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Groups which do not possess characters of nontrivial prime power degree

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Abstract. In this paper, we consider groups where the only prime power that occurs as a character degree is 1. We prove that the nonabelian composition factors of such a group also have this property or are the sporadic simple group M_{12}. We characterize these groups that have a normal nonabelian Sylow p-subgroup.

1 Introduction

Throughout this note, all groups are assumed to be finite. We write $\operatorname{Irr}(G)$ for the irreducible characters of G, and $\operatorname{cd}(G) = \{\chi(1) \mid \chi \in \operatorname{Irr}(G)\}$ are the irreducible characters of G. Isaacs and Passman studied groups where all the character degrees are primes in [6]. Groups where all the character degrees are prime powers have been studied by Manz in [11] and [12]. In those papers, they showed that the structure of those groups is highly restricted. In particular, Manz showed that if G is nonsolvable with this property, then $G = A \times S$ where A is an abelian group and S is either $\mathrm{PSL}(2,4)$ or $\mathrm{PSL}(2,8)$.

In this paper, we wish to study groups with the complementary property that the only character degree that is a prime power is 1 (which of course has to be a prime power). With this in mind, a nonabelian group G is said to be a *CDG* (that is, a composite degree group) if a has at least two distinct prime divisors for every $1 \neq a \in \operatorname{cd}(G)$. That is, a group G is a CDG if and only if 1 is the only prime power that occurs in $\operatorname{cd}(G)$. Observe that quotients of a CDG must be either abelian or CDGs. These groups have previously been studied in the paper [14].

It has been observed that the set of character degrees of solvable and nonsolvable groups tends to "look different." One reason for this is that solvable groups tend to have characters of prime power degree, while it is not that uncommon for a nonsolvable group to have no characters of prime power degree. Unlike the prime power degrees case studied by Manz, we will see that there are many non-

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solvable CDGs. It is known which simple groups are CDGs. From [10], we see that the only simple groups that are CDGs are the alternating groups A_n where $n-1$ is not a prime power and the following sporadic simple groups: M_{22}, M_{23}, J_1, J_2, J_3, J_4, Co_1, Fi_{22}, Fi_{23}, Fi_{24}', HS, McL, He, Ru, Suz, O′N, HN, Ly, Th, B, and M. (All the sporadic simple groups other than M_{11}, M_{12}, M_{24}, Co_2, and Co_3.) Note that every simple group of Lie type necessarily has a character degree that is a prime power, namely the degree of the Steinberg character. Hence, no simple group of Lie type is a CDG.

We now show that the only nonabelian composition factors of a CDG are simple groups that are CDGs or M_{12}. We note that M_{12} is a real exception. Using Gap [4], one can see that

$$\mathrm{cd}(\mathrm{Aut}(M_{12})) = \{1, 22, 32, 45, 54, 55, 66, 99, 110, 120, 144, 176\},$$

and so $\mathrm{Aut}(M_{12})$ is a CDG that has a composition factor M_{12} that is not a CDG. In particular, this shows that the character degree sets of nonsolvable groups having no composition factor that is a simple group of Lie type is different from the degree sets of nonsolvable groups where some composition factor is a simple group of Lie type.

Theorem 1.1. *Let G be a CDG. If the nonabelian simple group S is isomorphic to a composition factor of G, then S is a CDG or is isomorphic to M_{12}. That is, S is either an alternating group whose degree is one more than a number that is not a prime power, S is one of the* 21 *sporadic simple groups listed above, or $S = M_{12}$.*

Beyond this theorem, we do not see any other general restrictions that we can put on nonsolvable CDGs. We will give a couple of ways of building CDGs using wreath products, and using these constructions one can obtain nonsolvable CDGs with very complicated structures.

However, we will see that we can characterize the groups that are CDGs having a normal nonabelian Sylow p-subgroup for some prime p. In particular, we are able to prove the following theorem.

Theorem 1.2. *Suppose that G has a normal nonabelian Sylow p-subgroup P for some prime p, and let H be a Hall p-complement of G. Then G is a CDG if and only if the following conditions all hold:*

(1) $C_{P'}(H) = 1$.

