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Finite soluble groups in which the normalizer of every non-normal cyclic subgroup is maximal

Jianji Cao and Xiuyun Guo

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Abstract. In this paper a classification is given for finite soluble groups in which the normalizer of every non-normal cyclic subgroup is a maximal subgroup.

1 Introduction

All groups considered in this paper are finite.

It is well known that the normality of subgroups plays an important part in research in group theory. So it is reasonable to investigate the structure of a group by using normalizers of certain types of subgroups. For example, Bianchi, Gillio Berta Mauri and Hauck in [2] proved that a group is nilpotent if and only if the normalizer of every Sylow subgroup is nilpotent. Ballester-Bolinches and Shemetkov in [1] gave a beautiful criterion for nilpotent groups: a group is nilpotent if and only if the normalizer of every Sylow p-subgroup is p-nilpotent for every prime p. Zhang in [11] investigated the structure of groups using Sylow numbers (that is, the index of the normalizer of a Sylow subgroup). On the other hand, there is considerable research in the literature concerning the relationship between the structure of p-groups and the normalizers of certain kinds of subgroups (see [6, 8, 9]). For example, Parmeggiani investigates the p-groups with many subgroups that are 2-subnormal [9]. Ormerod investigates the p-groups in which every cyclic subgroup is 2-subnormal [8].

Inspired by the above research, we are interested in the class of groups in which the normalizer of every non-normal cyclic subgroup is a maximal subgroup. We note that Mann in [7] investigated non-soluble groups in which the normalizer of every non-normal subgroup is maximal and he also gave a classification for this kind of group. Although the class of groups in which the normalizer of every non-normal cyclic subgroup is maximal is different from the class of groups in which the normalizer of every non-normal subgroup is maximal (see the following

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The corresponding author is Xiuyun Guo.

Examples 2.1 and 2.2), we only investigate soluble groups in which the normalizer of every non-normal cyclic subgroup is a maximal subgroup in this paper and then give a classification for this kind of groups.

2 Preliminaries

In this section we list some basic properties of groups in which the normalizer of every non-normal cyclic subgroup is a maximal subgroup and other useful results for us. For convenience, we call a group an NCM-group if the normalizer of every non-normal cyclic subgroup is a maximal subgroup.

We begin with some examples which illustrate that there exist NCM-groups and that there also exists a non-normal subgroup in these NCM-groups such that its normalizer is not maximal.

Example 2.1. Let p be a prime and let M_i be a non-abelian group of order p^3 with exponent p for $i = 1, 2, \ldots, k$ and $k \geqslant 2$. Then the central product

$$P = M_1 * M_2 * \cdots * M_k$$

is an NCM-group and there exists a non-normal subgroup H in P such that the normalizer $N_P(H)$ is not maximal in P.

Proof. Let $\langle a \rangle$ be a non-normal cyclic subgroup of P. Then there exists an element $b \in P$ such that $[a, b] = c \neq 1$. Since P is an extra special p-group, we have $Z(P) = \langle c \rangle$, and therefore $\langle a, b \rangle \simeq M_i$. Thus we get $P = \langle a, b \rangle C_P(\langle a, b \rangle)$ by [4, Lemma 5.4.6]. It follows that $N_P(\langle a \rangle)$ is maximal in P, and therefore P is an NCM-group.

Let $M_i = \langle a_i, b_i \rangle$ for $i = 1, 2, \ldots, k$, and let $H = \langle a_1, a_2 \rangle$ with $\langle a_1 \rangle \ntrianglelefteq M_1$, $\langle a_2 \rangle \ntrianglelefteq M_2$. Thus $H = \langle a_1 \rangle \times \langle a_2 \rangle$. Since ${a_1}^{b_1} = a_1 c \notin H$ and ${a_2}^{b_2} = a_2 c \notin H$, we see b_1 and b_2 are not in $N_P(H)$. Noting that both b_1 and b_2 are not in $\Phi(P)$, we see $N_P(H)$ is not maximal in P. □

Example 2.2. Let

$$G = \langle a, b, c \mid a^3 = b^3 = c^4 = 1,\ a^c = ab^2,\ b^c = a^2 b^2,\ [a, b] = 1 \rangle.$$

Then G is an NCM-group and there exists a non-normal subgroup H in G such that $N_G(H)$ is not maximal in G.

Proof. Since $G = (\langle a \rangle \times \langle b \rangle) \rtimes \langle c \rangle$, we see that G has no non-trivial cyclic normal subgroup and that G is an NCM-group. Take $H = \langle a \rangle \langle c^2 \rangle$. Since $b \notin N_G(H)$ and $c \notin N_G(H)$, we see $N_G(H) < M = (\langle a \rangle \times \langle b \rangle) \rtimes \langle c^2 \rangle < G$. So $N_G(H)$ is not maximal in G. □

Lemma 2.3. *Let q be a prime dividing the order of a group G. If G is an* NCM-*group and A is a non-normal cyclic q-subgroup of G, then there exists a normal Hall q'-subgroup T in $C_G(A)$ and every subgroup of T is normal in $N_G(A)$. Particularly, if q is the smallest prime dividing the order of G, then $N_G(A) = T \rtimes Q$ with T an abelian q'-subgroup and Q a Sylow q-subgroup of $N_G(A)$.*

Proof. If $C_G(A)$ is a q-subgroup, then the lemma is true. Assume that $C_G(A)$ is not a q-subgroup and that B is a cyclic p-subgroup of $C_G(A)$ with $p \neq q$ a prime. Since A is a characteristic subgroup of AB, we see $AB \ntrianglelefteq G$ and $A \trianglelefteq N_G(AB)$. It follows from the maximality of $N_G(A)$ that $N_G(A) = N_G(AB)$. By the same reason, we have that $B \trianglelefteq N_G(AB)$ and therefore $N_G(B) \geqslant N_G(AB) = N_G(A)$. It follows that the Sylow p-subgroup in $C_G(A)$ is normal in $N_G(A)$ and therefore the Hall q'-subgroup T in $C_G(A)$ is normal in $N_G(A)$. By the above proof, we see that every subgroup in T is normal in $N_G(A)$.

In particular, if q is the smallest prime dividing the order of G, then the Hall q'-subgroup of $N_G(A)$ is contained in $C_G(A)$ since $N_G(A)/C_G(A)$ is a q-subgroup. Hence $N_G(A) = T \rtimes Q$ with T an abelian q'-subgroup and Q a Sylow q-subgroup of $N_G(A)$. □

Lemma 2.4. *Let N be a normal subgroup of a group G. If G is an* NCM-*group, then G/N is also an* NCM-*group.*

Proof. Let $\langle x\rangle N/N \ntrianglelefteq G/N$. Then $\langle x\rangle \ntrianglelefteq G$, therefore the normalizer $N_G(\langle x\rangle)$ is a maximal subgroup in G. It follows from $N_G(\langle x\rangle)N/N \leqslant N_{G/N}(\langle x\rangle N/N)$ that $N_{G/N}(\langle x\rangle N/N)$ is a maximal subgroup in G/N. □

Lemma 2.5 ([3, Theorem A.15.2]). *Let G be a primitive group with stabilizer M. Then exactly one of the following three statements holds:*

(1) *G has a unique minimal normal subgroup N, this subgroup N is self-centralizing (in particular, abelian), and N is complemented by M in G.*

(2) *G has a unique minimal normal subgroup N, this N is non-abelian, and N is supplemented by M in G.*

(3) *G has exactly two minimal normal subgroups N and N^*, and each of them is complemented by M in G. Also one has $C_G(N) = N^*$, $C_G(N^*) = N$, and $N \simeq N^* \simeq NN^* \cap M$. Moreover, if $V < G$ and $VN = VN^* = G$, then one has $V \cap N = V \cap N^* = 1$.*

Recall that an automorphism of a group is called a power automorphism if it leaves every subgroup invariant.

