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Finite 2-groups whose non-abelian subgroups have the same center

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Abstract. We classify the finite 2-groups that have the property that all their non-abelian subgroups have the same center.

1 Introduction

The center of a group is a very important concept in group theory. In recent years, some scholars have studied the structure of finite p-groups that satisfy conditions on their centers or on the centers of their subgroups. For example, Janko [6] studied finite non-abelian p-groups that have exactly one maximal subgroup with a non-cyclic center. Finogenov [5] studied finite p-groups with cyclic commutator group and cyclic center. Berkovich and Janko [2] asked:

Problem 1392. Does there exist a p-group G such that $A \cap Z(G) = 1$ for all minimal non-abelian $A < G$? A minimal non-abelian group is a non-abelian group all proper subgroups of which are abelian.

We prove the following more general result which shows that Problem 1392 has a negative answer.

Theorem 1.1. *Let G be a finite p-group. Then there exists a minimal non-abelian subgroup A of G such that $A' \leq Z(G)$.*

Proof. If the nilpotency class of G is n, then there exist $x_1, x_2, \dots, x_n \in G$ such that $o([x_1, x_2, \dots, x_n]) = p$. Hence the subgroup $A = \langle [x_1, x_2, \dots, x_{n-1}], x_n \rangle$ is minimal non-abelian by Lemma 2.2, and $A' \leq Z(G)$. □

We know that $1 \leq A \cap Z(G) \leq Z(A)$ for each subgroup A of a finite p-group G. Naturally, we consider the other extreme of Problem 1392.

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Problem 1.2. Does there exist a p-group G such that $A \cap Z(G) = Z(A)$ for all non-abelian $A < G$?

The answer is positive. For example,

$$\langle a, b, c \mid a^4 = b^4 = c^4 = 1,\ b^2 = a^2c^2,\ [a, b] = a^2,\ [a, c] = c^2,\ [b, c] = 1\rangle$$

satisfies the condition of Problem 1.2. Thus, it is natural to ask:

Problem 1.3. Is it possible to classify finite p-groups satisfying the condition in Problem 1.2?

In an email to the second author, Janko proposed a more specific problem.

Problem 1.4. Classify the finite 2-groups G such that $Z(A) = Z(G)$ for all non-abelian subgroups A of G.

After we finished this paper, Berkovich advised us that Janko had obtained some related and, as yet, unpublished results about Problem 1392 and Problems 1.2 and 1.3. However, Problem 1.4 remained open. The present paper is devoted to this problem and all such groups are classified.

For convenience, we introduce the following notation and concepts.

- $\mathcal{P}$-group: a finite 2-group in which the centers of all non-abelian subgroups coincide.
- $\mathcal{Q}$-group: a $\mathcal{P}$-group all of whose non-abelian subgroups are generated by two elements.
- $\mathcal{S}$-group: a $\mathcal{P}$-group which has at least one non-abelian subgroup H with $d(H) > 2$.

Obviously, $\mathcal{P} = \mathcal{Q} \cup \mathcal{S}$ and $\mathcal{Q} \cap \mathcal{S} = \emptyset$.

Suppose that G is a finite p-group. If all subgroups of index p^t of G are abelian and at least one subgroup of index p^{t-1} of G is not abelian, then G is called an $\mathcal{A}_t$-group. Obviously, an $\mathcal{A}_1$-group is a minimal non-abelian p-group.

2 Preliminaries

In this section, we give some lemmas which will be used in the proof of our result.

Lemma 2.1 ([1, Lemma 1.1]). *If a non-abelian p-group G has an abelian maximal subgroup, then $|G| = p|G'||Z(G)|$.*

Lemma 2.2 ([7, Lemma 2.2]). *Suppose that G is a finite non-abelian p-group. The following conditions are equivalent:*

(1) *G is minimal non-abelian,*

(2) *$d(G) = 2$ and $|G'| = p$,*

(3) *$d(G) = 2$ and $\Phi(G) = Z(G)$.*

The following simple lemma is often used.

Lemma 2.3. *If $G = \langle x, y\rangle$ is a minimal non-abelian 2-group, then*

$$Z(G) = \langle x^2, y^2, [x, y]\rangle.$$

Lemma 2.4. *Let G be a $\mathcal{P}$-group. If $x, y \in G\backslash Z(G)$ and $[x, y] = 1$, then*

$$C_G(x) = C_G(y).$$

Proof. If $C_G(x) \neq C_G(y)$, then without loss of generality, we can assume that $z \in C_G(x)\backslash C_G(y)$. By considering the non-abelian subgroup $H = \langle x, y, z\rangle$, we deduce that $x \in Z(H) = Z(G)$, which contradicts $x \notin Z(G)$. Therefore we obtain $C_G(x) = C_G(y)$. □

To classify $\mathcal{S}$-groups, we first give the following three lemmas.

Lemma 2.5. *Let*

$$M = \langle a, b, c \mid a^{2^n} = b^4 = c^4 = 1,\ [b, a] = c^2,\ [c, a] = c^2b^2,\ [c, b] = 1\rangle,$$

where $n \geq 1$. Then there is no group G in $\mathcal{P}$ such that M is a maximal subgroup of G.

Proof. Suppose there is a group G in $\mathcal{P}$ such that M is a maximal subgroup of G. It is clear that $Z(M) = \langle a^2, b^2, c^2\rangle$ and M has exactly one abelian maximal subgroup $A = \langle a^2, b, c\rangle$. Hence $A \trianglelefteq G$. Since $|G/A| = 4$, $G' \leq A$. By hypotheses, $Z(G) = Z(M) = \langle a^2, b^2, c^2\rangle$. Take $x \in G\backslash M$. Then $G = \langle x, a, b, c\rangle$.