(2) *H is abelian or H is a CDG.*

(3) *All nontrivial orbits of H on $\mathrm{Irr}(P/P')$ have sizes that are not prime powers.*

2 Normal nonabelian Sylow subgroups

In this section, we prove Theorem 1.2. If N is normal in G, then we write $\mathrm{Irr}(G \mid N)$ for the set of irreducible characters of G whose kernels do not contain N. It is easy to see that $\mathrm{Irr}(G)$ is partitioned by $\mathrm{Irr}(G/N)$ and $\mathrm{Irr}(G \mid N)$.

Proof of Theorem 1.2. First, suppose G is a CDG. By [7, Lemma 3.2], we have $\mathbf{C}_{P'}(H) = 1$. Next, we observe that $H \cong G/P$. Since quotients of CDGs are either CDGs or abelian, we conclude that H is either abelian or a CDG. Consider a character $\theta \in \mathrm{Irr}(G/P')$, and suppose λ is an irreducible constituent of θ_P. Let T be the stabilizer of λ in G, and let $\hat{\theta} \in \mathrm{Irr}(T \mid \lambda)$ be the Clifford correspondent for θ (the Clifford correspondence is [5, Theorem 6.11]). Since $(|T : P|, |P : P'|) = 1$, we see that λ is extendible to T ([5, Corollary 8.16]). By Gallagher's theorem ([5, Corollary 6.17]), every character in $\mathrm{Irr}(T \mid \lambda)$ is an extension of λ. This implies that

$$\theta(1) = \hat{\theta}^G(1) = |G : T|\hat{\theta}(1) = |G : T|\lambda(1) = |G : T|.$$

Since G is a CDG, we know that $\theta(1)$ is not a prime power. It follows that $|G : T|$ is not a prime power. Since the various $|G : T|$ are the sizes of the orbits of H on $\mathrm{Irr}(P/P')$, we obtain the conclusion.

Now, assume that G satisfies (1), (2), and (3). Since G has a nonabelian Sylow p-subgroup, G is not abelian. Consider a nonlinear character $\chi \in \mathrm{Irr}(G/P')$. Let λ be an irreducible constituent of χ_P. Then the kernel of λ contains P', so λ is a linear character. Let T be the stabilizer of λ in G. If $T = G$, then since $|G : P|$ and $|P|$ are coprime, λ extends to $\alpha \in \mathrm{Irr}(G)$, and by Gallagher's theorem, $\chi = \alpha\theta$ where θ is an irreducible character of H. Since λ is linear and α is an extension of λ, we have $\alpha(1) = 1$. This implies that $\theta(1) = \chi(1)$ is nonlinear. This forces H to be nonabelian, and so H is an CDG. Hence, $\theta(1)$ has at least two prime divisors by (2). On the other hand, if $T < G$, then $|G : T|$ is the size of an orbit of H on $\mathrm{Irr}(P/P')$, and so $|G : T|$ is not a prime power by (3). Since $|G : T|$ divides $\chi(1)$, we see that $\chi(1)$ is not a prime power. It follows that any nonlinear character in $\mathrm{Irr}(G/P')$ has degree that is not a prime power. On the other hand, any character in $\mathrm{Irr}(G \mid P')$ will have an irreducible constituent on P which does not have P' in its kernel. Thus, every character in $\mathrm{Irr}(G \mid P')$ has degree divisible by p. From (1) and [7, Lemma 3.2], we see that $\mathrm{Irr}(G)$ has no character whose degree is a power of p. Thus, no character in $\mathrm{Irr}(G \mid P')$ is a prime power. Since these two sets partition $\mathrm{Irr}(G)$, we conclude that G is a CDG. □

We now show that condition (3) in Theorem 1.2 can be replaced by another condition when G is solvable.