Lemma 2.6 ([10, 13.4.3]). *Let α be a power automorphism of an abelian group A. If A is a p-group, then there is a positive integer l such that $a^\alpha = a^l$ for all $a \in A$. If α is nontrivial and has order prime to p, then α is fixed-point-free.*

Lemma 2.7 ([5, Theorem 3.8.2]). *Let G be a p-group. If G has a unique cyclic subgroup of order p, then*

(1) *G is a cyclic group for $p > 2$.*

(2) *G is either a cyclic group or a generalized quaternion group for $p = 2$.*

Lemma 2.8 ([5, Theorem 6.14.4]). *Let G be a soluble group with all Sylow subgroups abelian, and let D be a system normalizer of G. Then D is the complement of G' in G.*

3 Main results

In this section the main results in this paper are given.

Lemma 3.1. *Let G be a soluble* NCM-*group. If there exists no non-trivial cyclic normal subgroup in G, then there is a maximal subgroup M in G such that* $\mathrm{core}_G(M) = 1$, *and therefore G has a unique minimal normal subgroup.*

Proof. Suppose that there exists no non-trivial cyclic normal subgroup in G and $\mathrm{core}_G(M) \neq 1$ for every maximal subgroup M in G. Then G is a non-nilpotent group. Let q be a minimal prime dividing the order of G and let Q be a Sylow q-subgroup of G. Then:

(1) One has $O_{q'}(G) \neq 1$.

In fact, if $O_{q'}(G) = 1$, then $F(G) = O_q(G)$. Let $\langle y \rangle$ be a cyclic p-subgroup of G with $p \neq q$. Then $N_G(\langle y \rangle)$ is maximal in G and $\mathrm{core}_G(N_G(\langle y \rangle)) \neq 1$. Thus there is a minimal normal subgroup N of G such that $N \leqslant N_G(\langle y \rangle)$ and N is an elementary abelian q-group. Let $\langle x \rangle \leqslant N \cap Z(Q)$ be a cyclic subgroup of order q. Then, by Lemma 2.3, $N_G(\langle x \rangle) = T_x \rtimes Q$ with T_x a Hall q'-subgroup of $N_G(\langle x \rangle)$. Noting that $O_q(G) \leqslant Q$, we see $[T_x, O_q(G)] \leqslant T_x \cap O_q(G) = 1$, and therefore $T_x \leqslant C_G(O_q(G)) = C_G(F(G)) \leqslant O_q(G)$, in contradiction to that $1 \neq y \in T_x$ is a p-element.

(2) Let $N_G(\langle x \rangle) = T_x \rtimes Q_x$ with Q_x a Sylow q-subgroup of $N_G(\langle x \rangle)$ and T_x a Hall q'-subgroup of $N_G(\langle x \rangle)$ for $x \in Q$. If $T_x \neq 1$, then $N_G(\langle x \rangle) \trianglelefteq G$ if and only if $\mathrm{core}_G(T_x) \neq 1$.

In fact, if $N_G(\langle x \rangle) \trianglelefteq G$, then $T_x \trianglelefteq G$. On the other hand, if $\mathrm{core}_G(T_x) \neq 1$, then

$$[N_G(\langle x \rangle)]^g = {T_x}^g \rtimes {Q_x}^g \geqslant \mathrm{core}_G(T_x)$$

for any $g \in G$. Now let $1 \neq y \in \mathrm{core}_G(T_x)$. Then $\langle y \rangle \ntrianglelefteq G$ by hypothesis. Thus, by Lemma 2.3,

$$[N_G(\langle x \rangle)]^g = N_G(\langle y \rangle) = N_G(\langle x \rangle).$$

So $N_G(\langle x \rangle) \trianglelefteq G$.

(3) $N_G(\langle x \rangle) \trianglelefteq G$ for any $1 \neq x \in Q$.

In fact, by (1), there is a minimal normal subgroup K in G such that K is an elementary abelian p-group with $p \neq q$. If there exists an element $x \in Q$ such that $N_G(\langle x \rangle) \ntrianglelefteq G$, then, by (2), $G = KN_G(\langle x \rangle)$. Thus by Lemma 2.3, we have $G = K \rtimes (T_x \rtimes Q)$ with T_x a Hall q'-subgroup of $N_G(\langle x \rangle)$. On the other hand, it follows from $\mathrm{core}_G(N_G(\langle x \rangle)) \neq 1$ and (2) that there exists a minimal normal subgroup N of G such that $N \leqslant Q$. Noting that G is a q-nilpotent group, we see $|G : C_G(N)| = q^i$ by Frobenius's criterion for q-nilpotent. So, for any element $y \in N \cap Z(Q)$, we have $\langle y \rangle \trianglelefteq G$, a contradiction. Hence $N_G(\langle x \rangle) \trianglelefteq G$ for any $1 \neq x \in Q$.

(4) $(|N_G(\langle x \rangle)|, |G : N_G(\langle x \rangle)|) = 1$ for any $1 \neq x \in Q$ and Q is a Dedekind group.

In fact, let $N_G(\langle x \rangle) = T_x \rtimes Q_x$ with Q_x a Sylow q-subgroup of $N_G(\langle x \rangle)$ and T_x a Hall q'-subgroup of $N_G(\langle x \rangle)$. If $T_x = 1$, then $N_G(\langle x \rangle) = Q_x$ is maximal in G, and therefore the conclusion is true. Thus we only need to consider $T_x \neq 1$. It is clear that we may assume that $Q_x \leqslant Q$ and $|G : N_G(\langle x \rangle)| = p$ is a prime. Let P_x be a Sylow p-subgroup of $N_G(\langle x \rangle)$ and let P be a Sylow p-subgroup of G with $P_x \leqslant P$. If $p \in \pi(T_x)$, then $P_x \trianglelefteq G$ by Lemma 2.3. Thus there exists an element h such that $1 \neq h \in Z(P) \cap P_x$. By Lemma 2.3, we have that $\langle h \rangle \trianglelefteq G$, a contradiction. If $p = q$, then we may assume $P = Q$. Take $a \in Z(Q)$. Then $Q \leqslant N_G(\langle a \rangle)$ and therefore $G = N_G(\langle a \rangle)N_G(\langle x \rangle)$. Let $N_G(\langle a \rangle) = T_a \rtimes Q$. The normality of $N_G(\langle a \rangle)$ implies $T_a \trianglelefteq G$ and therefore we obtain $T_a \leqslant T_x$. Since $|G : N_G(\langle a \rangle)| = r$ is a prime and $r \neq q$, we see $(r, |T_a|) = 1$ by the above proof, and therefore $(|T_x : T_a|, |T_a|) = 1$. Noting that both T_x and T_a are normal in G and T_x is abelian, we see that the complement of T_a in T_x is a cyclic normal subgroup of G, another contradiction. Hence $(|G : N_G(\langle x \rangle)|, |N_G(\langle x \rangle)|) = 1$, which also proves that Q is a Dedekind group.

(5) There exists a maximal subgroup M in G such that $\mathrm{core}_G(M) = 1$.

In fact, by (3) we can let $|G : N_G(\langle x \rangle)| = p$ for $1 \neq x \in Q$ and let H be a Sylow p-subgroup of G. By (4), $N_G(\langle x \rangle)$ is a Hall p'-subgroup of G with $p \neq q$ and H is a cyclic group of order p. Also by (3), we have $N_G(\langle x \rangle) \cap N_G(H) \trianglelefteq N_G(H)$ and $N_G(\langle x \rangle) \cap N_G(H)$ is a Hall p'-subgroup of $N_G(H)$. Hence

$$N_G(H) = H \times (N_G(\langle x \rangle) \cap N_G(H)).$$

It follows from Lemma 2.3 that every cyclic q'-subgroup of $N_G(\langle x \rangle) \cap N_G(H)$ is normal in G. Thus $N_G(\langle x \rangle) \cap N_G(H)$ is a q-subgroup. Noting that Q is a Dedekind group, we see that

$$N_G(\langle y\rangle) = \langle N_G(H), Q\rangle = G$$

for any $y \in N_G(\langle x\rangle)\cap N_G(H)$. So we get $N_G(\langle x\rangle)\cap N_G(H) = 1$. Consequently, $N_G(H) = H$ is a maximal subgroup of G and $\mathrm{core}_G(H) = 1$. By Lemma 2.5, there is a unique minimal normal subgroup in G. The proof is now complete. □

Lemma 3.2. *Let q be the smallest prime dividing the order of a group G. If G is a soluble* NCM*-group, then G is either q-closed or q-nilpotent.*

Proof. Suppose that the result is not true and let G be a counterexample of minimal order. Let Q be a Sylow q-subgroup of G. Then:

(1) There is no non-trivial cyclic normal q-subgroup in G.

