_3 + a_3^*)$$
$$= 2[a_2(a_1 + a_1^*)(c + c^*)(a_3 + a_3^*)$$
$$+ (a_3 + a_3^*)a_2(a_1 + a_1^*)(c + c^*)] = 0$$

because any two of $c + c^*$, $a_3 + a_3^*$, a_2 and $a_1 + a_1^*$ commute.

(4) If $\alpha = a_1 + a_1^*$, $\beta = a_2(c + c^*)$, $\gamma = a_3(c + c^*)$, then

$$\alpha \circ \beta = 2a_2(a_1 + a_1^*)(c + c^*)$$

as before, and so

$$\begin{aligned}(\alpha\circ\beta)\circ\gamma &= [2a_2(a_1+a_1^*)(c+c^*)]\circ[a_3(c+c^*)]\\ &= 2[a_2(a_1+a_1^*)(c+c^*)a_3(c+c^*)\\ &\qquad + a_3(c+c^*)a_2(a_1+a_1^*)(c+c^*)]\\ &= 2[a_2(a_1+a_1^*)a_3^*(c+c^*)^2\\ &\qquad + a_3a_2^*(c+c^*)(a_1+a_1^*)(c+c^*)]\end{aligned}$$

which is 0 because $c+c^*$ commutes with $a_1+a_1^*$ and $(c+c^*)^2 = 2c(c+c^*)$.

(5) The case $\alpha = a_1(c+c^*)$, $\beta = a_2+a_2^*$, $\gamma = a_3+a_3^*$ is the same as case (3) because $\alpha\circ\beta = \beta\circ\alpha$.

(6) The case $\alpha = a_1(c+c^*)$, $\beta = a_2+a_2^*$, $\gamma = a_3(c+c^*)$ is the same as case (4).

(7) If $\alpha = a_1(c+c^*)$, $\beta = a_2(c+c^*)$, $\gamma = a_3+a_3^*$, then

$$\begin{aligned}\alpha\circ\beta &= a_1(c+c^*)a_2(c+c^*) + a_2(c+c^*)a_1(c+c^*)\\ &= a_1a_2^*(c+c^*)^2 + a_2a_1^*(c+c^*)^2\\ &= 2(a+a^*)c(c+c^*), \qquad \text{with } a = a_1a_2^*,\end{aligned}$$

and so

$$\begin{aligned}(\alpha\circ\beta)\circ\gamma &= [2(a+a^*)c(c+c^*)]\circ(a_3+a_3^*)\\ &= 2[(a+a^*)c(c+c^*)(a_3+a_3^*)\\ &\qquad + (a_3+a_3^*)(a+a^*)c(c+c^*)] = 0\end{aligned}$$

because both c and $c+c^*$ commute with $a_3+a_3^*$.

(8) If $\alpha = a_1(c+c^*)$, $\beta = a_2(c+c^*)$, $\gamma = a_3(c+c^*)$, then setting $a = a_1a_2^*$ as in the previous case, we have

$$\begin{aligned}(\alpha\circ\beta)\circ\gamma &= [2(a+a^*)c(c+c^*)]\circ[a_3(c+c^*)]\\ &= 2[(a+a^*)c(c+c^*)a_3(c+c^*)\\ &\qquad + a_3(c+c^*)(a+a^*)c(c+c^*)]\\ &= 2[(a+a^*)ca_3^*(c+c^*)^2\\ &\qquad + a_3(a+a^*)(c+c^*)c(c+c^*)] = 0.\end{aligned}$$

We now attack the theorem in the main direction. The underlying assumption in everything that follows is that $(RG)^+$ is Jordan nilpotent of index 3, so $(\alpha\circ\beta)\circ\gamma = 0$ for any $\alpha,\beta,\gamma\in\mathcal{S} = \mathcal{S}_1\cup\mathcal{S}_2$. We have noted earlier that the characteristic of R must be 4 and that the result is clear if G is abelian.

Thus we assume that G is nonabelian and show that G satisfies the four conditions (a)–(d) of the theorem. Since $g = g^*$ implies $g \in \mathcal{S}_1$, which is central, we already have (a).

To establish properties (b), (c) and (d), we proceed by adapting some ideas which have proven helpful in the past [4].

Let $T = \{g \in G \mid (g^2)^* \neq g^2\}$. By assumption $T \neq \emptyset$. Let $g \in T$, $h \in T$ and suppose $gh \neq hg$. Thus $g \notin \mathcal{S}_1$, $h \notin \mathcal{S}_1$ (because $\mathcal{S}_1$ is central), so the pair g, h falls into one of the cases (1a), (2a), (3a) described in Lemma 3.1, but not (1a) because $g^2 \notin \mathcal{S}_1$ (or, equally, because $h^2 \notin \mathcal{S}_1$).