Corollary 2.1. *Suppose that the solvable group G has a normal nonabelian Sylow p-subgroup P for some prime p, and let H be a Hall p-complement of G. For every prime divisor q of $|H|$, let A_q be a Hall q-complement of H. Then the following statements are equivalent:*

(1) *All nontrivial orbits of H on $\mathrm{Irr}(P/P')$ have sizes that are not prime powers.*

(2) $C_{\mathrm{Irr}(P/P')}(A_q) = C_{\mathrm{Irr}(P/P')}(H)$ *for every prime q that divides $|H|$.*

(3) $[P, H]P' = [P, A_q]P'$ *for every prime q that divides $|H|$.*

Proof. Let $A = A_q$ for some prime q that divides $|H|$, and assume that there exists a character $\lambda \in C_{\mathrm{Irr}(P/P')}(A)$, but λ is not in $C_{\mathrm{Irr}(P/P')}(H)$. Let T be the stabilizer of λ in H. Since λ is stabilized by A and not by H, we have $A \le T < H$. This implies that $|H : T|$ is a nontrivial power of q. Thus, not all nontrivial orbits of H on $\mathrm{Irr}(P/P')$ have sizes that are not prime powers. This implies that (1) implies (2).

Conversely, assume that there is a character $\lambda \in \mathrm{Irr}(P/P')$ that lies in a nontrivial H-orbit that is a power of a prime q. Thus, if T is the stabilizer of λ in H, then $|H : T|$ is a nontrivial power of q. Since H is solvable, it T will contain a Hall q-complement A of H. It follows that $\lambda \in C_{\mathrm{Irr}(P/P')}(A)$, but $\lambda \notin C_{\mathrm{Irr}(P/P')(H)}$. This implies that $C_{\mathrm{Irr}(P/P')}(A) \ne C_{\mathrm{Irr}(P/P')}(H)$. Thus, we have (2) implies (1).

Finally, notice that $\lambda \in C_{\mathrm{Irr}(P/P')}(A_q)$ if and only if $[P, A_q]P' \le \ker\lambda$ and $\nu \in C_{\mathrm{Irr}(P/P')}(H)$ if and only if $[P, H]P' \le \ker\nu$. It follows that the identity $C_{\mathrm{Irr}(P/P')}(A_q) = C_{\mathrm{Irr}(P/P')}(H)$ holds if and only if $[P, A_q]P' = [P, H]P'$. This implies that (2) and (3) are equivalent. □

3 Nonsolvable CDGs

In this section, we gather several results regarding nonsolvable CDGs. We first prove Theorem 1.1. The key to proving this theorem is [1, Lemma 5].

Proof of Theorem 1.1. By the Jordan–Hölder Theorem, we can find normal subgroups $M < N$ of G so that N/M is a chief factor for G and $N/M = S \times \cdots \times S$. We now will refer to [1, Lemma 5]. From that lemma, if there exists a character $\sigma \in \mathrm{Irr}(S)$ that extends to $\mathrm{Aut}(S)$ and if $\theta = \sigma \times \cdots \times \sigma$ is an element of $\mathrm{Irr}(N/M) = \mathrm{Irr}(S \times \cdots \times S)$, then θ extends to G. Hence, if we can find such a character σ which is a prime power that extends to $\mathrm{Aut}(S)$, then $\theta(1) \in \mathrm{cd}(G)$ will be a prime power. If S is a simple group of Lie type, then we may use the Steinberg character for σ by Schmid's theorem (see [16] and [17]). Suppose now that S is an alternating group of degree $n \ge 9$ so that $n - 1$ is a prime power. (Note that if $n \le 4$, then A_n is solvable; if $n = 5, 6, 8$, then A_n is isomorphic to a group of Lie type and has already been handled; and if $n = 7$, then $n - 1 = 6$ is not a prime power.) Let χ be the irreducible character of S having degree $n - 1$. It is well known that $1 + \chi$ is the permutation character for the standard action of S. Notice that this character extends to the permutation character of S_n in its standard action. It follows that χ will extend to $\mathrm{Aut}(S) = S_n$, and so we can take σ to be χ in this case. Finally, we consider the sporadic simple groups. If S is M_{11}, we take σ so that $\sigma(1) = 11$ and if S is M_{24}, Co_2, or Co_3, we take σ so that $\sigma(1) = 23$ (these all come from [3] or look at [1, Table 1]). The only remaining simple group that is not a CDG is M_{12}, and so the theorem is proved. □

We now look at a semi-direct product involving CDGs. Recall that a group G is perfect if $G' = G$.