If there is an element $x \in G$ of order q such that $\langle x\rangle \trianglelefteq G$, then

$$\langle x\rangle \leqslant Z(G)\cap Q = K.$$

By Lemma 2.4, G/K is an NCM-group. The minimality of G implies that G/K is q-closed or q-nilpotent. It follows that G is q-closed or q-nilpotent, a contradiction.

(2) There is no non-trivial cyclic normal q'-subgroup in G.

In fact, suppose that there is a non-trivial cyclic normal q'-subgroup in G and that N is the product of all cyclic normal q'-subgroups in G. By Lemma 2.4, G/N is an NCM-group, and therefore G/N is q-closed or q-nilpotent. If G/N is q-nilpotent, then G is also q-nilpotent, a contradiction. So we may assume that G/N is q-closed and that QN is normal in G.

By Lemma 2.3, $N_G(\langle x\rangle) = T_x \rtimes Q_x$ with T_x a Hall q'-subgroup of $N_G(\langle x\rangle)$ and Q_x a Sylow q-subgroup of $N_G(\langle x\rangle)$ for every $1 \neq x \in Q$. If $\langle x\rangle \trianglelefteq Q$, then $Q_x = Q$. We conclude that $N \leqslant T_x$. Otherwise, the maximality of $N_G(\langle x\rangle)$ implies that $G = NN_G(\langle x\rangle)$ is q-nilpotent, a contradiction. So we get $x \in C_G(N)$. If $\langle x\rangle \ntrianglelefteq Q$, then T_x is a Hall q'-subgroup of G. Thus $N \leqslant T_x$, and therefore $x \in C_G(N)$. So $Q \leqslant C_G(N)$. Therefore Q is normal in G, a contradiction.

(3) Final contradiction.

By (1) and (2), we see that there is no non-trivial cyclic normal subgroup in G. Lemma 3.1 implies that there must be a maximal subgroup M of G such that $\mathrm{core}_G(M) = 1$ and G has a unique minimal normal subgroup K. So $G = MK$, $M \cap K = 1$ and $F(G) = K$.

Let K be an elementary abelian p-group for some prime p. Then M must be a p'-group. Otherwise, there exists an element $y \in M$ such that $\langle y\rangle$ is a p-group. If $K \leqslant N_G(\langle y\rangle)$, then we have $y \in C_G(K)$, in contradiction to $C_G(F(G)) = F(G)$. If $K \nleqslant N_G(\langle y\rangle)$, then the maximality of $N_G(\langle y\rangle)$ implies that $G = N_G(\langle y\rangle)K$. So $K \rtimes \langle y\rangle$ is a normal p-subgroup of G, in contradiction to $F(G) = K$.

So if $p = q$, then G is q-closed, a contradiction. Hence we assume $p \neq q$. Since G is not q-nilpotent, there exists a q'-element $x \in M^g$ for some element $g \in G$ such that $x \notin O_{q'}(G)$. If $O_{q'}(G) \leqslant N_G(\langle x \rangle)$, then, since $K \leq O_{q'}(G)$, we have $[K, x] \leqslant K \cap \langle x \rangle = 1$ and therefore $x \in C_G(K) = C_G(F(G)) = K$, a contradiction. If $O_{q'}(G) \nleq N_G(\langle x \rangle)$, then we get $G = O_{q'}(G) N_G(\langle x \rangle)$ and therefore $O_{q'}(G) \rtimes \langle x \rangle$ is normal q'-subgroup of G, a contradiction. These contradictions show that the result is true. □

Theorem 3.3. *Let G be a nilpotent group. Then G is an* NCM-*group if and only if all Sylow subgroups in G are* NCM-*groups and there is at most one Sylow subgroup in G is not a Dedekind group.*

Proof. It is easy to see that we only need to prove necessity. Let

$$G = P_1 \times P_2 \times \cdots \times P_s$$

with P_i a Sylow p_i-subgroup. If there exist elements $x_i \in P_i$ and $x_j \in P_j$ such that $\langle x_i \rangle \ntrianglelefteq P_i$ and $\langle x_j \rangle \ntrianglelefteq P_j$, then it follows from

$$N_G(\langle x_i \rangle \langle x_j \rangle) = N_G(\langle x_i x_j \rangle) = N_G(\langle x_i \rangle) \cap N_G(\langle x_j \rangle)$$

that

$$N_G(\langle x_i \rangle \langle x_j \rangle) = N_G(\langle x_i \rangle) = N_G(\langle x_j \rangle)$$

and therefore $P_i \times P_j \leqslant N_G(\langle x_i \rangle)$, in contradiction to $\langle x_i \rangle \ntrianglelefteq P_i$ and $\langle x_j \rangle \ntrianglelefteq P_j$. Now suppose $P \in \mathrm{Syl}_p(G)$ and P is not a Dedekind group. If $\langle x \rangle \ntrianglelefteq P$, then the maximality of $N_G(\langle x \rangle)$ implies that $N_G(\langle x \rangle) \trianglelefteq G$ and $|G : N_G(\langle x \rangle)| = p$. Hence $|P : N_P(\langle x \rangle)| = p$ and P is an NCM-group. □

Lemma 3.4. *Let q be the smallest prime dividing the order of a group G and let Q be a Sylow q-subgroup in G. If G is a soluble non-nilpotent* NCM-*group and G has a normal q-complement T, then:*

(1) *$T \leqslant C_G(x)$ if $\langle x \rangle \trianglelefteq G$ or $\langle x \rangle \ntrianglelefteq Q$ for $x \in Q$.*

(2) *There is a prime p such that $|G : N_G(\langle y \rangle)|$ is a power of p for every $y \in Q$ with $\langle y \rangle \trianglelefteq Q$ but $\langle y \rangle \ntrianglelefteq G$.*

(3) *$C_T(y) = C_T(Q)$ for every $y \in Q$ with $\langle y \rangle \trianglelefteq Q$ but $\langle y \rangle \ntrianglelefteq G$.*

(4) *P is normal in G if P is a Sylow p-subgroup of T.*

(5) *P is an abelian group and $C_P(Q)$ is in the center of G.*

(6) *If there is an element $t \in T$ with $\langle t \rangle \trianglelefteq T$ but $\langle t \rangle \ntrianglelefteq G$, then $[P, Q]$ is a minimal normal subgroup in G.*

Proof. (1) It is easy to find that $N_G(\langle x \rangle)/C_G(x)$ is a q-group for every $x \in Q$ by the minimality of q. It follows that (1) is true.

(2) Suppose the result is not true. Then there are two elements $y_i \in Q$ with $\langle y_i \rangle \trianglelefteq Q$ but $\langle y_i \rangle \ntrianglelefteq G$ for $i = 1, 2$ such that

$$|G : N_G(\langle y_1 \rangle)| = {p_1}^l \quad \text{and} \quad |G : N_G(\langle y_2 \rangle)| = {p_2}^m$$