Assume the pair g, h falls into case (2a). We shall make use, in particular, of these equations:

$$g^*h = hg \quad \text{and} \quad h^*g = g^*h^*.$$

Since gh and h do not commute (so neither is in $\mathcal{S}_1$), gh and h satisfy the properties of cases (1a), (2a) or (3a), but not (1a) because $h^2 \notin \mathcal{S}_1$. If gh and h satisfy the equations of (2a), then the first equation of (2a) says

$$gh^2 = h(gh)^* = hh^*g^* = g^*h^*h,$$

so $gh = g^*h^*$ (and $hg = h^*g^*$). But now

$$h^*g^2 = g^*h^*g = g(hg) = gg^*h = hgg^* = h^*(g^*)^2,$$

giving $g^2 = (g^*)^2$, which is not true. If gh and h satisfy the equations of (3a), then the first equation of (3a) reads $gh^2 = h^*gh = g^*h^*h$, once more we have $gh = g^*h^*$ and $hg = h^*g^*$ and, as just shown, this cannot happen.

Assume g, h fall into case (3a). We shall make use of the facts

$$gh = h^*g \quad \text{and} \quad gh^* = hg.$$

Since $g^2 \neq (g^2)^*$, the pair hg, g must satisfy the conditions of cases (2a) or (3a). If (2a), then the first equation of (2a) applied to hg, g says

$$hg^2 = g(hg)^* = gg^*h^* = h^*g^*g,$$

so $hg = h^*g^*$ (and $gh = g^*h^*$). But now

$$h^*g^2 = (h^*g)g = ghg = g^*(h^*g) = g^*gh = (hg)g^* = h^*(g^*)^2,$$

implying $g^2 = (g^2)^*$, contrary to fact. Similarly, if hg, g satisfy the first equation of (3a), we obtain $hg^2 = g^*(hg) = g^*gh^* = h^*g^*g$, so $hg = h^*g^*$ and $gh = g^*h^*$, which leads to the same falsehood as before.

The contradictions of the previous paragraphs show that T is commutative and so the subgroup $A = \langle T \rangle$ generated by T is abelian.

Lemma 3.3. *Let $A = \langle T \rangle$ be the abelian subgroup generated by T. Then for any $t \in T$ and $g \in G \setminus A$, the elements g and t do not commute and the pair g, t satisfies the conditions of case* (3a) *in Lemma* 3.1. *Moreover, A has index* 2 *in G.*

Proof. Let $t \in T$; thus $t^2 \neq (t^2)^*$. Let $g \in G \setminus A$. Then $g \notin T$, so $(g^2)^* = g^2$. It follows that $gt \notin T$ (else, $gt \in A$, $t \in A$ would imply $g \in A$), so

$$[(gt)^*]^2 = (gt)^2,$$

that is, $t^*g^*t^*g^* = gtgt$. Suppose $gt = tg$. Then $g^*t^* = t^*g^*$ too and we have $(g^*)^2(t^*)^2 = g^2t^2$. Since $(g^*)^2 = g^2$ while $(t^*)^2 \neq t^2$, this cannot happen. We conclude that $gt \neq tg$, so the pair g, t must satisfy the conditions of one of cases (1a), (2a) or (3a) specified in Lemma 3.1. Case (1a) is not a possibility because $t^2 \notin \mathcal{S}_1$. Assume g, t satisfy the conditions of (2a). Then

$$t^*g^*t^*g^* = t^*(g^*t^*)g^* = t^*t^*gg^* = (t^*)^2gg^* = gg^*(t^*)^2,$$

while $gtgt = g(tg)t = gg^*tt = gg^*t^2$. Since $t^2 \neq (t^*)^2$, this case also does not occur. Thus, as claimed, the pair g, t must satisfy the conditions of (3a).

Suppose now that $g \notin A$ and $h \notin A$. Thus each pair g, t and h, t satisfies the equations of case (3a). We show that $gh \in A$ by showing that this element fails the test just described for membership in the complement of A; namely, we show that the pair gh, t does not satisfy the equations of (3a), in particular, the first equation of (3a). To see why, we use the first equation of (3a) as it applies to h, t to obtain $(gh)t = g(ht) = gt^*h$, and then the third equation of (3a) as it applies to g, t to obtain $(gt^*)h = (tg)h = t(gh)$. Since $(gh)t = t(gh) \neq t^*(gh)$, the pair gh, t does not satisfy the first equation of (3a). It follows that $gh \in A$, as claimed.