Lemma 3.1. *Let N be a normal subgroup of a group G. If G/N is an abelian group or a CDG and N is a perfect CDG, then G is a CDG.*

Proof. Since N is perfect and a CDG, we have $1 < N = N' \leq G'$, and so G is not abelian. Let $\chi \in \mathrm{Irr}(G)$. If N is in the kernel of χ, then $\chi \in \mathrm{Irr}(G/N)$ and since G/N is either abelian or a CDG, we see that $\chi(1)$ is either 1 or not a prime power. If N is not in the kernel of χ, then χ_N has an irreducible constituent $\theta \neq 1_N$. Since N is perfect, we know that $\theta(1) \neq 1$, and since N is a CDG, we know that $\theta(1)$ is not a prime power. We see then that $\chi(1)$ is neither 1 nor a prime power, and we conclude that G is a CDG. □

We now consider wreath products involving CDGs. We begin with a wreath product where H is perfect.

Lemma 3.2. *Let S be a CDG, and let H be a perfect CDG. If $G = H \operatorname{wr} S$ is a wreath product of S on H, then G is a CDG.*

Proof. Let n be the degree of the permutation representation of S used to construct G, and let $N = H \times \cdots \times H$ (n copies of H) so that G is the semi-direct product of S acting on N. Observe that N will be a perfect CDG and $G/N \cong S$ is a CDG. Thus, we may apply Lemma 3.1 to see that G is a CDG. □

Applying Lemma 3.2 to iterated wreath products involving the simple groups that are CDGs yields CDGs with relatively complicated structure.

In the next lemma, we look at a different wreath product involving CDGs. Note that we are able to remove the hypothesis that H is perfect. In fact, H abelian or solvable can arise. However, to do this, we need to add an additional hypothesis on the permutation representation of S used to obtain the wreath product. We note that the condition on S in this next lemma forces S to be nonsolvable.

(This follows from the fact that a primitive solvable permutation group must have prime power degree, and an imprimitive solvable permutation group has a quotient that is primitive and we can take our subset to be one of the blocks from the quotient.) There are nonsolvable examples of groups S that satisfy the hypotheses of being a CDG and having that 1 is the only prime power that is the length of an orbit on the power set. One such example is A_{15}. To see this, if the stabilizer of some set is proper, then it must be contained in some maximal subgroup, and the index of every maximal subgroup is not a prime power. This was checked in Magma [2]. (In fact, we believe A_n will work whenever n and $n-1$ are not prime powers, but we have not checked this.)

Lemma 3.3. *Let S be a permutation group on Ω that is a CDG and assume that the only prime power that is the length of an orbit of S on the power set of Ω is* 1. *Suppose that H is either abelian or a CDG. If $G = H \text{ wr } S$ is a wreath product (using the action of S on Ω), then G is a CDG.*

Proof. Since S is a CDG, we see S is not abelian, and this implies G is not abelian. Let $n = |\Omega|$, and let $N = H_1 \times \cdots \times H_n$ where each $H_i \cong H$ so that G is the semi-direct product of S acting on N. Observe that N will be a CDG and $G/N \cong S$ is a CDG. Let $\chi \in \mathrm{Irr}(G)$. If N is in the kernel of χ, we know that $\chi \in \mathrm{Irr}(G/N) = \mathrm{Irr}(S)$, and $\chi(1)$ is either 1 or not a prime power. If χ_N has a non-linear irreducible constituent, then the degree of that constituent will divide $\chi(1)$, and thus, $\chi(1)$ is not a prime power. Hence, we may assume every irreducible constituent of χ_N is linear.