with primes p_1 and p_2 and $p_1 \neq p_2$. Let P_1 be a Sylow p_1-subgroup of T such that $P_1 \leqslant N_T(\langle y_2 \rangle)$ and let P_2 be a Sylow p_2-subgroup of T such that $P_2 \leqslant N_T(\langle y_1 \rangle)$. Then we get $[y_1 y_2, P_1] \neq 1$ and $[y_1 y_2, P_2] \neq 1$. If T is a nilpotent group, then $p_1 p_2$ divides $|G : N_G(\langle y_1 y_2 \rangle)|$, in contradiction to the fact that $N_G(\langle y_1 y_2 \rangle)$ is a maximal subgroup. So T is a non-nilpotent group. By Lemma 2.3, $N_G(\langle y_i \rangle) = C_T(y_i) \rtimes Q$ with $C_T(y_i)$ an abelian Hall q'-subgroup of $N_G(\langle y_i \rangle)$ for $i = 1, 2$, and hence we have $N_G(\langle y_2 \rangle) \leqslant N_G(P_1)$ and $N_G(\langle y_1 \rangle) \leqslant N_G(P_2)$. If $N_T(P_1) = C_T(P_1)$, then T is p_1-nilpotent. If $N_T(P_1) \neq C_T(P_1)$, then there exists an element $t \in N_T(P_1) \backslash C_T(P_1)$. Since $C_T(y_2) \leqslant C_T(P_1)$, we conclude that $t \notin N_G(\langle y_2 \rangle)$. The maximality of $N_G(\langle y_2 \rangle)$ implies that $P_1 \trianglelefteq G$. Thus T is p_1-nilpotent or $P_1 \trianglelefteq T$. Similarly, T is p_2-nilpotent or $P_2 \trianglelefteq T$. If P_1 and P_2 are normal in T, then, since the Hall $\{p_1, p_2\}'$-subgroup in T is contained in the center of T, it is clear that T is nilpotent. Similarly, T is also nilpotent if T is p_1-nilpotent and p_2-nilpotent. So we may assume that P_1 is normal in T and that T is p_2-nilpotent. In this case we may assume $T = H \times (P_1 \rtimes P_2)$ with H an abelian Hall $\{p_1, p_2\}'$-subgroup of T. Noting that every element of Q induces a power automorphism on P_1 and P_2 by Lemma 2.3, we see that y_1 induces a non-trivial ${p_1}'$-automorphism on P_1 since $P_1 \nleq C_T(y_1)$. By Lemma 2.6, y_1 can induce a fixed-point-free automorphism on P_1. Thus $C_{P_1}(y_1) = 1$, and therefore $C_T(y_1)$ is a Hall ${p_1}'$-subgroup of T.

Suppose that there exists an element $g \in P_1$ such that $[y_1 y_2, {P_2}^g] = 1$. By the above arguments, y_1 induces a fixed-point-free automorphism on $\langle g \rangle$. So we may assume $g^{y_1} = g^j$ with $(j, |\langle g \rangle|) = 1$ and $j \not\equiv 1 \pmod{|\langle g \rangle|}$. On the other hand, since $P_2 \ntrianglelefteq T$, there exists $\langle d \rangle \leqslant P_2$ such that $\langle d \rangle \ntrianglelefteq T$. Similarly, y_2 induces a power automorphism on $\langle d \rangle$ by conjugation. We may assume $d^{y_2} = d^k$ with $(k, |\langle d \rangle|) = 1$. Since $N_G(\langle y_1 \rangle^g)$ is maximal and $\langle d \rangle^g \ntrianglelefteq T$, we see

$$N_G(\langle y_1 \rangle^g) = N_G(\langle y_1 \rangle)^g = (H^g \times {P_2}^g) \rtimes Q^g = N_G(\langle d \rangle^g)$$

by Lemma 2.3. Noting that $g^{j-1} \notin N_T(\langle d^g \rangle) = N_T(\langle (d^g)^k \rangle)$, we see

$$\begin{aligned} d^g &= (d^g)^{y_1 y_2} = (d^{y_1 y_2})^{g^{y_1 y_2}} = (d^k)^{g^j} \\ &= (d^{g g^{j-1}})^k = ((d^g)^k)^{g^{j-1}} \notin \langle (d^g)^k \rangle = \langle d^g \rangle, \end{aligned}$$

a contradiction. Hence $[y_1 y_2, {P_2}^g] \neq 1$ for any $g \in G$, in contradiction to that $N_G(\langle y_1 y_2 \rangle)$ is a maximal subgroup. So (2) is true.

(3) If not, then there are two elements $y_i \in Q$ with $\langle y_i\rangle \trianglelefteq Q$ but $\langle y_i\rangle \ntrianglelefteq G$ for $i = 1, 2$ such that $C_T(y_1) \neq C_T(y_2)$. By Lemma 2.3, $N_G(\langle y_i\rangle) = C_T(y_i) \rtimes Q$ with $C_T(y_i)$ an abelian Hall q'-subgroup of $N_G(\langle y_i\rangle)$ for $i = 1, 2$. Then:

(i) T is a non-nilpotent group.

In fact, if T is a nilpotent group, then it follows from $C_T(y_1) \neq C_T(y_2)$ and (2) that $C_P(y_1) \neq C_P(y_2)$ if P is a Sylow p-subgroup of T. Noting that y_1 induces a p'-power automorphism on $C_P(y_2)$ by Lemma 2.3 and y_1 induces an identity automorphism on $C_P(y_1)$, we see $C_P(y_2) \leqslant C_P(y_1)$ by Lemma 2.6 if $C_P(y_1) \cap C_P(y_2) \neq 1$. Similarly, $C_P(y_1) \leqslant C_P(y_2)$ if $C_P(y_1) \cap C_P(y_2) \neq 1$. Therefore $C_P(y_1) = C_P(y_2)$, a contradiction. So $C_P(y_1) \cap C_P(y_2) = 1$ and P is an elementary abelian group, and furthermore T is an abelian group. By Lemma 2.3 again, Q normalizes every subgroup of $C_P(y_1)$ and $C_P(y_2)$. Thus every subgroup of $C_P(y_1)$ and $C_P(y_2)$ is normal in G. The maximality of $N_G(\langle y_1\rangle)$ and $N_G(\langle y_2\rangle)$ imply that $C_P(y_1) \neq 1$ and $C_P(y_2) \neq 1$, and therefore we conclude that $P = C_P(y_1) \times C_P(y_2)$ is an elementary abelian group of order p^2. Let $C_P(y_1) = \langle a\rangle$ and $C_P(y_2) = \langle b\rangle$. It is clear that $[y_1y_2, a] \neq 1$ and $[y_1y_2, b] \neq 1$. If $C_P(y_1y_2) = 1$, then the maximality of the normalizer $N_G(\langle y_1y_2\rangle)$ implies that $G = \langle N_G(\langle y_1y_2\rangle), a\rangle$, and therefore $|P| = p$, a contradiction. Thus there exists an element $a^mb^n \in P$ with $(mn, p) = 1$ such that $a^mb^n \in C_T(y_1y_2)$. On the other hand,

$$[y_1y_2, a^mb^n] = [y_1, a^mb^n][y_1, a^mb^n, y_2][y_2, a^mb^n].$$

Since $1 \neq [y_1, a^mb^n] = [y_1, b^n][y_1, a^m][y_1, a^m, b^n] = [y_1, b^n] \in \langle b^n\rangle = \langle b\rangle$ and $1 \neq [y_2, a^mb^n] = [y_2, b^n][y_2, a^m][y_2, a^m, b^n] = [y_2, a^m] \in \langle a^m\rangle = \langle a\rangle$, we see $[y_1y_2, a^mb^n] \neq 1$, a contradiction. So T is a non-nilpotent group.

(ii) One has $P \trianglelefteq T$.

Assume $C_T(y_1) = M \times P_1 \times \cdots \times P_l \times P*$, where M is the product of all normal Sylow subgroups of T which are contained in $C_T(y_1)$, P_k is a Sylow p_k-subgroup of T for $k = 1, 2, \ldots, l$ and $P*$ is a Sylow p-subgroup of $C_T(y_1)$. The maximality of $N_G(\langle y_1\rangle)$ implies that

$$G = \langle C_T(y_1), C_T(y_2), Q\rangle = \langle C_T(y_1), C_T(y_2)\rangle Q.$$

Thus $T \leqslant \langle C_T(y_1), C_T(y_2)\rangle$. By part (2), $M \leqslant C_T(y_1) \cap C_T(y_2)$. Since $C_T(y_1)$ and $C_T(y_2)$ are abelian groups, we see $M \leqslant Z(T)$. Then it follows from (i) that M is not a Hall p'-subgroup of T, and therefore $l \geqslant 1$. If there exists an element $t \in N_T(P_k)\backslash C_T(P_k)$, then, since $C_T(y_1) \leqslant C_T(P_k)$, we see $t \notin N_G(\langle y_1\rangle)$. It follows from Lemma 2.3 that $N_G(\langle y_1\rangle) < N_G(P_k)$, and therefore $P_k \trianglelefteq G$, a contradiction. Hence $N_T(P_k) = C_T(P_k)$ for $k = 1, 2, \ldots, l$, and therefore T has normal p_k-complement T'_{p_k}. It is clear that the intersection of all T'_{p_k} for $k = 1, 2, \ldots, l$ is equal to $M \times P$, which yields that P is normal in T.