Let $[a, x] = a^{2s}b^tc^u$. It follows from

$$1 = [a^2, x] = [a, x]^2[a, x, a] = a^{4s}b^{2(t+u)}c^{2t}$$

that $2^{n-2} \mid s$, $2 \mid t$ and $2 \mid u$. Hence $[a, x] = a^{2^{n-1}i_1}b^{2j_1}c^{2k_1}$. Since $[a, b] = c^2$ and $[a, c] = b^2c^2$,

$$[a, xc^{j_1}b^{j_1+k_1}] = a^{2^{n-1}i_1}.$$

Replacing x by $xc^{j_1}b^{j_1+k_1}$, we can assume that $[a, x] = a^{2^{n-1}i_1}$. If $[a, x] \neq 1$, then it follows from Lemma 2.2 that the subgroup $\langle a, x\rangle$ is minimal non-abelian.

Thus, by Lemma 2.3,

$$Z(\langle a, x\rangle) = \langle a^2, x^2\rangle \neq \langle a^2, b^2, c^2\rangle = Z(G),$$

a contradiction. Hence $[a, x] = 1$.

Let $[b, x] = a^{2s}b^t c^u$. It follows from $1 = [b^2, x] = [b, x]^2 = a^{4s}b^{2t}c^{2u}$ that $2^{n-2} \mid s$, $2 \mid t$ and $2 \mid u$. So we can assume that

$$[b, x] = a^{2^{n-1}i_2}b^{2j_2}c^{2k_2}.$$

Using the same argument, we assume that

$$[c, x] = a^{2^{n-1}i_3}b^{2j_3}c^{2k_3}.$$

It is easy to see that $[b, x], [c, x] \in Z(G)$. We obtain $G' \leq Z(G)$ and $\exp G' = 2$.

If $G/A \cong C_4$, then $G/A = \langle xA\rangle$ and $M/A = \langle aA\rangle = \langle x^2A\rangle$. It follows that $x^2a^{-1} \in A$. Let $x^2 = aa'$, where $a' \in A$. Then

$$1 = [b, x]^2 = [b, x^2] = [b, aa'] = [b, a] = c^2,$$

a contradiction. Hence $G/A \cong C_2 \times C_2$ and $x^2 \in A$.

Since $[a, x] = 1$, we can assume that $x^2 = b^t c^u$. Since

$$1 = [a, x]^2 = [a, x^2] = b^{2u}c^{2(t+u)},$$

it follows that $2 \mid t$ and $2 \mid u$. Assume that $x^2 = b^{2j}c^{2k}$.

Since $x \notin Z(G)$, it follows that

$$[b, x] = a^{2^{n-1}i_2}b^{2j_2}c^{2k_2} \neq 1, \quad [c, x] = a^{2^{n-1}i_3}b^{2j_3}c^{2k_3} \neq 1,$$

and

$$[bc, x] = a^{2^{n-1}(i_2+i_3)}b^{2(j_2+j_3)}c^{2(k_2+k_3)} \neq 1$$

by Lemma 2.4. It is easy to see that there exists at least one even number in the set $\{i_2, i_3, i_2 + i_3\}$. Without loss of generality, we can assume that i_2 is even. Hence $Z(\langle b, x\rangle) \leq \langle b^2, c^2\rangle$ by Lemma 2.3. Since G is a $\mathcal{P}$-group, it follows that $Z(G) = Z(\langle b, x\rangle)$. Thus, $a^2 = 1$, and $n = 1$.

Replacing x by xa^{k_2}, we still have $[a, x] = 1$. Hence $[b, x] = b^{2j_2} \neq 1$. Thus we get $[b, x] = b^2$. By Lemma 2.2, $\langle b, x\rangle$ is minimal non-abelian. It follows by Lemma 2.3 that $x^2 = b^{2j}c^2$.

By Lemma 2.4, we know that $C_G(b) = C_G(c) = C_G(bc)$. Hence we infer that $[c, x] \neq [b, x] = b^2$. Since $[c, ax] \neq 1$, we have that $[c, x] \neq [c, a] = b^2c^2$ and $[c, x] = c^2$. By Lemma 2.2, we deduce $\langle c, x\rangle$ is minimal non-abelian. It follows by Lemma 2.3 that $Z(\langle c, x\rangle) = Z(G) = \langle c^2, x^2\rangle$. Thus $x^2 = b^2c^2$. Since $[bc, x] = [b, x][c, x] = b^2c^2$, we conclude that

$$Z(\langle bc, x\rangle) = \langle b^2c^2\rangle \neq Z(G) = \langle b^2, c^2\rangle,$$

a contradiction. □

Lemma 2.6. *Let $G \in \mathcal{P}$ and*

$$M = \langle a, b, c \mid a^4 = b^4 = c^{2^n} = 1,\ b^2 = a^2c^{2^{n-1}},\ [b,a] = a^2,\ [a,c] = c^2,\ [c,b] = 1\rangle,$$

where $n \geq 3$. If M is a maximal subgroup of G, then

$$G = \langle a_1, b_1, c_1 \mid a_1^4 = b_1^4 = c_1^{2^{n+1}} = 1,\ b_1^2 = a_1^2c_1^{2^n},\ [b_1,a_1] = a_1^2,\ [a_1,c_1] = c_1^2,\ [c_1,b_1] = 1\rangle.$$

Proof. It is clear that $Z(M) = \langle a^2, b^2\rangle$ and M has exactly one abelian maximal subgroup $A = \langle b, c\rangle$. Hence $A \trianglelefteq G$. Since $|G/A| = 4$, $G' \leq A$. By hypothesis, $Z(G) = Z(M) = \langle a^2, b^2\rangle$. Take $x \in G\backslash M$. Then $G = \langle x, a, b, c\rangle$.