Let θ be an irreducible constituent of χ_N. Let T be the stabilizer of θ in G, and observe that $N \leq T$, so $T = N(T \cap S)$. By [5, Problem 6.18], we know that θ extends to $\hat{\theta} \in \mathrm{Irr}(T)$. Thus, it suffices to show that either $G = T$ or $|G : T|$ is not a prime power. This follows since if $G = T$, then Gallagher's theorem will apply to show that $\chi = \psi\hat{\theta}$ for some character $\psi \in \mathrm{Irr}(G/N) = \mathrm{Irr}(S)$. We then have

$$\chi(1) = \psi(1)\hat{\theta}(1) = \psi(1)\theta(1) = \psi(1) \cdot 1 = \psi(1)$$

and since S is a CDG, we know that $\psi(1)$ is either 1 or not a prime power. On the other hand, $|G : T|$ divides $\chi(1)$, so if $|G : T|$ is not a prime power, then neither is $\chi(1)$.

We can write $\theta = \lambda_1 \times \cdots \times \lambda_n$ where $\lambda_i \in \mathrm{Irr}(H_i)$. For each $\lambda \in \mathrm{Irr}(H)$, we define $\Omega_\lambda = \{i \in \Omega \mid \lambda_i = \lambda\}$. Notice that Ω is partitioned by the sets Ω_λ. It is not difficult to see that $s \in S$ is in T if and only if s stabilizes all the sets Ω_i. Hence,

$$S \cap T = \bigcap_{\lambda \in \mathrm{Irr}(H)} C_S(\Omega_\lambda).$$

It follows that $|S : C_S(\Omega_\lambda)|$ divides $|S : S \cap T| = |G : T|$. If $|S : C_S(\Omega_\lambda)| > 1$ for some $\lambda \in \mathrm{Irr}(H)$, then the hypothesis on the action of S on the power set of Ω implies that $|S : C_G(\Omega_\lambda)|$ is not a prime power, and so $|G : T|$ is a prime power. If $S = C_S(\Omega_\lambda)$ for all $\lambda \in \mathrm{Irr}(H)$, then it follows that $G = T$, and so we have $|G : T| = 1$. This proves the lemma. □

4 Examples

In this section, we enumerate several examples of groups that are CDGs. The easiest examples arise when G is a Frobenius group whose Frobenius complement C is cyclic with order not a prime power. By [5, Theorem 6.34], we see that $|C|$ will divide the degree of every nonlinear character in $\mathrm{Irr}(G)$, and so G will be a CDG. In this first example, we consider a particular family of solvable Frobenius groups that are CDGs, and we will see that we can find members of this family with arbitrarily large derived length.

Example 4.1. Let p be a prime that is not a Mersenne prime and $n > 6$ be an integer. Assume that e is a prime number so that $e > \max(p^2, n)$. Thus e is relatively prime to both $p^2 - 1$ and $n!$. These are the conditions required so that $P_n = P_n(p^2, e)$, the group of order $(p^2)^{en}$ and nilpotence class n, is defined in [15], and let

$$c = \frac{(p^2)^e - 1}{p^2 - 1}.$$

In [15, Corollary 2.12], it is shown that the group P_n has derived length equal to $\lceil \log_2(n+1) \rceil$. Take C to be the cyclic group of order c. By [15, Theorem 2.5 (ii)], we know that there is an action by C on P_n via automorphisms so that P_nC is a Frobenius group. From [15, p. 207 at the beginning of Section 5], we see that

$$\mathrm{cd}(P_nC) = \{1\} \cup \{c((p^2)^{(e-1)/2})^i \mid i = 0, 1, \ldots, n\}.$$

Notice that p and n have been chosen so that $p^n - 1$ and $(p^2)^n - 1$ are not exceptions to the Zsigmondy prime theorem. Hence, we can take q to be a Zsigmondy prime divisor of $p^n - 1$ and r to be a Zsigmondy prime divisor of $(p^2)^n - 1$. It follows that q and r are distinct primes that do not divide $p^2 - 1$, but do divide $(p^2)^n - 1 = (p^n - 1)(p^n + 1)$. We deduce that qr divides c, and so c is not a prime power. We conclude that P_nC is a CDG. Since we can choose n and e to be arbitrarily large, we see that there is no bound on the nilpotence class of P if P is a nonabelian normal Sylow p-subgroup of a CDG. Also, there is no bound on the derived length of P which implies that there is no bound on the derived length of G when G is a solvable group that is a CDG.

