

# Counting conjugacy classes of cyclic subgroups for fusion systems

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**Abstract.** Thévenaz [6] made an interesting observation that the number of conjugacy classes of cyclic subgroups in a finite group  $G$  is equal to the rank of the matrix of the numbers of double cosets in  $G$ . We give another proof of this fact and present a fusion system version of it. In particular we use finite groups realizing the fusion system  $\mathcal{F}$  as in our previous work [3].

## 1 Statements of the results

In [6], Thévenaz observed the ‘curiosity’ that a finite cyclic group  $G$  can be characterized by the nonsingularity of the matrix of the numbers of double cosets in  $G$ . In fact he proved a stronger proposition that for an arbitrary finite group  $G$  the number of the conjugacy classes of cyclic subgroups of  $G$  is equal to the rank of that matrix. This can be stated slightly more generally by introducing a subgroup  $H$  of  $G$  and considering the  $G$ -conjugacy classes of subgroups of  $H$  as follows.

**Theorem 1.1.** *Let  $G$  be a finite group and let  $H \leq G$ . The rank of the matrix  $(|P \backslash G / Q|)_{P, Q \leq_G H}$ , whose rows and columns are indexed by the  $G$ -conjugacy classes of subgroups of  $H$  and whose entries are the numbers of the corresponding double cosets in  $G$ , is equal to the number of the  $G$ -conjugacy classes of cyclic subgroups of  $H$ .*

In the above theorem, the matrix  $(|P \backslash G / Q|)_{P, Q \leq_G H}$  is determined by the  $(H, H)$ -biset  ${}_H G_H$ , i.e., the set  $G$  with left and right  $H$ -action (induced by multiplication in the group  $G$ ) which are compatible. In particular, when  $H = S$  is a Sylow  $p$ -subgroup of  $G$ , the  $(S, S)$ -biset  ${}_S G_S$ , viewed as a left  $S \times S$ -set, decomposes as

$${}_S G_S \cong \coprod_{x \in [S \backslash G / S]} S \times S / \Delta(c_x, S \cap S^x).$$

See Section 3 for the notation. Consequently the  $(S, S)$ -biset  ${}_S G_S$  determines the  $p$ -fusion of  $G$ , i.e., the conjugacy relation between  $p$ -subgroups of  $G$ . The es-

sential feature of the  $p$ -fusion in finite groups is generalized to categories called *saturated fusion systems*, which connect the  $p$ -local aspects of group theory, representation theory and homotopy theory. We refer the reader to the book [1] for an introduction to the subject.

In [3], we observed that every saturated fusion system  $\mathcal{F}$  on a finite  $p$ -group  $S$  can be realized by a finite group  $G$  containing  $S$  as a (not necessarily Sylow)  $p$ -subgroup. Thus the above theorem yields a fusion system version as follows.

**Theorem 1.2.** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$ . Let  $G$  be a finite group which contains  $S$  as a subgroup and realizes  $\mathcal{F}$ . Then the rank of the matrix  $(|P \backslash G / Q|)_{P, Q \leq_G S}$  is equal to the number of the  $\mathcal{F}$ -conjugacy classes of cyclic subgroups of  $S$ .*

By a result of Broto, Levi and Oliver [2, Proposition 5.5], every saturated fusion system  $\mathcal{F}$  on a finite  $p$ -group  $S$  has a (non-unique) characteristic biset  $\Omega$ . See Section 3 for a precise definition; in particular,  $\Omega$  is a finite  $(S, S)$ -biset. If  $\mathcal{F}$  is the fusion system of a finite group  $G$  on its Sylow  $p$ -subgroup  $S$ , then  $G$  is a characteristic biset for  $\mathcal{F}$  with the obvious  $S$ -action on the left and right. So we may well expect that the matrix of the above theorem, with  $G$  replaced by  $\Omega$ , has the same rank. Indeed this is the case.