Let $[b,x] = b^sc^t$. Since

$$1 = [b^2, x] = [b,x]^2 = b^{2s}c^{2t},$$

$2 \mid s$ and $2^{n-1} \mid t$. So we can assume that $[b,x] = b^{2i_2}c^{2^{n-1}j_2}$.

Let $[c,x] = b^sc^t$. Since

$$1 = [c^{2^{n-1}}, x] = c^{2^{n-1}t}$$

and

$$1 = [x,a,c][a,c,x][c,x,a] = [c^2,x][b^sc^t] = b^{2s}a^{2s},$$

$2 \mid t$ and $2 \mid s$. Assume that $[c,x] = b^{2i_3}c^{2t'}$.

If $G/A \cong C_4$, then $G/A = \langle xA\rangle$ and $M/A = \langle aA\rangle = \langle x^2A\rangle$. It follows that $x^2a^{-1} \in A$. Let $x^2 = ab^sc^t$. Then

$$1 = [b,x]^2 = [b,x^2] = [b, ab^sc^t] = [b,a] = a^2,$$

a contradiction. Hence $G/A \cong C_2 \times C_2$ and $x^2 \in A$.

By $1 = [c,x^2] = [c,x]^2[c,x,x] = c^{4t'}[b^{2i_3}c^{2t'}, x] = c^{4t'+4t'^2}$, we have that $2^{n-2} \mid t'(t'+1)$. It follows that $2^{n-2} \mid t'$ or $2^{n-2} \mid t'+1$. Hence

$$[c,x] = b^{2i_3}c^{2^{n-1}j_3} \quad \text{or} \quad [c,x] = b^{2i_3}c^{2^{n-1}j_3-2}.$$

If $[c,x] = b^{2i_3}c^{2^{n-1}j_3-2}$, then

$$[c,xa] = [c,a][c,x][c,x,a] = b^{2i_3}c^{2^{n-1}j_3}.$$

Replacing x by xa, we have $[c,x] = b^{2i_3}c^{2^{n-1}j_3}$.

If $[c,x] = b^{2i_3}c^{2^{n-1}j_3} \neq 1$, then the subgroup $\langle c, x\rangle$ is minimal non-abelian and $c^2 \in Z(\langle c, x\rangle)$. By hypothesis, $c^2 \in Z(G)$, a contradiction. So we obtain that $[c,x] = 1$. It follows from Lemma 2.4 that $[b,x] = 1$.

Let $[a,x]=b^s c^t$. Since

$$1=[a^2,x]=[a,x]^2[a,x,a]=b^{2s}a^{2s},$$

we have $2\mid s$. Hence $[a,x]=b^{2i_1}c^t$. Since $[a,xb^{i_1}]=c^{t+i_1 2^{n-1}}$, we can assume that $[a,x]=c^{j_1}$ by replacing x by xb^{i_1}.

If $2\mid j_1$, then $j_1=2j_1'$. Since

$$[a,xc^{-j_1'}]=[a,x][a,c^{-j_1'}]=c^{2j_1'}c^{-2j_1'}=1,$$

we have $xc^{-j_1'}\in Z(G)\le A$ and $x\in A$, a contradiction. Hence, $2\nmid j_1$. We can assume that $[a,x]=c$.

Let $x^2=b^s c^t$. Then $[a,x^2]=[a,x]^2[a,x,x]=c^2$. On the other hand, we have $[a,x^2]=[a,b^s c^t]=a^{2s}c^{2t}$. It follows that $2\mid s$ and $2^{n-1}\mid 1-t$. Assume that $x^2=b^{2i}c^{1-2^{n-1}k}$.

Let $H=\langle xbac, c^{2^{n-2}}\rangle$. Since $[xbac,c^{2^{n-2}}]=[a,c^{2^{n-2}}]=c^{2^{n-1}}$, it follows that H is minimal non-abelian from Lemma 2.2. By Lemma 2.3,

$$Z(H)=\langle (xbac)^2, c^{2^{n-1}}, [xbac,c]\rangle.$$

Since

$$(xbac)^2=b^{2(i+1)} \quad \text{and} \quad Z(H)=Z(G)=\langle b^2, c^{2^{n-1}}\rangle,$$

we have $2\mid i$ and $x^2=c^{1-2^{n-1}k}=ca^{2k}b^2k=[a,x]a^{2k}b^2k$.

Notice that

$$[a,xb^k]=x^2b^{2k}=(xb^k)^2.$$

Replacing x by xb^k, we have $[a,x]=x^2$. Hence

$$G=\langle a,b,x \mid a^4=b^4=x^{2^{n+1}}=1,\ b^2=a^2x^{2^n},$$
$$[b,a]=a^2,[a,x]=x^2,[b,x]=1\rangle$$

and we obtain the required group. □

Lemma 2.7. *Let $G\in\mathcal{P}$ and*

$$M=\langle a,b,c \mid a^{2^m}=b^4=c^4=1,\ b^2=c^2,\ [a,b]=b^2,$$
$$[c,a]=a^{2^{m-1}},\ [c,b]=1\rangle,$$

where $m\ge 3$. If M is a maximal subgroup of G, then

$$G=\langle a_1,b_1,c_1 \mid a_1^{2^m}=b_1^8=c_1^4=1,\ b_1^4=c_1^2,\ [a_1,b_1]=b_1^2,$$
$$[c_1,a_1]=a_1^{2^{m-1}},[c_1,b_1]=1\rangle.$$

Proof. It is clear that $Z(M)=\langle a^2,b^2\rangle$ and M has exactly one abelian maximal subgroup $A=\langle a^2,b,c\rangle$. Hence, $A\trianglelefteq G$. Since $|G/A|=4$, $G'\le A$. By hypothesis, $Z(G)=Z(M)=\langle a^2,b^2\rangle$. Take $x\in G\backslash M$. Then $G=\langle x,a,b,c\rangle$.