Now fix $c \notin A$. Thus $c^{-1} \notin A$. For any $g \notin A$, the previous paragraph says $gc^{-1} \in A$, that is, $g \in Ac$. It follows that $G = A \cup Ac$ and the lemma is established. □

This lemma shows that G satisfies statement (b) of Theorem 3.2. Moreover, for any $c \notin A$, we have $c^* \neq c$ because c cannot be central and $(c^2)^* = c^2$ because $c \notin A$ means $c \notin T$. This is property (c) of the theorem. Finally, for any $t \in T$, Lemma 3.3 says that the pair c, t satisfies the conditions of case (3a) of Lemma 3.1, in particular, $ct^* = tc$. So $t^* = c^{-1}tc$. Since T generates A, $a^* = c^{-1}ac$ for all $a \in A$. This is part (d) of the theorem and the proof is complete.

Remark 3.4. Certain dihedral groups provide a family of groups satisfying the conditions of Theorem 3.2. Indeed, let $D_{2n} = \langle a, c \mid a^n = c^2 = 1,\ ca = a^{-1}c \rangle$ with n even and define an involution by setting $a^* = a^{-1}$ and $c^* = a^{n/2}c$.

Squares fixed by *

We turn now to the case that $(g^2)^* = g^2$ for all $g \in G$. For $g \in G$, set $g^{-1}g^* = s_g$. Then $s_g^* = g(g^*)^{-1} = s_g$, so s_g is central, $s_g^2 = 1$ and $g^* = s_g g$. As noted in Section 2, when squares in a group are central, so are commutators and the square of any commutator is 1.

Theorem 3.5. *Let $g \mapsto g^*$ be an involution on a group G with the property that $(g^2)^* = g^2$ for all $g \in G$ and extend $*$ linearly to a group ring RG where* char $R \neq 2$. *Then $(RG)^+$ is Jordan nilpotent of index* 3 *if and only if* char $R = 4$ *and G is either abelian or nonabelian with central squares and the involution has the following properties:*

(a) *Any $g \in G$ with $g^* = g$ is central.*

(b) *For each $g \in G$, there exists a central element s_g with $g^* = s_g g$ and $s_g^2 = 1$.*

(c) *If $g, h \in G$ do not commute, then either $s_g = s_h$ or $s_g = (g, h)$ or $s_h = (g, h)$.*

(d) *For any $g, h, k \in G$ with $g^* \neq g$, $h^* \neq h$ and $k^* \neq k$, one of the following is the case:*

   (i) *three of the elements $s_g, s_h, s_k, (g, h), (gh, k)$ are equal,*

   (ii) *$(g, h) \in \mathcal{A} = \{1, s_g, s_h, s_k, s_g s_h, s_g s_k, s_h s_k, s_g s_h s_k\}$ and s_g is an element of the set $\mathcal{B} = \{s_h, s_k, s_h s_k, s_g s_h s_k\}$,*

   (iii) *$(gh, k) \in \mathcal{A}$ and $s_g \in \mathcal{B}$,*

   (iv) *$(g, h) \in \mathcal{A}$ and $(gh, k) \in \mathcal{A}$,*

   (v) *$(g, h) = (gh, k)x$ for some $x \in \mathcal{A}$, and $s_g \in \mathcal{B}$,*

   (vi) *$(g, h) = (gh, k)x$ for some $x \in \mathcal{A}$, and $s_g = (gh, k)y$ for some $y \in \mathcal{A}$.*

Proof. Assume that $(RG)^+$ is Jordan nilpotent of index 3 and that G is not abelian. As before, the ring must have characteristic 4. The elements of $\mathcal{S}_1$ are always central giving (a). We have already noted (b) and the three equations in part (c) correspond respectively to cases (1a), (2a), (3a) of Lemma 3.1. (Remember that the s_g and all commutators are central with square 1 so an equation like $s_g(g, h) = 1$ implies $s_g = (g, h)$.) It remains to establish (d).

For this, apply the condition $(\alpha \circ \beta) \circ \gamma = 0$ to elements $\alpha, \beta, \gamma \in \mathcal{S}_2$ and obtain immediately

$$(1 + s_g)(1 + s_h)(1 + s_k)(1 + c_1)(1 + c_2) = 0 \tag{3.2}$$

for all $g, h, k \in G$ satisfying $g^* \neq g$, $h^* \neq h$, $k^* \neq k$ (and so $s_g \neq 1$, $s_h \neq 1$ and $s_k \neq 1$) where here, and in everything that follows, we write $c_1 = (g, h)$ and $c_2 = (gh, k)$. There are thirty-two terms in the expansion of the left side of (3.2).