**Theorem 1.3.** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$ . Let  $\Omega$  be a characteristic biset for  $\mathcal{F}$ . Then the rank of the matrix  $(|P \backslash \Omega / Q|)_{P, Q \leq_{\mathcal{F}} S}$  of the number of  $(P, Q)$ -orbits of  $\Omega$  indexed by the  $\mathcal{F}$ -conjugacy classes of subgroups of  $S$  is equal to the number of the  $\mathcal{F}$ -conjugacy classes of cyclic subgroups of  $S$ .*

Finally, one can replace the characteristic biset  $\Omega$  in the above theorem by the characteristic idempotent  $\omega_{\mathcal{F}}$  (which is a virtual  $(S, S)$ -biset; see Section 3) with  $|P \backslash \omega_{\mathcal{F}} / Q|$  as the linearized number of  $(P, Q)$ -orbits.

**Theorem 1.4.** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$ . Let  $\omega_{\mathcal{F}}$  be the characteristic idempotent for  $\mathcal{F}$ . The rank of the matrix  $(|P \backslash \omega_{\mathcal{F}} / Q|)_{P, Q \leq_{\mathcal{F}} S}$  is equal to the number of the  $\mathcal{F}$ -conjugacy classes of cyclic subgroups of  $S$ .*

The bisets appearing in Theorems 1.3 and 1.4 play an important role in the theory of fusion systems. See [4] for more details.

We will give a proof of Theorem 1.1 (and hence obtain Theorem 1.2 as a corollary), which is slightly different from that of [6]. This new proof uses (at least explicitly) only the Burnside ring  $B(G)$  of  $G$ , not the rational representation ring  $R_{\mathbb{Q}}(G)$  as in [6]. Therefore it is better suited for adapting to the fusion system case (Theorems 1.3 and 1.4), which we do subsequently.

## 2 The group case

We prove Theorem 1.1. As remarked in Section 1, Theorem 1.2 then immediately follows as a corollary by [3].

Let  $G$  be a finite group. Let  $B(G)$  be the Burnside ring of  $G$ , i.e., the Grothendieck ring of the isomorphism classes  $[X]$  of finite  $G$ -sets  $X$ . As an additive group,  $B(G)$  is a free abelian group with the canonical basis  $\{[G/P] \mid P \leq_G G\}$ . Let  $\mathbb{Q}B(G) = \mathbb{Q} \otimes_{\mathbb{Z}} B(G)$  and regard  $B(G)$  as a subring of  $\mathbb{Q}B(G)$ . In particular the canonical basis for  $B(G)$  is a  $\mathbb{Q}$ -basis for  $\mathbb{Q}B(G)$ .

It is a well-known fact that for each  $P \leq G$  the fixed-point map

$$\chi_P: B(G) \rightarrow \mathbb{Z}, \quad [X] \mapsto |X^P|,$$

is a ring homomorphism which depends only on the  $G$ -conjugacy class of  $P$ , and the product of these homomorphisms (tensored with  $\mathbb{Q}$ ),

$$\chi = \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{P \leq_G G} \chi_P: \mathbb{Q}B(G) \rightarrow \prod_{P \leq_G G} \mathbb{Q},$$

is a  $\mathbb{Q}$ -algebra isomorphism. For each subgroup  $P \leq G$ , let  $e_P^G$  denote the element of  $\mathbb{Q}B(G)$  such that

$$\chi_Q(e_P^G) = \begin{cases} 1, & P =_G Q, \\ 0, & \text{otherwise.} \end{cases}$$

Then again the element  $e_P^G$  depends only on the  $G$ -conjugacy class of  $P$  and  $\{e_P^G \mid P \leq_G G\}$  is a set of pairwise orthogonal primitive idempotents of  $\mathbb{Q}B(G)$  whose sum is equal to 1; in particular it is a  $\mathbb{Q}$ -basis for  $\mathbb{Q}B(G)$ . Furthermore, for  $H \leq G$ , let  $B(G)_H$  be the subgroup of  $B(G)$  generated by the elements  $[G/P]$  with  $P \leq_G H$ . Then  $\mathbb{Q}B(G)_H = \mathbb{Q} \otimes_{\mathbb{Z}} B(G)_H$  is a subalgebra of  $\mathbb{Q}B(G)$  with  $\mathbb{Q}$ -basis  $\{[G/P] \mid P \leq_G H\}$ . Note that the elements  $e_P^G$  with  $P \leq_G H$  belong to  $\mathbb{Q}B(G)_H$  and hence  $\{e_P^G \mid P \leq_G H\}$  is another basis for  $\mathbb{Q}B(G)_H$ .