Let $[a,x] = a^{2s}b^t c^u$. Since

$$1 = [a^2, x] = [a,x]^2[a,x,a] = a^{4s+2^{m-1}u}c^{2u},$$

we have $2^{m-2} \mid s$ and $2 \mid u$. Since $b^2 = c^2$, it follows that $[a,x] = a^{2^{m-1}i_1}b^{j_1}$. Since $[a, xc^{i_1}] = b^{j_1}$, we can assume that $[a,x] = b^{j_1}$ by replacing x by xc^{i_1}.

Since $[M, A, G] \le [M', G] \le [Z(G), G] = 1$ and $[G, M, A] \le [G', A] = 1$, it follows that $[A, G, M] = 1$ by the Three Subgroups Lemma. Hence,

$$[A, G] \le C_A(M) = Z(M) = \langle a^2, b^2\rangle.$$

Let $[b,x] = a^{2s}b^{2t}$. Since $1 = [b^2, x] = [b,x]^2 = a^{4s}$, we obtain that $2^{m-2} \mid s$. It follows that

$$[b,x] = a^{2^{m-1}i_2}b^{2j_2}.$$

Since $[b, xa^{j_2}] = a^{2^{m-1}i_2}$, we can assume that $[b,x] = a^{2^{m-1}i_2}$ by replacing x by xa^{j_2}.

If $[b,x] \ne 1$, then $[b,x] = a^{2^{m-1}}$. It follows from Lemma 2.2 that $\langle b, x\rangle$ is minimal non-abelian. By Lemma 2.3,

$$x^2 \in Z(\langle b,x\rangle) = \langle b^2, x^2, [b,x]\rangle = Z(G) = \langle a^2, b^2\rangle.$$

Since $m \ge 3$, we have that $x^2 = a^2b^{2j}$. Now, since $[b, xa^{-1}] = a^{2^{m-1}}b^2$, we deduce that $\langle b, xa^{-1}\rangle$ is minimal non-abelian by Lemma 2.2. By Lemma 2.3,

$$Z(\langle b, xa^{-1}\rangle) = \langle b^2, (xa^{-1})^2, [b, xa^{-1}]\rangle \le \langle b, a^{2^{m-1}}\rangle.$$

Since $m \ge 3$,

$$a^2 \notin Z(\langle b, xa^{-1}\rangle) \quad \text{and} \quad Z(\langle b, xa^{-1}\rangle) \ne \langle a^2, b^2\rangle = Z(G),$$

a contradiction. Hence $[b,x] = 1$. It follows that $[c,x] = 1$ by Lemma 2.4.

Notice that $[a,x] = b^{j_1}$. If $2 \mid j_1$, then $j_1 = 2j_1'$. Since

$$[a, xb^{j_1'}] = [a,x][a, b^{j_1'}] = b^{2j_1'}b^{2j_1'} = 1,$$

we have $xb^{j_1'} \in Z(G) \le A$ and $x \in A$, a contradiction. Hence $2 \nmid j_1$. We can assume that $[a,x] = b$.

If $G/A \cong C_4$, then $G/A = \langle xA\rangle$ and $M/A = \langle aA\rangle = \langle x^2A\rangle$. It follows that $x^2a^{-1} \in A$. Let $x^2a^{-1} = a^{2s}b^tc^u$. Then

$$x^2 = a^{1+2s}b^tc^u$$

and

$$1 = [c,x]^2 = [c,x^2] = [c, a^{1+2s}b^tc^u] = [c,a] = a^{2^{m-1}},$$

a contradiction. Hence $G/A \cong C_2 \times C_2$ and $x^2 \in A$.

Let $x^2 = a^{2s}b^tc^u$. Then $[a, x^2] = [a, a^{2s}b^tc^u] = b^{2t}a^{2^{m-1}u}$. On the other hand, $[a, x^2] = [a, x]^2[a, x, x] = b^2$. We deduce that $2 \mid u$ and $2 \mid 1 - t$. Since $b^2 = c^2$, we can assume that $x^2 = a^{2i}b^{1-2k}$.

Let $H = \langle xa, b\rangle$. By $[xa, b] = [a, b] = b^2$, it follows from Lemma 2.2 that H is minimal non-abelian. By Lemma 2.3,

$$Z(H) = \langle (xa)^2, b^2, [xa, b]\rangle = Z(G) = \langle a^2, b^2\rangle.$$

Since $(xa)^2 = a^{2(i+1)}b^{2k}$, we have that $2 \mid i$. Let $i = 2i'$. Then $x^2 = a^{4i'}b^{1-2k}$.