We now consider solvable CDGs that are not Frobenius groups.

Example 4.2. In [14, Example 3.4], it was shown that if P is the Suzuki 2-group of order 2^{12}, then P has an automorphism C of order 21 so that PC is a CDG and PC is not a Frobenius group.

Example 4.3. Another example can be found in Section 6 of [8]. In that example, G is a solvable group with distinct primes p and q, and G has a normal nonabelian Sylow p-subgroup P of index $3q$. It was shown in [8] that

$$\mathrm{cd}(G) = \{1, 3q, p^2q, p^3 3\},$$

so G is a CDG.

In all of the examples we have seen so far of CDGs, the quotient modulo the Fitting subgroup is abelian. We next present an example of a solvable group whose quotient modulo its Fitting subgroup is not abelian.

Example 4.4. We start by taking F to be the field of order 2^{30}. Let N be isomorphic to the additive group of F, so N is an elementary abelian group of order 2^{30}. Notice that $49981 = 151 \cdot 331$ is a divisor of $2^{30} - 1$. Take C to be a cyclic group of order 49981 and we view C as a subgroup of the multiplicative group of F. Using the multiplication from F, we get an action of C on N by automorphisms. Notice that $M = NC$ will be a Frobenius group of order $2^{30}49981$ and that $\mathrm{cd}(M) = \{1, 49981\}$. Take S to be a cyclic group of order 15. We identify S with the subgroup of order 15 in the Galois group of F over the field of order 2. The action of S on F naturally yields an action of S by automorphisms on N and on the multiplicative group of F. Since C is characteristic in the multiplicative group of F, this yields an action of S on C, and so we obtain an action of S on M. Take G to be the semi-direct product of S acting on M. Note that 49981 will divide every degree in $\mathrm{cd}(G|N)$, so no degree in $\mathrm{cd}(G|N)$ will be a prime power.

We now consider $\mathrm{cd}(G/N)$ and notice that G/N is isomorphic to the semi-direct product of S acting on C. By Itô's theorem, every degree in $\mathrm{cd}(G/N)$ divides $|G : M| = |S| = 15$. Let A be the subgroup of order 3 in C and let B be the subgroup of order 5. We know that the fixed subfield for A in F has order 2^{10}. This implies that the centralizer for A in the multiplicative group of F has order $2^{10} - 1$. Since $2^{10} - 1$ and 49981 are relatively prime, we conclude that A centralizes no elements in C. Similarly, the fixed subfield for B in F has order 2^6. We see that the centralizer for B in the multiplicative group of F has order $2^6 - 1$, and as $2^6 - 1$ and 49981 are relatively prime, we see that B centralizes no element in C. We conclude that S acts Frobeniusly on C, and so we obtain that $\mathrm{cd}(G/N) = \mathrm{cd}(CS) = \{1, 15\}$. We conclude that 1 is the only prime power in $\mathrm{cd}(G)$, and so G is a CDG. It is not difficult to see that $G' = M$ and $G'' = N = F(G)$, so G has derived length 3 and $G/F(G)$ has derived length 2.

We note that this example does not depend on $p = 2$ and the exponent being 30. This example will be generated for any prime p, and take F to be the field of order p^{2qr} where q and r are distinct primes that are distinct from p. Then one can find s to be a Zsigmondy prime for $p^{2qr} - 1$ and t a Zsigmondy prime for $p^{qr} - 1$. Take C to be the cyclic group of order st and S the cyclic group of order qr.