(iii) $C_T(y_1)$ and $C_T(y_2)$ are Carter subgroups of T.

If there exists an element $t \in T \backslash C_T(y_1)$ such that t normalizes $C_T(y_1)$, then the maximality of $N_G(\langle y_1 \rangle)$ implies that $C_T(y_1)$ is normal in G and therefore T is nilpotent by (ii), a contradiction. Thus $C_T(y_i)$ is self-normalizing in T, and furthermore $C_T(y_i)$ for $i = 1, 2$ are Carter subgroups of T.

(iv) Final contradiction.

By (iii) there exists an element $g \in P$ such that $C_T(y_1) = C_T(y_2)^g$. It is clear that $\langle y_2 \rangle^g \trianglelefteq Q^g$ but $\langle y_2 \rangle^g \ntrianglelefteq G$. By Lemma 2.3,

$$N_G(\langle y_2 \rangle^g) = N_G(\langle y_2 \rangle)^g = C_T(y_2)^g \rtimes Q^g = C_T(y_1) \rtimes Q^g.$$

If $Q^g \leqslant N_G(\langle y_1 \rangle)$, then there exists an element $t \in C_T(y_1)$ such that $Q^g = Q^t$ by Sylow's theorem. Thus $gt^{-1} \in N_T(Q)$ and so $gt^{-1} \in C_T(Q) \leqslant C_T(y_1)$ by Frobenius's criterion for q-nilpotency. Hence $g \in C_T(y_1)$, a contradiction. So we may assume $Q^g \nleqslant N_G(\langle y_1 \rangle)$. Since both Q^g and $N_G(\langle y_1 \rangle)$ normalize $C_T(y_1)$, we see that $C_T(y_1)$ is normal in G by the maximality of $N_G(\langle y_1 \rangle)$, in contradiction to (iii). So (3) is true.

(4) It is clear that the result is true if T is a nilpotent group. Now we may assume that T is a non-nilpotent group and that $C_T(Q) = C_P(Q) \times P_1 \times P_2 \times \cdots \times P_s$ with P_r a Sylow p_r-subgroup of T for $r = 1, 2, \ldots, s$.

The maximality of $Q \times C_T(Q)$ implies $C_G(P_r) = G$ or $C_G(P_r) = Q \times C_T(Q)$ for $r = 1, 2, \ldots, s$. If there is a Sylow p_r-subgroup P_r such that $P_r \trianglelefteq G$ and $C_G(P_r) = Q \times C_T(Q)$, then, since $C_G(P_r)$ is normal in G, $Q \trianglelefteq G$, a contradiction. Thus $P_r \leqslant Z(G)$ if $P_r \trianglelefteq G$. If $P_r \ntrianglelefteq G$, then $N_G(P_r) = C_G(P_r)$ and therefore G is p_r-nilpotent. Let M be the intersection of all normal p_r-complements of non-normal Sylow p_r-subgroups P_r for $r \in \{1, 2, \ldots, s\}$. Then $M \cap T = N \times P$ with $N \leqslant Z(G)$, and therefore $P \trianglelefteq G$. So (4) is true.

(5) By (1)–(3), $G = \langle Q, C_T(Q), P \rangle$. Lemma 2.3 implies that $C_T(Q)$ is an abelian group. Now we claim $C_P(Q) \leqslant Z(G)$. In fact, if $\Phi(P) = 1$, then we have $C_P(Q) \leqslant Z(G)$. If $\Phi(P) \neq 1$, then, by (4), $\Phi(P) \leqslant \Phi(G) \leqslant Q \times C_T(Q)$ and therefore $\Phi(P) \leqslant C_P(Q)$. It follows from $Q \times C_T(Q) \leqslant C_G(C_P(Q))$ that $C_P(Q) \trianglelefteq G$. If $C_G(C_P(Q)) = Q \times C_T(Q)$, then it is easy to see $Q \trianglelefteq G$, a contradiction. Thus we still have $C_P(Q) \leqslant Z(G)$.

If P is a non-abelian group, then, since $p > 2$, there exists a cyclic subgroup $\langle z \rangle$ of P such that $\langle z \rangle \ntrianglelefteq P$. Thus $|G : N_G(\langle z \rangle)|$ is a power of p and $N_G(\langle z \rangle)$ contains a Hall p'-subgroup of G. It follows from $C_P(Q) \leqslant Z(G)$ that there is an element $g \in G$ such that $(Q \times C_T(Q))^g \leq N_G(\langle z \rangle)$. Since z is not contained in $C_T(Q)$, $N_G(\langle z \rangle) = G$, a contradiction. So (5) is true.

(6) It is clear that we may assume that t is an r-element with r a prime such that $\langle t \rangle \trianglelefteq T$ but $\langle t \rangle \ntrianglelefteq G$. In this case $\langle t \rangle \nleqslant C_T(Q)$. By (2) and (3), $|T : C_T(Q)|$ is a power of p and $[P, Q] \neq 1$. Since P is a normal abelian Sylow p-subgroup,

then, by [4, Theorem 5.2.3], $P = C_P(Q) \times [P, Q]$. It is clear that $C_T(Q)$ normalizes $[P, Q]$ and therefore $[P, Q]$ is a normal subgroup in G. The maximality of $Q \times C_T(Q)$ implies that $[P, Q]$ is a minimal normal subgroup in G. The proof is now complete. □

Theorem 3.5. *Let q be the smallest prime dividing the order of a group G, and let Q and T be a Sylow q-subgroup and a Hall q'-subgroup in G respectively. Then G is a soluble non-nilpotent* NCM-*group if and only if G is one of the following type groups:*

(I) *$G = Q \times T$ with Q a Dedekind group and T a soluble non-nilpotent* NCM-*group.*

(II) *$G = H \times (Q \ltimes P)$ with H an abelian Hall $\{p, q\}'$-subgroup, Q a Dedekind group and P a Sylow p-subgroup. Furthermore, $C_P(Q)$ is an abelian maximal subgroup in P and $HP \times C_Q(P)$ is maximal in G. If P is a non-abelian group, then there is an element $c \in P$ such that $P = C_P(Q)\langle c\rangle$ with $P' \leq \langle ac\rangle$ for any $a \in C_P(Q)$.*

(III) *$G = H \times (K \ltimes Q)$ with H an abelian Hall $\{p, q\}'$-group and K a minimal normal subgroup of order p^m in G, and there is a maximal subgroup Q_1 of Q and an element $b \in Q \backslash Q_1$ such that $Q = Q_1\langle b\rangle$ with $\langle ab\rangle \trianglelefteq Q$ but $\langle ab\rangle \ntrianglelefteq G$ for any $a \in Q_1$. Furthermore, every element of Q_1 induces a power automorphism on K and $N_G(\langle x\rangle) = N_G(\langle z\rangle)$ for any $z \in K - \{1\}$ and for any $x \in Q$ with $\langle x\rangle \ntrianglelefteq Q$.*

(IV) *$G = H \times S \times (K \ltimes Q)$ with H an abelian Hall $\{p, q\}'$-group, S an abelian p-group, K a minimal normal subgroup of order p^m in G, $C_Q(T)$ a Dedekind group and $C_Q(T)$ maximal in Q. Furthermore, there exists an element $b \in Q \backslash C_Q(T)$ such that $Q = C_Q(T)\langle b\rangle$ with $\langle ab\rangle \trianglelefteq Q$ but $\langle ab\rangle \ntrianglelefteq G$ for any $a \in C_Q(T)$.*

(V) *$G = H \times (P \ltimes Q)$ with H an abelian Hall $\{p, q\}'$-subgroup, P a cyclic Sylow subgroup of order p and Q an* NCM-*group. Furthermore, $C_Q(P)$ is normal in G and there exists an element $c \in Q \backslash C_Q(P)$ with the property that $Q = C_Q(P)\langle c\rangle$ and $\langle y\rangle \trianglelefteq Q$ but $\langle y\rangle \ntrianglelefteq G$ for any $y \in Q \backslash C_Q(P)$.*

