J. Group Theory **18** (2015), 655–684
DOI 10.1515/jgth-2014-0050

On the embedding problem for generalized Baumslag–Solitar groups

Fedor A. Dudkin

Communicated by Alexander Olshanskii

Abstract. A finitely generated group G acting on a tree so that all vertex and edge stabilizers are infinite cyclic groups is called a generalized Baumslag–Solitar group (GBS group). Such an action is described by a labeled graph. One GBS group can be presented by many labeled graphs. We study the embedding problem for GBS groups: to determine algorithmically when two given labeled graphs $\mathbb{A}_1$ and $\mathbb{A}_2$ define GBS groups G_1 and G_2 such that G_1 is embeddable into G_2. We prove that if G_1 can be presented by only a finite number of reduced labeled graphs, then the embedding problem is solvable. Moreover, we describe the algorithm in this situation.

1 Introduction

Call a finitely generated group G a *generalized Baumslag–Solitar group* or a GBS group if G can act on a tree so that the stabilizers of vertices and edges are infinite cyclic groups. By the Bass–Serre Theorem, G is representable as $\pi_1(\mathbb{A})$, the fundamental group of a graph of groups $\mathbb{A}$ (see [13]).

Given a GBS group G, we can present the corresponding graph of groups $\mathbb{A}$ by a labeled graph (A, λ), where A is a finite connected graph and $\lambda\colon E(A) \to \mathbb{Z} \setminus \{0\}$ labels the edges of A. The label λ_e of an edge e with the source vertex v defines an embedding $\alpha_e\colon e \to v^{\lambda_e}$ of the cyclic edge group $\langle e\rangle$ into the cyclic vertex group $\langle v\rangle$ (for more details about labeled graphs see Section 2). Using the notion of expansion for labeled graphs, we can easily see that every GBS group can be presented by infinitely many labeled graphs.

Note that GBS groups have been the subject of much recent study [2, 4, 9]. In particular, the isomorphism problem for GBS groups has been discussed: to determine algorithmically when two given labeled graphs define isomorphic GBS groups. Despite that, the isomorphism problem has been solved only in several special cases [3, 10, 11], and the general solution has not been established. We study the embedding problem for GBS groups: to determine algorithmically when

The work was partially supported by RFBR (Grant No. 14-01-90013-Bel_a).

two given labeled graphs $\mathbb{A}_1$ and $\mathbb{A}_2$ define GBS groups G_1 and G_2 such that G_1 is embeddable into G_2.

Let graph A be a loop (a graph with one vertex and one edge) and let p, q be nonzero integers – labels of A. Then the corresponding GBS group is isomorphic to the Baumslag–Solitar group

$$\mathrm{BS}(p,q) = \langle a, t \mid t^{-1}a^p t = a^q \rangle.$$

The embedding problem for Baumslag–Solitar groups has been investigated from two directions. On the one hand, for prime numbers p and q all labeled graphs whose fundamental groups embed into $\mathrm{BS}(p,q)$ were described in [5]. Also, for coprime p and q (with $p \neq \pm 1$ and $q \neq \pm 1$), the labeled graphs corresponding to the finite index subgroups of $\mathrm{BS}(p,q)$ were described in [6]. On the other hand, for coprime integers p and q (denoted by $p \perp q$) other than 0, 1 and -1, it is proved in [7, 12] that $\mathrm{BS}(p,q)$ embeds into a GBS group G if and only of the equation $x^{-1}y^p x = y^q$ with $y \neq 1$ is solvable in G (i.e. $\frac{p}{q} \in \Delta(G)$, where Δ is the modular homomorphism).