At this time, we do not know how to construct any solvable CDGs that have Fitting height larger than 3. Also, we would be interested in seeing an example of a solvable CDG with $G/F(G)$ having derived length at least 3 or proving that such a group cannot exist. It is worth noting that the problem of finding a solvable CDG group G such that $G/F(G)$ has derived length at least 3 is closely related to finding an action of a solvable group of derived length bigger than 2 on a finite module without orbits of prime power size. The easy way to build these actions without orbits of prime power size is to consider Frobenius actions (or more generally half transitive actions), but in these cases, it is difficult to find situations where the group acting has at derived length more than 2. (Obviously, we would be assuming that the group acting is solvable.)

Notice that we may apply Lemma 3.3 with A_{15} acting on any of our examples of a CDG with a normal nonabelian Sylow p-subgroup with a large enough prime p to obtain a nonsolvable CDG with a normal nonabelian Sylow p-subgroup.

5 Critical CDGs

Finally, we want to look more closely at the groups studied in [14]. Following the definition there, a group G is called a *critical CDG* if G is a CDG but any proper subgroup of G is not a CDG. In [14], they considered critical CDGs. In particular, they showed in [14, Theorem 3.13] that if G is a critical CDG with a normal (non-abelian) Sylow p-subgroup P for some prime p, then either $\Delta(G)$ is a complete graph or $G' = P$ has nilpotence class 2 where $\Delta(G)$ is the prime-vertex character degree graph of G. That is, $\Delta(G)$ is the graph whose vertex set is $\rho(G)$ which is the set of primes dividing degrees in $\mathrm{cd}(G)$ and there is an edge between p and q if pq divides a for some $a \in \mathrm{cd}(G)$. We now show that if $\Delta(G)$ is not a complete graph, its structure is still highly limited in the case where G is a CDG with a non-abelian normal Sylow p-subgroup. If n is an integer, we use $\pi(n)$ to denote the set of primes that divide n.

Theorem 5.1. *Suppose G is a critical CDG with a normal nonabelian p-Sylow group P. Then the diameter of $\Delta(G)$ is at most* 2.

Proof. Let H be a p-complement of P in G. Then H is abelian by Theorem 1.2. Because $F(G) = P \times C_H(P)$ and the group $G/F(G)$ is abelian, we can see that $|G : F(G)| \in \mathrm{cd}(G)$ by [13, Lemma 18.1]. Then we have

$$\rho(G) = \pi(|G : C_H(P)|) = \{p\} \cup \pi(|G : F|)$$

by Itô's theorem. Since P is not abelian, there exists a prime $q \in \pi(|G : F(G)|)$ which is connected with p. Therefore, the diameter of $\Delta(G)$ is at most 2. □

We now present an example of a solvable group G that is a critical CDG with a normal nonabelian Sylow p-subgroup where $\Delta(G)$ has diameter 2.

Example 5.2. Let P be a Sylow 5-subgroup of $\mathrm{GL}(3, 25)$ so that P is a Heisenberg group which implies that $Z(P) = P' = \Phi(P)$. Using Magma, we can see that P has automorphisms σ and τ of order 3 and 13, respectively, so that σ and τ commute, σ acts Frobeniusly on P, and τ centralizes $Z(P)$ and acts Frobeniusly on $P/Z(P)$. We take G to be the semi-direct product of $\langle \sigma, \tau \rangle$ acting on the subgroup P. Using Magma, we compute $\mathrm{cd}(G) = \{1, 39, 75\}$ so G is a CDG. Also, notice that 5 and 13 have distance 2 in $\Delta(G)$, so $\Delta(G)$ has diameter 2. Finally, we use Magma to compute the subgroups of G and their character degrees, and we see that no proper subgroup of G is a CDG. Thus, G is a critical CDG.

We note that this previous example can be generalized to any prime p and integer $a \geq 1$ where $p^{2a} - 1$ has a Zsigmondy prime divisor. We note that to find the correct automorphisms, one will need to make use of the automorphism group of the Heisenberg group of order p^{3a} which is described in [9, Theorem 4.1].

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