In order for this element to be 0 in characteristic 4, these thirty-two elements must be the union of eight multisets each consisting of four equal elements.

We simplify the situation by introducing the element

$$A = 1 + s_g + s_h + s_k + s_g s_h + s_g s_k + s_h s_k + s_g s_h s_k$$

and rewriting (3.2) as

$$A + c_1 A + c_2 A + c_1 c_2 A = 0, \quad \text{equivalently,} \quad (1 + c_2)(A + c_1 A) = 0. \tag{3.3}$$

Note that the set $\mathcal{A}$ of the theorem is supp A, the support of A, which in this case is a group. We consider four exhaustive possibilities.

Case I: Four elements in $\mathcal{A}$ are all equal to s_g. Here, since $s_g \neq 1, s_g \neq s_g s_h$, $s_g \neq s_g s_k$ and $s_g = s_g s_h s_k$ implies $s_h = s_k$, it must be that $s_g = s_h = s_k$, scenario (i).

Case II: $s_g = c_1 x$ for some $x \in \mathcal{A}$. Here $c_1 = s_g x \in \mathcal{A}$, so

$$c_1 A = A \quad \text{and} \quad (1 + c_2)(A + c_1 A) = 2(1 + c_2)A = 0.$$

There are two subcases. In the first subcase, $s_g = y$ for some $y \in \operatorname{supp} A$ with $y \notin \{1, s_g, s_g s_h, s_g s_k\}$ putting us in scenario (ii). In the second subcase, $s_g = c_2 y$ for some $y \in \mathcal{A}$ so $c_2 = s_g y \in \mathcal{A}$, as in (iv).

Case III: $s_g = c_2 x$ for some $x \in \mathcal{A}$. Here $c_2 \in \mathcal{A}$, so

$$c_2 A = A \quad \text{and} \quad (1 + c_2)(A + c_1 A) = 2(1 + c_1)A = 0.$$

It follows that either $s_g \in \{s_h, s_k, s_h s_k, s_g s_h s_k\}$ (so the situation is as described in (iii)) or $s_g = c_1 y$, $y \in \mathcal{A}$, so $c_1 = s_g y \in \mathcal{A}$, as described by (iv).

Case IV: $s_g = c_1 c_2 x$ for some $x \in \mathcal{A}$. In this case, we have $c_1 = c_2 y$ with $y = s_g x \in \mathcal{A}$. Thus

$$c_1 A = c_2 A \quad \text{and} \quad (1 + c_2)(A + c_1 A) = 2(A + c_2 A) = 0$$

and we have either $s_g \in \{s_h, s_k, s_h s_k, s_g s_h s_k\}$ (which is (v)) or $s_g = c_2 w$, $w \in \mathcal{A}$, which is (vi).

The converse. Now we show that if char $R = 4$ and G is a nonabelian group satisfying properties (a)–(d) of the theorem, then $(RG)^+$ is Jordan. It suffices to show that

$$(\alpha \circ \beta) \circ \gamma = 0 \tag{3.4}$$

for all $\alpha, \beta, \gamma \in \mathcal{S}_1 \cup \mathcal{S}_2$. In characteristic 4, it is easy to see that (3.4) holds if two or three of α, β, γ are in $\mathcal{S}_1$ since elements of $\mathcal{S}_1$ are central by (a). If just one of α, β, γ is in $\mathcal{S}_1$, there is no loss of generality if we assume this is α. Thus let $\alpha = z \in \mathcal{Z}(G)$, $\beta = g + g^*$ and $\gamma = h + h^*$ with $g, h \in G$, $g^* \neq g$, $h^* \neq h$. We have $\beta = g + s_g g = (1+s_g)g$ with $1 \neq s_g \in \mathcal{Z}(G)$ and $s_g^2 = 1$ and, similarly, $\gamma = (1 + s_h)h$ with $1 \neq s_h \in \mathcal{Z}(G)$ and $s_h^2 = 1$. Since $\alpha \circ \beta = 2(1 + s_g)zg$, the left side of (3.4) becomes

$$2(1 + s_g)zg \circ (1 + s_h)h = 2z(1 + s_g)(1 + s_h)(gh + hg)$$
$$= \begin{cases} 4z(1 + s_g)(1 + s_h)gh & \text{if } gh = hg, \\ 2z(1 + s_g)(1 + s_h)(1 + (g, h))gh & \text{if } gh \neq hg. \end{cases}$$

In view of part (c) of the theorem and noting that $(1 + s)^2 = 2(1 + s)$ for any central s with $s^2 = 1$, $(\alpha \circ \beta) \circ \gamma = 0$ in either case.