For each  $P \leq G$  consider the  $\mathbb{Q}$ -linear map

$$\rho_P: \mathbb{Q}B(G) \rightarrow \mathbb{Q}, \quad [X] \mapsto |P \setminus X|,$$

which counts the  $P$ -orbits. By Burnside's orbit counting lemma, we have

$$\rho_P(x) = \frac{1}{|P|} \sum_{u \in P} \chi_{\langle u \rangle}(x), \quad x \in \mathbb{Q}B(G).$$

Thus

$$\rho_P(e_Q^G) \neq 0 \iff Q \text{ is cyclic and } Q \leq_G P. \tag{2.1}$$

Now the given matrix in Theorem 1.1 is equal to  $(\rho_P(G/Q))_{P,Q \leq_G H}$ . By change of basis, this matrix has the same rank as  $(\rho_P(e_Q^G))_{P,Q \leq_G H}$ . List the subgroups of  $H$  (up to  $G$ -conjugacy) in two families, the first consisting of cyclic subgroups and the second of noncyclic subgroups, and with nondecreasing order in each family. Then by (2.1) the above matrix has the form  $\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$  where  $A$  is a lower triangular matrix with nonzero diagonal entries. Thus Theorem 1.1 follows.

### 3 The fusion system case

We first prove Theorem 1.3. In fact, we prove a slightly generalized version of it.

**Proposition 3.1.** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$ . Let  $\Omega$  be a finite  $(S, S)$ -biset which is  $\mathcal{F}$ -stable and  $\mathcal{F}$ -generated and which contains the obvious  $(S, S)$ -biset  $S$ . Then the rank of the matrix  $(|P \backslash \Omega / Q|)_{P,Q \leq_{\mathcal{F}} S}$  is equal to the number of the  $\mathcal{F}$ -conjugacy classes of cyclic subgroups of  $S$ .*

We first explain the terminology. Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$ . An  $S$ -set  $X$  is  $\mathcal{F}$ -stable if, for all  $P \leq S$  and all morphisms  $\varphi: P \rightarrow S$  in  $\mathcal{F}$ , the restrictions of the  $S$ -action on  $X$  to  $P$  via the inclusion  $P \hookrightarrow S$  and via  $\varphi: P \rightarrow S$  give isomorphic  $P$ -sets. We say that an  $(S, S)$ -biset is  $\mathcal{F}$ -stable if it is  $\mathcal{F} \times \mathcal{F}$ -stable viewed as a left  $S \times S$ -set by inverting the right action of  $S$ . An  $(S, S)$ -biset is  $\mathcal{F}$ -generated if, viewed as a left  $S \times S$ -set, all its isotropy subgroups are of the form  $\Delta(\varphi, P) = \{(\varphi(u), u) \mid u \in P\}$  with  $P \leq S$ ,  $\varphi: P \rightarrow S$  in  $\mathcal{F}$ . A finite  $(S, S)$ -biset  $\Omega$  is called a *characteristic biset* for  $\mathcal{F}$  if it is  $\mathcal{F}$ -stable and  $\mathcal{F}$ -generated and such that  $|\Omega|/|S|$  is not divisible by  $p$ . It is easy to see that every characteristic biset  $\Omega$  contains the  $(S, S)$ -biset  $S$ .

Define

$$B(\mathcal{F}) = \{x \in B(S) \mid \chi_P(x) = \chi_{P'}(x) \text{ for all } P, P' \leq S \text{ with } P =_{\mathcal{F}} P'\}.$$