The aim of this article is to begin studying the embedding problem for GBS groups. Our main tool is the covering theory for graphs of groups discovered by H. Bass [1]. More precisely, we use a statement that easily follows from [1, Proposition 2.7, Corollary 4.6] (for the definition of immersion see Section 2):

Remark 2.15. Given labeled graphs $\mathbb{A}$ and $\mathbb{B}$, we can embed $\pi_1(\mathbb{A})$ into $\pi_1(\mathbb{B})$ if and only if there exists a labeled graph $\mathbb{C}$ with $\pi_1(\mathbb{C}) \cong \pi_1(\mathbb{A})$ immersible into $\mathbb{B}$.

If two labeled graphs $\mathbb{A}$ and $\mathbb{B}$ define isomorphic GBS groups $\pi_1(\mathbb{A}) \cong \pi_1(\mathbb{B})$ and $\pi_1(\mathbb{A})$ is not isomorphic to $\mathbb{Z}, \mathbb{Z}^2$ or the Klein bottle group, then there exists a finite sequence of *expansion* and *collapse* moves (see Figure 1) connecting $\mathbb{A}$ and $\mathbb{B}$ (see [8]). A labeled graph is said to be *reduced* if it admits no collapse move (equivalently, the labeled graph contains no edges with distinct endpoints and labels ± 1).

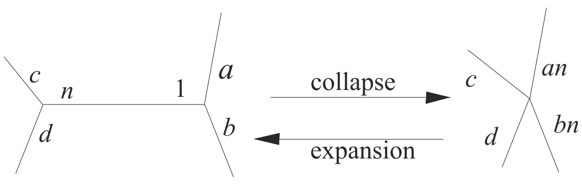

Figure 1. Expansion and collapse moves.

Given a labeled graph $\mathbb{A}$ (a GBS group G), we shall denote the set of reduced labeled graphs with fundamental group isomorphic to $\pi_1(\mathbb{A})$ (resp. G) by $R(\mathbb{A})$ (resp. $R(G)$).

Definition 3.5. Consider a labeled graph $\mathbb{A} = (A, \lambda)$. A vertex a of A is called *special* if and only if $|\mathrm{St}_A\, a| > 2$ or there exists an $e \in \mathrm{St}_A\, a$ such that $\lambda_e \neq 1$. A vertex v is *non-special* if and only if $|\mathrm{St}_A\, v| = 2$ and $\lambda_e = 1$ for all $e \in \mathrm{St}_A\, v$.

Definition 2.16. An expansion is *unnatural* if $n = 1$ and either $\mathrm{St}_A\, \partial_0(e) = \{e\}$ or $\mathrm{St}_A\, \partial_1(e) = \{\bar{e}\}$, where e is a new edge.

If an expansion is not unnatural, then we call it a *natural* expansion.

A labeled graph $\mathbb{C}$ is called an *extension of* $\mathbb{A}$ if there exists a sequence of natural expansions from $\mathbb{A}$ to $\mathbb{C}$.

Definition 3.7. Consider a reduced labeled graph $\mathbb{A}$. Denote by $\Omega_N(\mathbb{A})$ the set of labeled graphs such that every $\mathbb{B} \in \Omega_N(\mathbb{A})$ is an extension of $\mathbb{A}$ and every reduced (i.e. without $e, \bar{e}$ inclusions) path $e_1, e_2, \dots, e_{N+1}$ without self-intersections in B contains at least one special vertex.

Now we can state our main result.

Theorem 1.1. *Consider labeled graphs $\mathbb{A}$ and $\mathbb{B}$ such that there exists an immersion $\mathbb{A} \to \mathbb{B}$. Suppose that $R(\mathbb{A}) = \{\mathbb{A}_1, \dots, \mathbb{A}_R\}$. Then there exists a computable natural number $N = N(R(\mathbb{A}), \mathbb{B})$ such that some $\mathbb{C} \in \bigcup_{i=1}^{R} \Omega_N(\mathbb{A}_i)$ can be immersed into $\mathbb{B}$.*

Using Theorem 1.1 we can easily prove

Theorem 1.2. *Consider labeled graphs $\mathbb{A}$ and $\mathbb{B}$. If $|R(\mathbb{A})| < \infty$, then there is an algorithm to decide whether $\pi_1(\mathbb{A})$ embeds into $\pi_1(\mathbb{B})$.*

If $\pi_1(\mathbb{A}) \ncong \mathrm{BS}(1, n)$, then we can test the property $|R(\mathbb{A})| < \infty$ algorithmically [3]. The embedding problem for $\pi_1(\mathbb{A}) \cong \mathrm{BS}(1, n)$ is described in [7, 12].