Let $x' = xa^{-2i'}$. Then

$$x'^2 = b^{1-2k'}, \quad [a, x'] = b \quad \text{and} \quad [b, x'] = [c, x'] = 1.$$

Finally, let $a_1 = a$, $b_1 = x'a^{2^{m-2}k}b^kc^k$ and $c_1 = c$. Now

$$b_1^2 = x'^2a^{2^{m-1}k}b^{2j_1k}c^{2k} = b^{(1+2k)}a^{2^{m-1}k},$$

$$[a, b_1] = [a, x'a^{2^{m-2}k}b^kc^k] = [a, x'][a, b^k][a, c^k] = b^{(1+2k)}a^{2^{m-1}k},$$

$$[a, b_1] = b_1^2.$$

Hence, we obtain the required group

$$G = \langle a_1, b_1, c_1 \mid a_1^{2^m} = b_1^8 = c_1^4 = 1,\ b_1^4 = c_1^2,\ [a_1, b_1] = b_1^2,\ [c_1, a_1] = a_1^{2^{m-1}},\ [c_1, b_1] = 1\rangle.$$

□

3 Main result

In this section, we give the classification of $\mathcal{P}$-groups. We know that $\mathcal{P} = \mathcal{Q} \cup \mathcal{S}$ and $\mathcal{Q} \cap \mathcal{S} = \emptyset$. Next, we classify $\mathcal{Q}$-groups and $\mathcal{S}$-groups.

Theorem 3.1. *Let G be a finite non-abelian 2-group. Then G is a $\mathcal{Q}$-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups:*

(1) $\langle a, b \mid a^{2^c} = b^{2^m} = 1,\ [a, b] = a^{-2}\rangle$, $m \geq 1$, $c \geq 3$, $|G| = 2^{m+c}$,

(2) $\langle a, b \mid a^{2^c} = 1,\ b^{2^m} = a^{2^{c-1}},\ [a, b] = a^{-2}\rangle$, $m \geq 1$, $c \geq 3$, $|G| = 2^{m+c}$,

(3) $\langle a, b \mid a^{2^c} = b^{2^m} = 1,\ [a, b] = a^{-2+2^{c-1}}\rangle$, $m \geq 1$, $c \geq 3$, $|G| = 2^{m+c}$.

Proof. Let G be a $\mathcal{Q}$-group. Then $Z(H) = Z(G)$ and $d(H) = 2$ for any non-abelian subgroup H of G.

Assume that G has no abelian maximal subgroup. It follows that G is metacyclic by [7, Theorem 5.2]. Metacyclic 2-groups G are classified by [8], and G is isomorphic to one of the following groups:

(I) $G = \langle a, b \mid a^{2^{r+s+u}} = 1,\ b^{2^{r+s+t}} = a^{2^{r+s}},\ [a, b] = a^{2^r}\rangle$, where r, s, t, u are non-negative integers with $r \geq 2$ and $u \leq r$,

(II) $G = \langle a, b \mid a^{2^{r+s+v+t'+u}} = 1, b^{2^{r+s+t}} = a^{2^{r+s+v+t'}}, [a,b] = a^{-2+2^{r+v}} \rangle$, where r, s, v, t, t', u are non-negative integers with $r \geq 2$, $t' \leq r$, $u \leq 1$, $tt' = sv = tv = 0$, and if $t' \geq r-1$, then $u = 0$.

If G is the group (I), then let

$$H = \langle a^{2^{s+u-1}}, b \rangle.$$

If G is the group (II), then let

$$H = \langle a^{2^{r+s+v+t'+u-2}}, b \rangle.$$

In either case, H is a minimal non-abelian subgroup of G. Since $b^2 \in Z(H)$ and $Z(H) = Z(G)$, $b^2 \in Z(G)$ and G has an abelian maximal subgroup $M = \langle a, b^2 \rangle$, a contradiction. Hence, G has an abelian maximal subgroup. Checking the list of groups in [7, Theorem 3.12, Lemma 3.1 (4)], we obtain the groups listed in the theorem. □

For $\mathcal{S}$-groups G, it is easy to know that $|G| \geq 2^5$. Using Magma [4] to check the SmallGroups database [3], we prove the following

Lemma 3.2. *Let G be a finite non-abelian 2-group and $|G| \leq 2^7$. Then G is an $\mathcal{S}$-group if and only if G is isomorphic to one of the following non-isomorphic groups:*

(I) $|G| = 2^5$.

(I-1) SmallGroup$(2^5, 32)$*, which corresponds to the group*

$$\langle a, b, c \mid a^4 = b^4 = c^4 = 1,\ b^2 = a^2c^2,\ [b,a] = a^2,\ [a,c] = c^2,\ [b,c] = 1 \rangle.$$

(I-2) SmallGroup$(2^5, 33)$*, which corresponds to the group*

$$\langle a, b, c \mid a^2 = b^4 = c^4 = 1,\ [a,b] = c^2,\ [a,c] = b^2c^2,\ [b,c] = 1 \rangle.$$

(II) $|G| = 2^6$.