Clearly  $B(\mathcal{F})$  is a subring of  $B(S)$ , which is called the Burnside ring of the fusion system  $\mathcal{F}$ . For a finite  $S$ -set  $X$ , we have  $[X] \in B(\mathcal{F})$  if and only if  $X$  is  $\mathcal{F}$ -stable. The elements

$$e_P^{\mathcal{F}} := \sum_{P' =_{\mathcal{F}} P} e_{P'}^S,$$

where  $P \leq S$  and the sum is over the  $S$ -conjugacy classes of subgroups  $P'$  of  $S$  which are  $\mathcal{F}$ -conjugate to  $P$ , belong to  $\mathbb{Q}B(\mathcal{F})$ . The set  $\{e_P^{\mathcal{F}} \mid P \leq_{\mathcal{F}} S\}$  is a set of pairwise orthogonal primitive idempotents of  $\mathbb{Q}B(\mathcal{F})$  whose sum is equal to 1; in particular it is a  $\mathbb{Q}$ -basis for  $\mathbb{Q}B(\mathcal{F})$ . By (2.1), we have

$$\rho_P(e_Q^{\mathcal{F}}) \neq 0 \iff Q \text{ is cyclic and } Q \leq_{\mathcal{F}} P. \tag{3.1}$$

Let  $\Omega$  be the  $(S, S)$ -biset given in the above proposition. By the  $\mathcal{F}$ -stability of  $\Omega$ , the left  $S$ -set  $\Omega/P$  of the right  $P$ -orbits of  $\Omega$  is also  $\mathcal{F}$ -stable for  $P \leq S$ . Moreover

$$\chi_Q([\Omega/P]) \neq 0 \implies Q \leq_{\mathcal{F}} P; \quad \chi_P([\Omega/P]) \geq |N_S(P)/P|.$$

The former implication follows from the fact that  $\Omega$  is  $\mathcal{F}$ -generated and the latter inequality comes from the fact that  $\Omega$  contains  $S$ . Hence  $\{[\Omega/P] \mid P \leq_{\mathcal{F}} S\}$  is a  $\mathbb{Q}$ -basis for  $\mathbb{Q}B(\mathcal{F})$ . Thus the matrix

$$(|P \setminus \Omega / Q|)_{P, Q \leq_{\mathcal{F}} S} = (\rho_P([\Omega/Q]))_{P, Q \leq_{\mathcal{F}} S}$$

has the same rank as  $(\rho_P(e^{\mathcal{F}}_Q))_{P, Q \leq_{\mathcal{F}} S}$ , which is equal to the number of the  $\mathcal{F}$ -conjugacy classes of cyclic subgroups of  $S$  by (3.1).

**Remark.** Note that the finite group  $G$  in Theorem 1.2, viewed as an  $(S, S)$ -biset, satisfies the hypotheses for  $\Omega$  in Proposition 3.1. Thus Theorem 1.2 can also be obtained from Proposition 3.1.

Now we address Theorem 1.4. In Proposition 3.1, the condition that the finite  $(S, S)$ -biset  $\Omega$  contains the  $(S, S)$ -biset  $S$  is equivalent to that  $\chi_P(\Omega/P) \neq 0$  for all  $P \leq S$ , given the other conditions on  $\Omega$ . Proposition 3.1 then applies to all virtual  $(S, S)$ -bisets  $\omega$  with coefficients in  $\mathbb{Q}$  which are  $\mathcal{F}$ -stable,  $\mathcal{F}$ -generated and such that  $\chi_P(\omega/P) \neq 0$  for all  $P \leq S$ , where  $\omega/P$  denotes the linearized right  $P$ -orbits of  $\omega$ . The proof is identical to the one given above. In particular, Reeh [5, Proposition 4.5, Corollary 5.8] shows that if  $\omega$  is the *characteristic idempotent* of  $\mathcal{F}$ , i.e., the unique virtual  $(S, S)$ -biset with coefficients in  $\mathbb{Z}_{(p)}$  which is  $\mathcal{F}$ -stable,  $\mathcal{F}$ -generated and which is an idempotent in the double Burnside ring  $\mathbb{Z}_{(p)}B(S, S)$  of  $S$  (i.e., the Burnside ring of finite  $(S, S)$ -bisets), then the elements  $\omega/P = \omega \circ_S [S/P] = \beta_P$  with  $P \leq_{\mathcal{F}} S$  form a basis of  $\mathbb{Z}_{(p)}B(\mathcal{F})$  such that  $\chi_P(\omega/P) \neq 0$ . This proves Theorem 1.4.

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