This paper is organized as follows. In Section 2 we discuss the results of Bass [1] in relation to labeled graphs. Section 3 contains some technical statements. In Section 4 we prove the main theorem. Implementation of the algorithm is described in Section 5.

2 Preliminaries

We begin with some definitions from the paper of H. Bass [1].

Definition 2.1. A *graph* A is a set $V(A)$ (vertices), a set $E(A)$ (edges), maps $\partial_0, \partial_1 \colon E(A) \to V(A)$ (endpoints) and a fixed point free involution $e \to \bar{e}$ of $E(A)$ (reversal of orientation) such that $\partial_i \bar{e} = \partial_{1-i} e$. Denote by A_0 the undirected graph corresponding to graph A (we can get A_0 from A by identifying e and $\bar{e}$ for all $e \in E(A)$). We define

$$\mathrm{St}(a) \text{ (or } \mathrm{St}_A(a)) = \partial_0^{-1}(a) = \{e \in E(A) : \partial_0 e = a\},$$

the *star* of a. An *edge path of length n* is either a vertex a_0 (if $n = 0$) or, if $n > 0$, a sequence $\gamma = (e_1, \dots, e_n)$ of edges with $a_i = \partial_1 e_i = \partial_0 e_{i+1}$ $(1 \leqslant i \leqslant n)$.

Definition 2.2. A *morphism* $\phi\colon A \to B$ carries vertices to vertices, edges to edges, and, for $e \in E(A)$, $\phi(\partial_i e) = \partial_i \phi(e)$ $(i = 0, 1)$ and $\phi(\bar{e}) = \overline{\phi(e)}$. For $a \in V(A)$ we then have the *local map*

$$\phi_{(a)}\colon \mathrm{St}_A(a) \to \mathrm{St}_B(\phi(a)).$$

Definition 2.3. A *Graph of groups* is a pair $\mathbb{A} = (A, \mathscr{A})$, where A is a connected graph, and $\mathscr{A}$ assigns groups $\mathscr{A}_a (a \in V(A))$, $\mathscr{A}_e = \mathscr{A}_{\bar{e}} (e \in E(A))$, and monomorphisms $\alpha_e\colon \mathscr{A}_e \to \mathscr{A}_a$, where $a = \partial_0 e$. We further write

$$\mathscr{A}_{a/e} = \mathscr{A}_a / \alpha_e \mathscr{A}_e \quad \text{when } \partial_0 e = a.$$

Definition 2.4. The *path group* $\pi(\mathbb{A})$ is defined by

$$\pi(\mathbb{A}) = \left[\left(\mathop{\Large *}_{a \in V(A)} \mathscr{A}_a \right) * F(E(A)) \right] / R,$$

where $F(E(A))$ denotes the free group with basis $E(A)$, and R is the normal subgroup which imposes the relations $\bar{e} = e^{-1}$ and $e\alpha_{\bar{e}}(s)e^{-1} = \alpha_e(s)$ for all $e \in E(A)$ and $s \in \mathscr{A}_e (= \mathscr{A}_{\bar{e}})$.

Definition 2.5. By a *path of length* $n \geqslant 0$ *in* $\mathbb{A}$ we mean a sequence

$$\gamma = (g_0, e_1, g_1, e_2, \dots, g_{n-1}, e_n, g_n),$$

where $(e_1, e_2, \dots, e_n)$ is an edge path of length n in A, say with vertex sequence $a_0, a_1, \dots, a_n$, and where $g_i \in \mathscr{A}_{a_i}$ $(i = 0, \dots, n)$. We say that γ is a *path from* a_0 *to* a_n, and we put

$$|\gamma| = g_0 e_1 g_1, \dots, g_{n-1} e_n g_n \in \pi(\mathbb{A}).$$

For $a, b \in A$ we write $\pi[a, b]$ for the set of $|\gamma|$ where γ is a path from a to b.