(II-1) SmallGroup$(2^6, 64)$*, which corresponds to the group*

$$\langle a, b, c \mid a^4 = b^4 = c^4 = 1,\ [b,a] = c^2,\ [c,a] = c^2b^2,\ [c,b] = 1 \rangle.$$

(II-2) SmallGroup$(2^6, 82)$*, which corresponds to the group*

$$\langle a, b, c \mid a^4 = b^4 = c^4 = 1,\ [b,a] = c^2,\ [c,a] = c^2b^2,\ [c,b] = b^2a^2,\ [a^2,b] = [a^2,c] = [b^2,a] = [b^2,c] = [c^2,a] = [c^2,b] = 1 \rangle.$$

(II-3) SmallGroup($2^6, 113$), *which corresponds to the group*

$$\langle a, b, c \mid a^8 = b^4 = c^4 = 1,\ b^2 = c^2,\ [a, b] = b^2,\ [c, a] = a^4,\ [c, b] = 1\rangle.$$

(II-4) SmallGroup($2^6, 180$), *which corresponds to the group*

$$\langle a, b, c \mid a^4 = b^4 = c^8 = 1,\ b^2 = a^2c^4,\ [b, a] = a^2,\ [a, c] = c^2,\ [c, b] = 1\rangle.$$

(II-5) SmallGroup($2^6, 245$), *which corresponds to the group*

$$\langle a, b, c, d \mid a^4 = b^4 = c^4 = d^4 = 1,\ a^2 = d^2,\ b^2 = c^2,\ [b, a] = a^2,\ [c, a] = c^2,\ [c, b] = [d, a] = a^2b^2,\ [d, b] = a^2,\ [d, c] = 1\rangle.$$

(III) $|G| = 2^7$.

(III-1) SmallGroup($2^7, 300$), *which corresponds to the group*

$$\langle a, b, c \mid a^8 = b^8 = c^4 = 1,\ b^4 = c^2,\ [a, b] = b^2,\ [c, a] = a^4,\ [c, b] = 1\rangle.$$

(III-2) SmallGroup($2^7, 571$), *which corresponds to the group*

$$\langle a, b, c \mid a^8 = b^4 = c^4 = 1,\ [b, a] = c^2,\ [c, a] = c^2b^2,\ [c, b] = 1\rangle.$$

(III-3) SmallGroup($2^7, 895$), *which corresponds to the group*

$$\langle a, b, c \mid a^{16} = b^4 = c^4 = 1,\ b^2 = c^2,\ [a, b] = b^2,\ [c, a] = a^8,\ [c, b] = 1\rangle.$$

(III-4) SmallGroup($2^7, 985$), *which corresponds to the group*

$$\langle a, b, c \mid a^4 = b^4 = c^{16} = 1,\ b^2 = a^2c^8,\ [b, a] = a^2,\ [a, c] = c^2,\ [c, b] = 1\rangle.$$

Here SmallGroup(o, n) *denotes the n-th group of order o in the* SMALLGROUPS *database.*

Theorem 3.3. *Let G be a finite non-abelian 2-group and $|G| \geq 2^7$. Then G is an $\mathcal{S}$-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups:*

(1) $G = \langle a, b, c \mid a^{2^n} = b^4 = c^4 = 1, [b, a] = c^2, [c, a] = c^2 b^2, [c, b] = 1\rangle$, *where* $n \geq 3$, $|G| = 2^{n+4}$ *and* $Z(G) = \langle a^2, b^2, c^2\rangle$,

(2) $G = \langle a, b, c \mid a^4 = b^4 = c^{2^n} = 1, b^2 = a^2 c^{2^{n-1}}, [b, a] = a^2, [a, c] = c^2, [c, b] = 1\rangle$, *where* $n \geq 4$, $|G| = 2^{n+3}$ *and* $Z(G) = \langle a^2, b^2\rangle$,

(3) $G = \langle a, b, c \mid a^{2^m} = b^{2^n} = c^4 = 1, b^{2^{n-1}} = c^2, [a, b] = b^2, [c, b] = 1, [c, a] = a^{2^{m-1}}\rangle$, *where* $m \geq 3$, $n \geq 2$, $|G| = 2^{m+n+1}$ *and* $Z(G) = \langle a^2, c^2\rangle$.

Proof. Since G is an $\mathcal{S}$-group, we discuss two cases.

Case 1: $d(H) = 2$ for every non-abelian subgroup H of G. Since $|G| \geq 2^7$, we can assume that G has an abelian maximal subgroup by [7, Theorem 5.2]. Since G is a $\mathcal{S}$-group, we obtain that $d(G) = 3$. It follows that G is an $\mathcal{A}_2$-group by [7, Theorem 4.1]. These are classified by [9]. By hypothesis, the required groups are listed in [9, Theorem 3.6]. By checking the list of groups, we get the groups of type (1) and of type (3) with $n = 2$.

Case 2: there exists an $H < G$ such that $H' \neq 1$ and $d(H) \geq 3$. If $|G| = 2^7$, then the theorem holds by Lemma 3.2. Assume that $|G| \geq 2^8$.

Let M be a maximal subgroup of G such that $H \leq M$. Then M is an $\mathcal{S}$-group. By induction, M is isomorphic to one of groups listed in the theorem.

By Lemma 2.5, M is not group (1) in the theorem. If M is group (2), then, by Lemma 2.6, we get (2). If M is group (3) with $n = 2$, then we get (3) with $n = 3$ by Lemma 2.7. If M is group (3) with $n \geq 3$, then assume that

$$M = \langle a, b, c \mid a^{2^m} = b^{2^n} = c^4 = 1,\ b^{2^{n-1}} = c^2,\ [a, b] = b^2,\ [c, a] = a^{2^{m-1}},\ [c, b] = 1\rangle.$$

It is clear that $Z(M) = \langle a^2, c^2\rangle$ and M has exactly one abelian maximal subgroup $A = \langle a^2, b, c\rangle$. Hence, $A \trianglelefteq G$ and $G' \leq A$ since $|G/A| = 4$. By hypotheses, $Z(G) = Z(M) = \langle a^2, c^2\rangle$. Take $x \in G \backslash M$. Then $G = \langle x, a, b, c\rangle$.