(VI) *$G = (T' \ltimes C_T(Q)) \ltimes Q$ with T' a cyclic subgroup of order p, $C_T(Q)$ an abelian Hall $\{p, q\}'$-subgroup of G and Q a Dedekind group.*

(VII) *$G = (T' \ltimes C_T(Q)) \ltimes Q$ with T' a minimal normal subgroup of order p^m in G, $C_T(Q)$ an abelian Hall $\{p, q\}'$-subgroup of G and Q a Dedekind group. Furthermore, there is a maximal subgroup T_1 of T and an r-element $d \in T \backslash T_1$ with r a prime and $r \neq p$ such that $T = T_1\langle d\rangle$, every element of Q and every element of T_1 induces a power automorphism on T'.*

(VIII) $G = (T' \rtimes C_T(Q)) \rtimes Q$ *with* T' *a minimal normal subgroup of order* p^m *in* G*,* $C_T(Q)$ *an abelian Hall* $\{p,q\}'$*-subgroup of* G *and* Q *a Dedekind group. Furthermore, there is a maximal subgroup* Q_1 *of* Q *and an element* $e \in Q \backslash Q_1$ *such that* $Q = Q_1\langle e \rangle$*, every element of* Q_1 *and every element of* T *induces a power automorphism on* T'*.*

Proof. We first assume that G is a soluble non-nilpotent NCM-group. In case that $T \leqslant C_G(Q)$, then $G = Q \times T$ with T a soluble non-nilpotent group. By Lemma 2.4, T is also an NCM-group. If there exists a subgroup $\langle x \rangle$ in Q such that $\langle x \rangle \ntrianglelefteq Q$, then the maximality of $N_G(\langle x \rangle)$ and Lemma 2.3 imply that T is an abelian group, and therefore G is a nilpotent group, a contradiction. Thus Q is a Dedekind group, and so G is the type (I).

Now we assume $T \nleqslant C_G(Q)$. By Lemma 3.2, we may consider the following two cases.

Case 1: $Q \trianglelefteq G$. If every cyclic subgroup $\langle x \rangle$ in Q is normal in G, then the minimality of q implies that $T \leqslant C_G(Q)$. Thus there exists a subgroup $\langle y \rangle$ in Q such that $\langle y \rangle \ntrianglelefteq G$. If Q is a Dedekind group, then, by Lemma 2.3, $N_G(\langle y \rangle) = T_y \times Q$ with T_y an abelian Hall q'-subgroup of $N_G(\langle y \rangle)$. Since $T \nleqslant C_G(Q)$, we see $C_G(Q) = T_y \times Z(Q)$ and T_y is normal in G, and therefore $N_G(\langle y \rangle) = T_y \times Q$ is normal in G. The maximality of $N_G(\langle y \rangle)$ implies that T_y is maximal in T and $T_y = C_T(Q)$. Thus there is a prime p such that $|G : N_G(\langle y \rangle)| = p$ and there is a p-element $c \in T$ such that $T = T_y\langle c \rangle$. If $Q \leq N_G(\langle c \rangle)$, then $[Q, c] = 1$ and therefore $Q \leqslant C_G(T)$, a contradiction. Hence $T \leq N_G(\langle c \rangle)$. It follows that $N_G(\langle c \rangle) = TC_Q(c)$, and therefore the Hall $\{p,q\}'$-subgroup H in G is contained in the center of G. Let $G = H \times (Q \rtimes P)$ with $P = C_P(Q)\langle c \rangle$ a Sylow p-subgroup in G. If P is a non-abelian group, then we have $p^n = |\langle c \rangle| \geqslant p^2$. Noting that $P' = [C_P(Q), \langle c \rangle] \leqslant C_P(Q) \cap \langle c \rangle \leqslant Z(P)$, we see that there is an element $g \in C_P(Q)$ such that $P' = \langle [g, c] \rangle$. Now for any $a \in C_P(Q)$, we see $\langle ac \rangle$ is normal in P by the above proof. Thus $[g, c] = [g, ac] \in P' \cap \langle ac \rangle$ and the group is the type (II).

Now we assume that Q is not a Dedekind group and that $Q \nleqslant N_G(\langle y \rangle)$. The maximality of $N_G(\langle y \rangle)$ implies that $N_G(\langle y \rangle)$ contains a Hall q'-subgroup K in G. Without loss of generality, we may assume $K = T$. By Lemma 2.3 and the normality of Q, we have $N_G(\langle y \rangle) = T \times Q_y$ with T an abelian group and Q_y a Sylow q-subgroup of $N_G(\langle y \rangle)$ and therefore $Q_y = C_Q(T)$. Noting that

$$C_Q(T)^G = \langle C_Q(T), [C_Q(T), Q] \rangle \quad \text{and} \quad [C_Q(T), Q] \leqslant \Phi(Q),$$

we see $C_Q(T)^G < Q$. The maximality of $C_Q(T) \times T$ implies $C_Q(T)^G = C_Q(T)$. Let $\langle x \rangle$ be a cyclic subgroup of Q. If $\langle x \rangle$ is normal in G, then the minimality of q implies that $N_G(\langle x \rangle)/C_G(x)$ is a q-group and therefore $x \in C_Q(T)$.

If $Q \not\leq N_G(\langle x\rangle)$, then there is an element $g \in G$ such that $N_G(\langle x\rangle) = T^g \times Q_x$. The maximality of $N_G(\langle x\rangle)$ implies $Q_x = C_Q(T^g)$. It follows that $Q_x = C_Q(T)$ and therefore $x \in C_Q(T)$. Hence $C_Q(T)$ is a Dedekind group and $\langle x\rangle$ is not normal in G with $Q \leq N_G(\langle x\rangle)$ if $x \in Q\backslash C_Q(T)$.

Let $c \in Q\backslash C_Q(T)$. Then, by the above arguments and Lemma 2.3, we have $N_G(\langle c\rangle) = T_c \times Q$ with T_c a Hall q'-subgroup in $N_G(\langle c\rangle)$. It follows from the commutativity of T that $N_G(\langle c\rangle) \trianglelefteq G$ and therefore T_c is maximal in T. Then $T_c = C_T(Q)$ and $|G : N_G(\langle c\rangle)| = p$ is a prime. Thus $QC_T(Q)/C_Q(T)C_T(Q)$ is a minimal normal subgroup in $G/C_Q(T)C_T(Q)$ and $c^q \in C_Q(T)$. In case that $[C_Q(T), c] = 1$, then, for any cyclic subgroup $\langle x\rangle$ of $C_Q(T)$, we have

$$N_G(\langle x\rangle) = \langle T, C_Q(T), c\rangle = G,$$

in contradiction to that Q is not a Dedekind group. Thus $[C_Q(T), c] \neq 1$ and therefore the order of c is at least q^2. On the other hand, there is a p-element t in T such that $T = C_T(Q)\langle t\rangle$. Then

$$\langle c\rangle^G = \langle c\rangle^{\langle t\rangle} = \prod_{j=0}^{p-1} \langle c\rangle^{t^j}.$$

The maximality of $T \times C_Q(T)$ implies that $Q = C_Q(T)\langle c\rangle^G$.