Definition 2.6. Choose a base point $a_0 \in V(A)$. The *fundamental group of* $\mathbb{A}$ *at* a_0 is $\pi_1(\mathbb{A}, a_0) = \pi[a_0, a_0]$. For different base points a_0 and a_1 the group $\pi_1(\mathbb{A}, a_0)$ is isomorphic to the group $\pi_1(\mathbb{A}, a_1)$(we will denote it by $\pi_1(\mathbb{A})$).

Definition 2.7. Let $\mathbb{A} = (A, \mathscr{A})$ and $\mathbb{A}' = (A', \mathscr{A}')$ be two graphs of groups. By a *morphism* $\Phi = (\phi, (\gamma)) \colon \mathbb{A} \to \mathbb{A}'$ we understand:

(1) a graph morphism $\phi \colon A \to A'$,

(2) group homomorphisms

$$\phi_a \colon \mathscr{A}_a \to \mathscr{A}'_{\phi(a)} \quad (a \in V(A)),$$
$$\phi_e = \phi_{\bar{e}} \colon \mathscr{A}_e \to \mathscr{A}'_{\phi(e)} \quad (e \in E(A)),$$

(3) families of elements $(\gamma_a)_{a \in V(A)}$ and $(\gamma_e)_{e \in E(A)}$ in $\pi(\mathbb{A}')$. The above data are assumed to satisfy the following conditions:

   (a) For $a \in V(A)$, $\gamma_a \in \pi_1(\mathbb{A}', \phi(a))$.

   (b) For $e \in E(A)$, put $a = \partial_0 e$ and $\delta_e = \gamma_a^{-1}\gamma_e$. Then one has $\delta_e \in \mathscr{A}'_{\phi(a)}$ and the following relation holds in $\pi(\mathbb{A}')$:

$$\phi_a(\alpha_e(s)) = \delta_e \alpha'_{\phi(e)}(\phi_e(s))\delta_e^{-1},$$

   for all $s \in \mathscr{A}_e$.

Definition 2.8. Let $\mathbb{A} = (A, \mathscr{A})$ and $\mathbb{A}' = (A', \mathscr{A}')$ be graphs of groups, $a = \partial_0 e$, $s \in \mathscr{A}_a$ and $\Phi \colon \mathbb{A} \to \mathbb{A}'$ be a morphism of graphs of groups. Then s defines a coset $[s]_e \in \mathscr{A}_a/\alpha_e\mathscr{A}_e$. If $\phi(e) = f$, we can define

$$\Phi_{a/f} \colon \mathscr{A}_a/\alpha_e\mathscr{A}_e \to \mathscr{A}'_{\phi(a)}/\alpha'_f\mathscr{A}'_f$$

by the rule $\Phi_{a/f}([s]_e) = [\phi_a(s)\delta_e]_f$.

Definition 2.9. Let $\mathbb{A} = (A, \mathscr{A})$ and $\mathbb{A}' = (A', \mathscr{A}')$ be two graphs of groups. The morphism $\Phi \colon \mathbb{A} \to \mathbb{A}'$ of graph of groups is called an *immersion* if the following conditions hold:

(1) each $\phi_a \colon \mathscr{A}_a \to \mathscr{A}'_{\phi(a)}$ is injective,

(2) each $\Phi_{a/f} \colon (\coprod_{e \in \phi_{(a)}^{-1}(f)} \mathscr{A}_{a/e}) \to \mathscr{A}'_{\phi(a)/f}$ is injective.