Let $[b, x] = a^{2s} b^t c^u$. Since

$$1 = [b^{2^{n-1}}, x] = [b, x]^{2^{n-1}} = a^{2^n s} b^{2^{n-1} t}$$

and

$$[a, b, x][b, x, a][x, a, b] = 1,$$

we have $2^{m-2} \mid s$, $2 \mid t$ and $2 \mid u$. Since $b^{2^{n-1}} = c^2$, we can assume that

$$[b, x] = a^{2^{m-1} i_2} b^{2t'}.$$

Let $[c,x]=a^{2s}b^tc^u$. Since

$$1=[c^2,x]=[c,x]^2=a^{4s}b^{2t}c^{2u}$$

and

$$[a,c,x][c,x,a][x,a,c]=1,$$

we have $2^{m-2}\mid s$, $2^{n-1}\mid t$ and $2\mid u$. Since $b^{2^{n-1}}=c^2$, we can assume that

$$[c,x]=a^{2^{m-1}i_3}b^{2^{n-1}j_3}.$$

If $G/A\cong C_4$, then $G/A=\langle xA\rangle$ and $M/A=\langle aA\rangle=\langle x^2A\rangle$. It follows that $x^2a^{-1}\in A$. Let $x^2a^{-1}=a^{2s}b^tc^u$. Then

$$x^2=a^{1+2s}b^tc^u$$

and

$$1=[c,x]^2=[c,x^2]=[c,a^{1+2s}b^tc^u]=[c,a]=a^{2^{m-1}},$$

a contradiction. Hence $G/A\cong C_2\times C_2$ and $x^2\in A$.

Since $1=[b,x^2]=b^{4t'+4t'^2}$, we have $2^{n-2}\mid t'(t'+1)$. It follows that $2^{n-2}\mid t'$ or $2^{n-2}\mid t'+1$. Hence,

$$[b,x]=a^{2^{m-1}i_2}b^{2^{n-1}j_2}\quad\text{or}\quad[b,x]=a^{2^{m-1}i_2}b^{2^{n-1}j_2-2}.$$

If $[b,x]=a^{2^{m-1}i_2}b^{2^{n-1}j_2}\neq 1$, then $\langle b,x\rangle$ is minimal non-abelian by Lemma 2.2 and $b^2\in Z(\langle b,x\rangle)$. By hypothesis, $b^2\notin Z(G)$, a contradiction. So

$$[b,x]=1\quad\text{or}\quad[b,x]=a^{2^{m-1}i_2}b^{2^{n-1}j_2-2}.$$

If $[b,x]\neq 1$, then $[b,x]=a^{2^{m-1}i_2}b^{2^{n-1}j_2-2}$. Thus

$$[b^2,x]=[b,x]^2=b^{-4}\quad\text{and}\quad[xa,b^2]=1.$$

It follows from Lemma 2.4 that $[xa,b]=1$ and $[xa,c]=1$. Replacing x by xa, we can assume that $[b,x]=[c,x]=1$.

Let $[a,x]=a^{2s}b^tc^u$. Since

$$1=[a^2,x]=[a,x]^2[a,x,a]=a^{4s+2^{m-1}u}c^{2u},$$

$2^{m-2}\mid s$ and $2\mid u$. Since $b^{2^{n-1}}=c^2$, we can assume that $[a,x]=a^{2^{m-1}i_1}b^{j_1}$. Since $[a,xc^{i_1}]=b^{j_1}$, we can assume that $[a,x]=b^{j_1}$ by replacing x by xc^{i_1}.

If $2\mid j_1$, then $j_1=2j_1'$. Since

$$[a,xb^{-j_1'}]=[a,x][a,b^{-j_1'}]=b^{2j_1'}b^{-2j_1'}=1,$$

we have that $xb^{-j_1'}\in Z(G)\leq A$ and $x\in A$, a contradiction. Hence, $2\nmid j_1$.

Let $x^2 = a^{2s}b^tc^u$. Then $[a, x^2] = [a, x]^2[a, x, x] = b^{2j_1}$. On the other hand, we have $[a, x^2] = [a, a^{2s}b^tc^u] = b^{2t}a^{2^{m-1}u}$. Also $2 \mid u$ and $2^{n-1} \mid j_1 - t$. Since $b^{2^{n-1}} = c^2$, we can assume that $x^2 = a^{2i}b^{j_1-2^{n-1}k}$.

Let $H = \langle xa, b^{2^{n-2}}\rangle$. Since

$$[xa, b^{2^{n-2}}] = [a, b^{2^{n-2}}] = b^{2^{n-1}},$$

it follows from Lemma 2.2 that H is minimal non-abelian. By Lemma 2.3, we have $Z(H) = \langle (xa)^2, b^{2^{n-1}}, [xa, b]\rangle$. Since

$$(xa)^2 = a^{2(i+1)-2^{m-1}i_1}b^{2^{n-1}k} \quad \text{and} \quad Z(H) = Z(G) = \langle a^2, b^2\rangle,$$

it follows that $2 \mid i$. Let $i = 2i'$. Then

$$x^2 = a^{4i'}b^{j_1-2^{n-1}k}.$$

Let $x' = xa^{-2i'}$. We have that

$$x'^2 = b^{j_1-2^{n-1}k}.$$

Finally, let $a_1 = a, b_1 = x'a^{2^{m-2}k}b^{2^{n-2}j_1k}c^k$ and $c_1 = c$. Thus

$$\begin{aligned} b_1^2 &= x'^2a^{2^{m-1}k}b^{2^{n-1}j_1k}c^{2k} \\ &= b^{j_1-2^{n-1}k}a^{2^{m-1}k}, \\ [a, b_1] &= [a, x'a^{2^{m-2}k}b^{2^{n-2}j_1k}c^k] = [a, x'][a, b^{2^{n-2}j_1k}][a, c^k] \\ &= b^{j_1-2^{n-1}k}a^{2^{m-1}k}. \end{aligned}$$

So $[a, b_1] = b_1^2$. Hence we obtain

$$\begin{aligned} G = \langle a_1, b_1, c_1 \mid a_1^{2^m} = b_1^{2^{n+1}} = c_1^4 = 1,\ b_1^{2^n} = c_1^2, \\ [a_1, b_1] = b_1^2,\ [c_1, a_1] = a_1^{2^{m-1}},\ [c_1, b_1] = 1\rangle, \end{aligned}$$

which is group (3).