Noting that $c^t \in Q\backslash C_Q(T)$, we see that $\langle c^t\rangle$ is normal in Q. If there exists a positive integer k with $1 \leq k \leq q-1$ such that $c^t = xc^k$ with $x \in C_Q(T)$, then it is easy to see that $\langle t\rangle$ normalizes $C_Q(T)\langle c\rangle$ and therefore $C_Q(T)\langle c\rangle$ is normal in G. Hence $Q = C_Q(T)\langle c\rangle$ and $QC_T(Q)/C_Q(T)C_T(Q)$ is a cyclic group of order q. By the minimality of q, $QC_T(Q)/C_Q(T)C_T(Q)$ is contained in the center of $G/C_Q(T)C_T(Q)$ and therefore T is normal in G, in contradiction to $T \not\leq C_G(Q)$. Thus we may assume that there is a positive integer l with $1 \leq l \leq p-1$ such that

$$c^t \notin C_Q(T)\langle c\rangle,\ c^{t^2} \notin C_Q(T)\langle c, c^t\rangle,\ \ldots,\ c^{t^l} \notin C_Q(T)\langle c, c^t, \ldots, c^{t^{l-1}}\rangle$$

but

$$c^{t^{l+1}} \in C_Q(T)\langle c, c^t, \ldots, c^{t^l}\rangle.$$

For convenience, we set $R = \langle c, c^t, \ldots, c^{t^l}\rangle$. Then it is easy to see that $C_Q(T)R$ is normal in G and therefore $Q = C_Q(T)R$. Furthermore we see $C_Q(T) \cap R = \langle c^q\rangle$ from the choice of l. In fact, if $C_Q(T) \cap R \neq \langle c^q\rangle$, then there is an element $r \in C_Q(T)$ such that $r \notin \langle c^q\rangle$ and $r = c^{i_0}(c^t)^{i_1} \cdots (c^{t^{l-1}})^{i_{l-1}}(c^{t^l})^{i_l}$. It follows that there exists an i_k such that $(i_k, q) = 1$. Without loss of generality, we may assume $k = l$. In this case

$$(c^{t^l})^{i_l} = (c^{t^{l-1}})^{-i_{l-1}} \cdots c^{-i_0} r \in C_Q(T)\langle c, c^t, \ldots, c^{t^{l-1}}\rangle,$$

and therefore

$$c^{t^l} \in C_Q(T)\langle c, c^t, \ldots, c^{t^{l-1}}\rangle,$$

in contradiction to $c^{t^l} \notin C_Q(T)\langle c, c^t, \ldots, c^{t^{l-1}}\rangle$. Noting that the order of every element in $Q \backslash C_Q(T)$ is at least q^2, we see that every element of order q in R is contained in $C_Q(T) \cap R$ and therefore there is a unique subgroup of order q in R. By Lemma 2.7, R is either a cyclic group or a generalized quaternion group. If R is a cyclic group, then T is normal in G by using the above proof, a contradiction. If R is a generalized quaternion group, then $|Z(R)| = 2$. Since $[c^{t^m}, c^{t^n}] \leqslant \langle c^2\rangle \leqslant Z(R)$ for any $0 \leqslant m, n \leqslant l$, we see that $R' \leqslant \langle c^2\rangle \leqslant Z(R)$ and therefore $|R'| = 2$. So R is a quaternion group of order 8. Noting that $C_Q(T)$ is a Dedekind 2-group, we may assume that $C_Q(T) = Q_8 \times L$ with Q_8 a quaternion group and L an elementary abelian 2-group. If there exists a cyclic subgroup $\langle d\rangle$ in $C_Q(T)$ of order 4 such that $\langle d\rangle \ntrianglelefteq Q$, then $dc \in Q\backslash C_Q(T)$ and so $\langle dc\rangle \trianglelefteq Q$. Since $\langle c^2\rangle = C_Q(T) \cap R \leqslant Z(Q)$ and $N_G(\langle d\rangle) = T \times C_Q(T)$, we see $c^{-1}(dc)c = cd \in \langle dc\rangle$, $cd \neq dc$ and $d^{-1}cd = c^{-1}$. Thus $(dc)^2 = d^2$, $d^{-1}c = (dc)^3 = cd$ and $c^2 = d^2$. It follows that

$$Q' = \langle [C_Q(T), C_Q(T)], [C_Q(T), R], [R, R]\rangle = \langle c^2\rangle$$

and therefore $\langle d\rangle \trianglelefteq Q$, a contradiction. Hence there is a cyclic subgroup $\langle u\rangle$ in $C_Q(T)$ of order 2 such that $\langle u\rangle \ntrianglelefteq Q$. Since $uc \in Q\backslash C_Q(T)$, we see that the order of uc is at least 4 by the above proof. On the other hand, $uc \neq cu$ implies that $u^{-1}cu = c^{-1}$ and therefore uc is an element of order 2, a contradiction. These contradictions tell us that Q must be a Dedekind group.

Case 2: G is a q-nilpotent group. By Lemma 3.4 (1)–(3), there is a prime p such that $|T : C_T(Q)|$ is a power of p and $Q \times C_T(Q)$ is maximal in G. Lemma 2.3 implies that $C_T(Q)$ is an abelian group. In order to complete the proof, we consider the following two subcases:

(a) T is a nilpotent group.

In this case, it is clear that $C_T(Q)$ contains a Hall p'-subgroup H of T. The commutativity of $C_T(Q)$ implies $H \leqslant Z(G)$. Let P be a Sylow p-subgroup of G. Then $C_T(Q) \leqslant Z(G)$ by Lemma 3.4 (5).

By Lemma 3.4 (5) again, we have that every subgroup of P is normal in T. If there exists an element $z \in P$ such that $\langle z\rangle \ntrianglelefteq G$, then, by Lemma 3.4 (6), we have $G = H \times C_P(Q) \times (K \rtimes Q)$ with K a minimal normal subgroup in G and K a p-group. Since $N_G(\langle z\rangle) = T \rtimes N_Q(\langle z\rangle)$ is maximal in G for any $z \in K$ with $z \neq 1$, we see that $N_Q(\langle z\rangle) = Q_1$ is maximal in Q and therefore $N_G(\langle z\rangle) \trianglelefteq G$. It follows that $N_G(\langle z\rangle) = N_G(\langle z^g\rangle)$ for any $g \in Q$. Hence every element of Q_1 induces a power automorphism on K. Let $b \in Q\backslash Q_1$. Then we have $Q = Q_1\langle b\rangle$. Lemma 3.4 (1) implies that $\langle ab\rangle \trianglelefteq Q$ but $\langle ab\rangle \ntrianglelefteq G$ for any $a \in Q_1$. If there is an element $x \in Q$ with $\langle x \rangle \ntrianglelefteq Q$, then, by Lemma 3.4 (1), $K \leqslant C_G(x)$ and therefore $N_G(\langle x \rangle) = N_G(\langle z \rangle)$ by Lemma 2.3. If $C_P(Q) = 1$, then G is a group of type (III).

Now we assume that $C_P(Q) \neq 1$. Let $s \in C_P(Q)$ with order p, and $\langle sz \rangle$ be a subgroup of order p for any $z \in K$. If $g \in N_Q(\langle sz \rangle)$, then there is a positive integer i such that $(sz)^i = (sz)^g = sz^g$. It follows that $s^{i-1} \in C_P(Q) \cap K = 1$ and therefore $g \in C_Q(z)$. By Lemma 2.6, we have $C_Q(z) = C_Q(K) \leqslant C_Q(T)$. Thus $N_Q(\langle sz \rangle) \leqslant C_Q(T)$. If $N_Q(\langle sz \rangle) = Q$, then $Q \leqslant C_G(T)$, a contradiction. So we may have $N_G(\langle sz \rangle) = T \rtimes N_Q(\langle sz \rangle)$ with $N_Q(\langle sz \rangle)$ maximal in Q. Thus

$$N_Q(\langle sz \rangle) = C_Q(T) = N_Q(\langle z \rangle)$$

for any $z \in K$. It is easy to find that there is an element $b \in Q \backslash C_Q(T)$ such that $Q = C_Q(T)\langle b \rangle$ with $\langle ab \rangle \trianglelefteq Q$ but $\langle ab \rangle \ntrianglelefteq G$. We have proved that

$$N_G(\langle x \rangle) = N_G(\langle z \rangle) = T \times C_Q(T)$$

for any $\langle x \rangle \ntrianglelefteq Q$ and any $z \in K$. Therefore $C_Q(T)$ is a Dedekind group. So G is a group of type (IV).