A finitely generated group G is said to be a *generalized Baumslag–Solitar group* or a GBS group if G can act on a tree such that the stabilizers of vertices and edges are infinite cyclic groups. By the Bass–Serre Theorem, G is representable as $\pi_1(\mathbb{A})$, the fundamental group of a graph of groups $\mathbb{A}$ (see [13]) whose vertex and edge groups are infinite cyclic groups.

To avoid excessive notation we use the symbols a and e for the generators of infinite cyclic groups $\mathscr{A}_a$ and $\mathscr{A}_e$ (i.e. $\mathscr{A}_a = \langle a \rangle$ and $\mathscr{A}_e = \langle e \rangle$, for all $a \in V(A)$ and $e \in E(A)$).

Definition 2.10. If $\mathscr{A}_a = \langle a\rangle$ and $\mathscr{A}_e = \langle e\rangle$, for all $a \in V(A)$ and $e \in E(A)$, then the structure of the graph of groups for A is completely determined by a *set* $\{\lambda_e : e \in E(A)\}$ *of nonzero labels* such that $\alpha_e(e) = a^{\lambda_e}$. Henceforth we consider only GBS groups and the corresponding *labeled graphs* $\mathbb{A} = (A, \lambda)$.

Now we give an interpretation of morphisms and immersions of graphs of groups for labeled graphs.

Definition 2.11. Given labeled graphs $\mathbb{A} = (A, \lambda)$ and $\mathbb{B} = (B, \mu)$, say that the tuple $(\phi, (\sigma_a)_{a\in V(A)}, (\theta_e)_{e\in E(A)}, (\delta_e)_{e\in E(A)})$ is a *morphism* $\Phi\colon \mathbb{A} \to \mathbb{B}$ *of labeled graphs* if the following conditions hold:

(1) $\phi\colon A \to B$ is a graph morphism.

(2) The group homomorphisms $\phi_a\colon \mathscr{A}_a \to \mathscr{B}_{\phi(a)}$ and $\phi_e = \phi_{\bar{e}}\colon \mathscr{A}_e \to \mathscr{B}_{\phi(e)}$ are defined by the integers σ_a and θ_e, i.e. $\phi_a\colon a \to \phi(a)^{\sigma_a}$ and $\phi_e\colon e \to \phi(e)^{\theta_e}$.

(3) Using the notion of $\delta\Phi$ ([1, Section 2.9]), we can choose the set (γ_a) in Definition 2.7 (3, a) to be trivial ($\gamma_a = 1$ for all $a \in V(A)$). Since $\mathscr{B}_{\phi(\partial_0 e)}$ is abelian, the equations in Definition 2.7 (3, b) simplify:

$$\begin{aligned}\phi(a)^{\sigma_a\cdot\lambda_e} = \phi(a)^{\theta_e\cdot\mu_{\phi(e)}} &\iff \sigma_a\cdot\lambda_e = \theta_e\cdot\mu_{\phi(e)}\\ &\iff \frac{\sigma_a\cdot\lambda_e}{\mu_{\phi(e)}} = \theta_e = \theta_{\bar{e}} = \frac{\sigma_b\cdot\lambda_{\bar{e}}}{\mu_{\phi(\bar{e})}}\end{aligned} \tag{2.1}$$

for all $e \in E(A)$, $a = \partial_0(e)$, and $b = \partial_0(\bar{e})$.

Denote the sets $(\sigma_a)_{a\in V(A)}$, $(\theta_e)_{e\in E(A)}$, and $(\delta_e)_{e\in E(A)}$ by (σ), (θ), and (δ) respectively. We can see that there is no condition on (δ) in Definition 2.11.

Definition 2.12. Given labeled graphs $\mathbb{A} = (A, \lambda)$ and $\mathbb{B} = (B, \mu)$, a morphism $\Phi\colon \mathbb{A} \to \mathbb{B}$ of labeled graphs is called an *immersion* if

(1) the integers σ_a for $a \in V(A)$ and θ_e for $e \in E(A)$ are nonzero.