Next, we prove that the groups listed in the theorem are pairwise non-isomorphic. Obviously, $d(Z(G)) = 3$ for the group (1), and $d(Z(G)) = 2$ for the groups (2) and (3). Thus (1) is not isomorphic to either (2) or (3). Moreover, $|Z(G)| = 2^2$ for the group (2), and $|Z(G)| = 2^m \geq 2^3$ for the group (3). Thus (2) is not isomorphic to (3).

Finally, we prove that the groups listed in the theorem are $\mathcal{S}$-groups. Suppose that G is (1). Since G is an $\mathcal{A}_2$-group, every non-abelian proper subgroup M of G is maximal. Since $Z(G) = \langle a^2, b^2, c^2\rangle = \Phi(G)$, we know that $Z(G) \leq M$. Since $|M : Z(G)| = 2^2$, $Z(G) = Z(M)$. Hence G is an $\mathcal{S}$-group.

Suppose that G is group (2). Then we have $\Omega_1(G) = Z(G) = \langle a^2, b^2 \rangle$ and $M = \langle a^2, b, c \rangle = \langle b, c \rangle$ is the unique abelian maximal subgroup of G. Let H be a non-abelian subgroup of G. Then $G = HM$ and there exists an element $h \in H$ such that $h \notin M$. Hence, $G = \langle h \rangle M$ and $H = \langle h \rangle (H \cap M)$. We prove that $Z(H) = Z(G)$.

First, we claim that $Z(G) \leq H$. Since $H = \langle h \rangle (H \cap M)$ is non-abelian, there exists an element $h' \in H \cap M$ such that $[h, h'] \neq 1$. Assume that $h = ab^i c^j$ and $h' = b^{i'} c^{j'}$. Then

$$[h, h'] = b^{2i'} c^{2^{n-1} i' + 2j'} = h'^2 c^{2^{n-1} i'} \neq 1.$$

If $2 \nmid i'$, then $c^{2^{n-1}} = h'^{-2}[h, h'] \in H$. If $2 \mid i'$, since $h' \notin Z(G) = \Omega_1(G)$,

$$h'^2 = c^{2j'} \neq 1.$$

So $c^{2^{n-1}} \in H$.

Since $h^2 = b^2 c^{2^{n-1}(i+1)}$, we deduce that $b^2 = h^2 c^{2^{n-1}(i+1)} \in H$. Hence we have $Z(G) = \Omega_1(G) \leq H$.

Since $H \cap M$ is an abelian maximal subgroup of H, we deduce that $Z(H)$ is a subgroup of $H \cap M$. Since $G = HM$ and M is abelian, $Z(H) \leq Z(G)$. Thus $Z(G) = Z(H)$ and G is an $\mathcal{S}$-group.

Suppose that G is group (3) with $n = 2$. Since G is an $\mathcal{A}_2$-group, every non-abelian proper subgroup M of G is maximal. Since $Z(G) = \langle a^2, b^2 \rangle = \Phi(G)$, we deduce that $Z(G) \leq M$. Since $|M : Z(G)| = 2^2$, it follows that $Z(G) = Z(M)$. Hence G is an $\mathcal{S}$-group.

Suppose that G is group (3) with $n > 2$. Then we have $Z(G) = \langle a^2, c^2 \rangle$ and $M = \langle a^2, b, c \rangle$ is an abelian maximal subgroup of G. Let H be a non-abelian subgroup of G. Then $G = HM$ and there exists an element $h \in H$ such that $h \notin M$. Hence, $G = \langle h \rangle M$ and $H = \langle h \rangle (H \cap M)$. Since $H' \neq 1$, there exists an element $h' \in H \cap M$ such that $[h, h'] \neq 1$. Let $h = ab^j c^k$ and $h' = a^{2i'} b^{j'} c^{k'}$. Then

$$h^2 = a^{2 + 2^{m-1} k} c^{2k}, \quad h'^2 = a^{4i'} b^{2j'} c^{2k'}, \quad [h, h'] = b^{2j'} a^{2^{m-1} k'} \neq 1.$$

Since $h^2 = a^{2+2^{m-1}k} c^{2k}$, we obtain that $a^4 \in H$. Hence each of $b^{2j'}, c^{2k'} \in H$. Since $[h, h'] = b^{2j'} a^{2^{m-1} k'} \neq 1$, we deduce that $2^{n-1} \nmid j'$ or $2 \nmid k'$. Hence we have $c^2 \in H$. Since $h^2 = a^{2+2^{m-1}k} c^{2k}$, $a^2 \in H$ and $Z(G) \leq H$.

Since $H \cap M$ is an abelian maximal subgroup of H, we have $Z(H) \leq H \cap M$. Notice that $G = HM$ and M is abelian. Thus $Z(G) = Z(H)$. □

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