It remains to verify (3.4) with $\alpha = g + g^*$, $\beta = h + h^*$ and $\gamma = k + k^*$ all in $\mathcal{S}_2$. As observed previously, for such elements, (3.4) is equivalent to either of equations (3.2) or (3.3), which we proceed to verify in each of the six cases identified in part (d) of the theorem. We again set $(g, h) = c_1$ and $(gh, k) = c_2$.

In case (i), (3.2) is satisfied because $(1 + s)^3 = 4(1 + s)$ whenever $s^2 = 1$.

In case (ii), $c_1 \in \mathcal{A} = \operatorname{supp} A$, which is a group, so $c_1 A = A$ and $A + c_1 A = 2A$. With $s_g \in \{s_h, s_k, s_h s_k, s_g s_h s_k\}$, it is simple to check that for each possible s_g, $A = 2B$ for some B. So (3.3) holds.

In case (iii), s_g is again in the set $\{s_h, s_k, s_h s_k, s_g s_h s_k\}$, so $A = 2B$ for some B. Furthermore, $c_2 \in \operatorname{supp} A$, so $c_2 A = A = 2B$ and

$$A + c_1 A + c_2 A + c_1 c_2 A = A + c_1 A + A + c_1 A = 2(1 + c_1)A = 0$$

because $A = 2B$.

In case (iv), both c_1 and c_2 are in the group $\operatorname{supp} A$, so $c_1 A = c_2 A = A$ and (3.3) is clear.

In case (v), we have $A = 2B$ for some B and, with $c_1 = c_2 x$ and $x \in \operatorname{supp} A$, $c_1 A = c_2 A$, so

$$A + c_1 A + c_2 A + c_1 c_2 A = A + c_1 A + c_2 A + A = 2(1 + c_1)A = 0$$

(using $c_1^2 = 1$).

Finally, in case (vi), we again have $c_1 A = c_2 A$ and $c_2 A = s_g A = A$, so

$$A + c_1 A + c_2 A + c_1 c_2 A = A + c_1 A + c_1 A + A = 4A = 0. \qquad \square$$

We conclude with an example which shows that $(RG)^+$ Jordan nilpotent of index 3 does not imply that RG is Jordan nilpotent of index 3. Indeed, the example is a group G satisfying the conditions labelled (a)–(d) in Theorem 3.5 (so, in characteristic 4, $(RG)^+$ is Jordan nilpotent of index 3), but G is not abelian (so RG is not Jordan nilpotent of index 3 in characteristic 4), nor is there a unique commutator.

Example 3.6. Let $G = G_1 \times G_2$ be the direct product of SLC groups G_1 and G_2 with $G_1' = \{1, s_1\}$ and $G_2' = \{1, s_2\}$ and define $(a,b)^* = (a^*, b^*)$ for $(a,b) \in G$ where the involutions in G_1 and G_2 are canonical. We claim that the group G satisfies the conditions of Theorem 3.5.

(a) For any $g = (a,b) \in G$, $g^* = g$ implies $a^* = a$ and $b^* = b$, so g is central.

(b) For $g = (a,b) \in G$, there are four possibilities for s_g; namely, $(1,1)$, $(s_1, 1)$, $(1, s_2)$ and (s_1, s_2). Each of these is central of order 2.

(c) Suppose $g = (a,b)$ and $h = (x,y)$ do not commute. Then one and only one of the following holds:
  - $ax \neq xa$ and $by = yb$, in which case $s_g = (s_1, 1) = (g,h)$,
  - $by \neq yb$ and $ax = xa$, in which case $s_g = (1, s_2) = (g,h)$,
  - $ax \neq xa$ and $by \neq yb$, in which case $s_g = (s_1, s_2) = (g,h)$.

(d) If $g, h, k \in G$ with $g^* \neq g$, $h^* \neq h$ and $k^* \neq k$, then $s_g \neq 1$, $s_h \neq 1$ and $s_k \neq 1$. If $s_g = s_h = s_k$, we have (i). Otherwise, the central subgroup $\mathcal{A}$ has order 4. Thus $\mathcal{A} = \mathcal{Z}(G)$ and we have (iv).

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