(2) for all pairs (a, f), with $a \in V(A)$ and $f \in E(B)$ satisfying $\phi(a) = \partial_0 f$ the map $\Phi_{a/f}\colon (\coprod_{e\in\phi^{-1}_{(a)}(f)} \mathscr{A}_{a/e}) \to \mathscr{B}_{\phi(a)/f}$ is injective.

Denote the greatest common divisor of two integers m and n by (m, n).

Given a morphism $\Phi\colon \mathbb{A} \to \mathbb{B}$ of labeled graphs, a pair (a, f) with $a \in V(A)$ and $f \in E(B)$ is *valid* if $\phi(a) = \partial_0(f) = b$ and $\{e_1, e_2, \dots, e_k\} = \phi^{-1}_{(a)}(f)$. For valid pairs we will discuss the condition

$$\lambda_{e_i} = \frac{\mu_f}{(\sigma_a, \mu_f)} \quad \text{for } i = 1, \dots, k \text{ and } k \leqslant (\sigma_a, \mu_f). \tag{2.2}$$

Remark 2.13. If $\Phi\colon \mathbb{A} \to \mathbb{B}$ is an immersion of labeled graphs, then condition (2.2) holds for all valid pairs (a, f).

Proof. Take $\mathscr{A}_{a/e_i} = \{[e]_{e_i}, [a]_{e_i}, \dots, [a^{\lambda_{e_i}-1}]_{e_i}\}$ for $i = 1, 2, \dots, k$. Then, by Definitions 2.8 and 2.12, the map

$$\Phi_{a/f}([a^l]_{e_i}) = [b^{\sigma_a \cdot l} \cdot \delta_{e_i}]_f$$

for $l = 0, 1, \dots, \lambda_{e_i} - 1$ and $i = 1, \dots, k$ is injective.

Recall that $\delta_{e_i} \in \mathscr{B}_b$; hence, $\delta_{e_i} = b^{r(e_i)}$ for appropriate integers $r(e_i)$. Therefore, using additive notation in the infinite cyclic groups $\mathscr{B}_f$, for $f \in E(B)$ we can see that Definition 2.12 (2) for a pair (a, f) is equivalent to the existence of integers $r(e_i)$ with $0 \leqslant r(e_i) < \mu_f$ such that all integers

$$\begin{gathered} r(e_1), \sigma_a + r(e_1), 2\sigma_a + r(e_1), \dots, (\lambda_{e_1} - 1)\sigma_a + r(e_1), \\ r(e_2), \sigma_a + r(e_2), 2\sigma_a + r(e_2), \dots, (\lambda_{e_2} - 1)\sigma_a + r(e_2), \\ \vdots \\ r(e_k), \sigma_a + r(e_k), 2\sigma_a + r(e_k), \dots, (\lambda_{e_k} - 1)\sigma_a + r(e_k) \end{gathered}$$

are distinct modulo μ_f.

The order of the element σ_a in additive group $\mathbb{Z}_{\mu_f}$ is equal to $\mu_f/(\sigma_a, \mu_f)$. If $\lambda_{e_i} > \operatorname{ord}(\sigma_a)$, then $j\sigma_a + r(e_i) \equiv_{\mu_f} r(e_i)$ for some j. Therefore, $\Phi_{a/f}$ cannot be an immersion. Thus we have $\lambda_{e_i} \leqslant \mu_f/(\sigma_a, \mu_f)$ for all $i = 1, \dots, k$. Recall that $\sigma_a \cdot \lambda_{e_i} = \theta_{e_i} \cdot \mu_f$ for all $i = 1, \dots, k$ by Definition 2.11 (3). Then

$$\begin{aligned} \mu_f \mid \sigma_a \cdot \lambda_{e_i} &\iff \frac{\mu_f}{(\sigma_a, \mu_f)} \,\Big|\, \frac{\sigma_a}{(\sigma_a, \mu_f)} \cdot \lambda_{e_i} \\ &\iff \frac{\mu_f}{(\sigma_a, \mu_f)} \,\Big|\, \lambda_{e_i}, \quad i = 1 \dots, k. \end{aligned}$$

Comparing $\lambda_{e_i} \leqslant \mu_f/(\sigma_a, \mu_f)$ and $\mu_f/(\sigma_a, \mu_f) \mid \lambda_{e_i}$ we obtain

$$\lambda_{e_i} = \frac{\mu_f}{(\sigma_a, \mu_f)}$$

for all $i = 1, \dots, k$.