If $\langle z \rangle \trianglelefteq G$ for any $z \in P$, then every element of Q induces a p'-power automorphism on P. Applying Lemma 2.6, we see $C_P(Q) = P$ or $C_P(Q) = 1$. It is clear that the former case implies that G is nilpotent. Thus $C_T(Q)$ is a Hall p'-subgroup of T. The maximality of $Q \times C_T(Q)$ and the normality of every subgroup of P in G imply that P is a cyclic subgroup of order p. Noting that there is an element $g \in G$ such that $C_G(P) = T \times C_Q(P)^g$, we see that $C_Q(P)$ is normal in G. Since $G/C_G(P)$ is a subgroup of a cyclic group of order $p - 1$, there is an element c in Q such that $Q = C_Q(P)\langle c \rangle$. By Lemma 3.4 (1), we have $\langle y \rangle \trianglelefteq Q$ but $\langle y \rangle \ntrianglelefteq G$ for any $y \in Q \backslash C_Q(P)$. It is clear that Q is an NCM-group by Lemma 2.4. So G is a group of type (V).

(b) T is a non-nilpotent group

Since T is not nilpotent, there is a prime $r \in \pi(T)$ with $r \neq p$ such that a Sylow r-subgroup of $C_T(Q)$ is not normal in T. It follows from the maximality of $Q \times C_T(Q)$, normality of P and commutativity of $C_T(Q)$ that $C_T(Q)$ is just a system normalizer of T.

It is clear that all Sylow subgroups of T are abelian. By Lemma 2.8, we have $G = (T' \rtimes C_T(Q)) \rtimes Q$ with T' a minimal normal subgroup. If there exists an element $x \in Q$ with $\langle x \rangle \ntrianglelefteq Q$, then $N_G(\langle x \rangle) = T \rtimes N_Q(\langle x \rangle)$. By Lemma 2.3, T is an abelian group, a contradiction. So Q is a Dedekind group.

If $C_P(Q) \neq 1$, then $\langle sz \rangle$ is a subgroup of order p for any $z \in T'$ and $s \in C_P(Q)$ with order p. If $g \in N_G(\langle sz \rangle)$, then there is a positive integer i with the property that $(sz)^i = (sz)^g = sz^g$. It follows that $s^{i-1} \in C_P(Q) \cap T' = 1$ and therefore $g \in C_G(z)$. If there exists an element $u \in G$ such that $Q^u \leqslant N_G(\langle sz \rangle)$, then we get $z \in C_T(Q^u)$, a contradiction. Thus we have $T \leqslant N_G(\langle sz\rangle) \leqslant C_G(z)$ for any $z \in T'$, and therefore T is a nilpotent group, a contradiction. So $C_T(Q)$ is a Hall $\{p,q\}'$-subgroup of G.

If $|T'| = p$, then G is a group of type (VI). Now we assume that $|T'| > p$. If there exists an element $z \in T'$ such that $\langle z\rangle \ntrianglelefteq T$, then there is a prime $r \in \pi(T)$ with $r \neq p$ such that $|G : N_G(\langle z\rangle)|$ is a power of r. Thus there is an element $g \in G$ such that $Q^g \leqslant N_G(\langle z\rangle)$. Noting that $T' \leq N_G(\langle z\rangle)$ and $T = T'C_T(Q)$, we see that $N_T(\langle z\rangle)$ is normal in T and therefore $N_G(\langle z\rangle)$ is normal in G. Thus $N_T(\langle z\rangle) = T_1$ is maximal in T and there is an r-element $d \in T$ with the property that $T = T_1\langle d\rangle$. Since $N_G(\langle z^h\rangle) = N_G(\langle z\rangle)$ for any $h \in G$, we see that every element of T_1 and every element of Q induce a power automorphism on T'. So G is a group of type (VII).

If there exists an element $z \in T'$ such that $\langle z\rangle \trianglelefteq T$, then

$$\langle z^g\rangle^{C_T(Q)} = \langle z^{C_T(Q)}\rangle^g = \langle z^g\rangle$$

for any $g \in Q$. Thus every element of T induces a power automorphism on T'. Furthermore, the maximality of $N_G(\langle z\rangle)$ implies that $N_Q(\langle z\rangle)$ is maximal in Q and therefore $N_G(\langle z\rangle)$ is normal in G. Thus

$$N_Q(\langle z\rangle) = Q_1 \leqslant N_G(\langle z\rangle) = N_G(\langle z^g\rangle)$$

for any $g \in Q$ and so every element of Q_1 induces a power automorphism on T'. Let e be an element of Q with $e \in Q\backslash Q_1$. Then $Q = Q_1\langle e\rangle$. So G is a group of type (VIII).

Conversely, it is clear that G is a soluble non-nilpotent group if G is one of the types from (I) to (VIII). Let $\langle n\rangle$ be any non-normal cyclic subgroup of G. If $\langle n\rangle$ is a q-group or a q'-group and G is one of the types from (I) to (VIII), then it is easy to see that $N_G(\langle n\rangle)$ is maximal in G. So we may assume that $\langle n\rangle = \langle n_q\rangle \times \langle n_{q'}\rangle$ with $\langle n_q\rangle \leqslant Q$ and $\langle n_{q'}\rangle \leqslant T$. It is clear that at least one of groups $\langle n_q\rangle$ and $\langle n_{q'}\rangle$ is not normal in G. If one of the groups $\langle n_q\rangle$ and $\langle n_{q'}\rangle$ is normal in G, then, without loss of generality, we may assume that $\langle n_q\rangle \ntrianglelefteq G$ and $\langle n_{q'}\rangle \trianglelefteq G$. Since $\langle n_q\rangle$ is characteristic in $\langle n\rangle$, we see $N_G(\langle n\rangle) \leqslant N_G(\langle n_q\rangle)$ and therefore $N_G(\langle n\rangle) = N_G(\langle n_q\rangle)$ is maximal in G. Now assume that neither $\langle n_q\rangle$ nor $\langle n_{q'}\rangle$ is normal in G. Noting that $N_G(\langle n\rangle) = N_G(\langle n_q\rangle) \cap N_G(\langle n_{q'}\rangle)$ and $\langle n_q\rangle \leqslant C_G(n_{q'})$, we see that $N_G(\langle n_q\rangle) = N_G(\langle n_{q'}\rangle)$ is maximal in G if G is one of the types from (I) to (VIII). Thus $N_G(\langle n\rangle) = N_G(\langle n_q\rangle) = N_G(\langle n_{q'}\rangle)$ is maximal in G. The proof is now complete. □

We immediately have the following corollary from Theorem 3.5.

Corollary 3.6. *Soluble* NCM-*groups have Fitting length at most* 2.

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Bibliography

[1] A. Ballester-Bolinches and L. A. Shemetkov, On normalizers of Sylow subgroups in finite groups, *Sib. Math. J.* **40** (1999), 1–2.

[2] M. Bianchi, A. Gillio Berta Mauri and P. Hauck, On finite groups with nilpotent Sylow-normalizers, *Arch. Math.* **47** (1986), 193–196.

[3] K. Doerk and T. Hawkes, *Finite Soluble Groups*, de Gruyter Exp. Math. 4, Walter de Gruyter, Berlin, 1992.

[4] D. Gorenstein, *Finite Groups*, Chelsea Publishing Company, New York, 1980.

[5] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, 1967.

[6] S. K. Mahdavianary, A classification of 2-generator p-groups, $p \geqslant 3$, with many subgroups 2-subnormal, *Arch. Math.* **43** (1984), 97–107.

[7] A. Mann, Finite groups with maximal normalizers, *Illinois J. Math* **12** (1968), 67–75.

[8] E. A. Ormerod, Finite p-groups in which every cyclic subgroup is 2-subnormal, *Glasg. Math. J.* **44** (2002), 443–453.

[9] G. Parmeggiani, On finite p-groups of odd order with many subgroups 2-subnormal, *Comm. Algebra* **24** (1996) 2707-2719.

[10] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1982.

[11] J. Zhang, Sylow numbers of finite groups, *J. Algebra* **176** (1995), 111–123.

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Author information

Jianji Cao, Department of Mathematics, Shanghai University,
Shanghai 200444, P. R. China;
and Department of Mathematics, Shanxi Datong University,
Datong Shanxi 037009, P. R. China.

Xiuyun Guo, Department of Mathematics, Shanghai University,
Shanghai 200444, P. R. China.
E-mail: `xyguo@staff.shu.edu.cn`