For each edge $e_i \in \phi_{(a)}^{-1}(f)$ the image of $(A)_{a/e_i}$ under the injection $\Phi_{a/f}$ consists of $\lambda_{e_i} = \mu_f/(\sigma_a, \mu_f)$ elements of $\mathscr{B}_{b/f} \cong \mathbb{Z}_{\mu_f}$. Thus, if $k > (\sigma_a, \mu_f)$, then $\Phi_{a/f}$ cannot be injective. □

Remark 2.14. Consider a morphism $\Phi\colon \mathbb{A} \to \mathbb{B}$ of labeled graphs. Suppose that conditions (2.2) and Definition 2.12 (1) hold for all valid pairs (a, f). Then Φ is an immersion for some choice of (δ).

Proof. In the notation of the previous proof we need to find integers $r(e_i)$ for all valid pairs (a, f) and all $e_i \in \phi_{(a)}^{-1}(f)$. It is easy to check that if we choose

$$r(e_i) = i - 1,$$

then $\Phi_{a/f}$ becomes an injection and Φ becomes an immersion. □

Using [1, Corollary 4.6 and Proposition 2.7], by analogy with [5, Proposition 2] we can prove the following statement.

Remark 2.15. Given labeled graphs $\mathbb{A}$ and $\mathbb{B}$, we can embed $\pi_1(\mathbb{A})$ into $\pi_1(\mathbb{B})$ if and only if there exists a labeled graph $\mathbb{C}$ with $\pi_1(\mathbb{A}) \cong \pi_1(\mathbb{C})$ immersible into $\mathbb{B}$.

Definition 2.16. An expansion is *unnatural* if $n = 1$ and either $\mathrm{St}_A\, \partial_0(e) = \{e\}$ or $\mathrm{St}_A\, \partial_1(e) = \{\bar{e}\}$, where e is a new edge.

If an expansion is not unnatural, then we call it a *natural* expansion.

Remark 2.17. Consider an immersion $\Phi\colon \mathbb{C} \to \mathbb{B}$ with $\mathbb{C}$ obtained from a reduced labeled graph $\mathbb{A}$ using expansion. Then there exists a labeled graph $\mathbb{D}$ which can be obtained from $\mathbb{A}$ using only natural expansions and which immerses into $\mathbb{B}$.

Proof. Denote the set of edges of C obtained by unnatural expansions by $U(\mathbb{C})$. Deleting $U(\mathbb{C})$ from $E(C)$, we obtain $\mathbb{D}$. □

Henceforth we use only natural expansions.

3 Auxiliary statements

Definition 3.1. Given labeled graphs $\mathbb{A} = (A, \lambda)$ and $\mathbb{B} = (B, \mu)$ and a graph morphism $\phi\colon A \to B$, say that a set $(\sigma), (\theta)$ is a *minimal set adjusted to* ϕ if the following conditions hold:

(1) the tuple $(\phi, (\sigma), (\theta), (\delta))$ defines a morphism of labeled graphs for some (δ),

(2) if $\Psi\colon \mathbb{A} \to \mathbb{B}$ is a morphism of labeled graphs corresponding to the tuple $(\phi, (\sigma'), (\theta'), (\delta'))$, then $\sigma_a \mid \sigma'_a$ for all $a \in V(A)$ and $\theta_e \mid \theta'_e$ for all $e \in E(A)$.

If only Definition 3.1 (1) holds for the set $(\sigma), (\theta)$, then it is called *a set adjusted to* ϕ.

Given a graph A, denote by A_0 the undirected graph obtained from A by identifying the edges e and $\bar{e}$. Denote by T_{A_0} some maximal subtree of the A_0. Denote by T_A subgraph of A such that $(T_A)_0$ coincides with T_{A_0}.

Definition 3.2. Let $\mathbb{A} = (A, \lambda)$ be a labeled graph and let $p = (e_1, e_2, \ldots, e_n)$ be an edge path in A. Put

$$\Delta(p) = \prod_{i=1}^{n} \frac{\lambda_{\bar{e}_i}}{\lambda_{e_i}}.$$

Lemma 3.3. *Given labeled graphs $\mathbb{A} = (A, \lambda)$ and $\mathbb{B} = (B, \mu)$ and a morphism $\Phi: \mathbb{A} \to \mathbb{B}$ of labeled graphs. Let $p = (e_1, \ldots, e_n)$ be an edge path in A. Put $q = \phi(p) = (f_1 = \phi(e_1), \ldots, f_n = \phi(e_n))$. Then*

$$\Delta(p) \cdot \sigma_{\partial_1(e_n)} = \Delta(q) \cdot \sigma_{\partial_0(e_1)}.$$

Proof. By Definition 2.11 (3),

$$\frac{\sigma_{\partial_0(e_i)} \cdot \lambda_{e_i}}{\mu_{f_i}} = \frac{\sigma_{\partial_1(e_i)} \cdot \lambda_{\bar{e}_i}}{\mu_{\bar{f}_i}} \quad \text{for all } i = 1, 2, \ldots, n.$$

Now it is easy to see that

$$\sigma_{\partial_1(e_n)} = \prod_{i=1}^{n} \frac{\mu_{\bar{f}_i}}{\mu_{f_i}} \cdot \frac{\lambda_{e_i}}{\lambda_{\bar{e}_i}} \cdot \sigma_{\partial_0(e_1)} = \frac{\Delta(q)}{\Delta(p)} \cdot \sigma_{\partial_0(e_1)}. \qquad \square$$

Lemma 3.4. *Given labeled graphs $\mathbb{A}, \mathbb{B}$ and a graph morphism $\phi: A \to B$, the following claims hold:*

(1) *if A_0 is a tree, then there exists (a unique up to a sign change) minimal set $(\sigma), (\theta)$ adjusted to ϕ,*

(2) *if A_0 is not a tree, then a minimal set adjusted to ϕ exists if and only if*

$$\frac{\sigma_{\partial_0 e} \cdot \lambda_e}{\mu_{\phi(e)}} = \frac{\sigma_{\partial_1 e} \cdot \lambda_{\bar{e}}}{\mu_{\phi(\bar{e})}} \quad \textit{for all } e \in E(A) \setminus E(T_A),$$

where $\sigma_{\partial_0 e}$ and $\sigma_{\partial_1 e}$ for T_A, $\phi|_{T_A}$ are defined in claim (1),

(3) *if a set adjusted to ϕ exists, then there exists (a unique up to a sign change) minimal set adjusted to ϕ,*

(4) *if $(\sigma), (\theta)$ is a minimal set adjusted to ϕ, then*

$$\max\{|\sigma_a|, |\theta_e| : a \in V(A),\ e \in E(A)\} \leqslant (M(\mathbb{A}) \cdot M(\mathbb{B}))^{2 \cdot |V(A)|},$$

where $M(\mathbb{A}) = \max\{|\lambda_e| : e \in E(A)\}$.

Proof. (1) It is easy to see that $|(\theta) \cup (\sigma)|$ is equal to

$$|V(A_0)| + |E(A_0)| = 2|E(A_0)| + 1.$$

Definition 2.11 (3) is equivalent to a system of $2|E(A_0)|$ linear Diophantine equations. It is easy to see that this system is homogeneous and its rank is equal to $2|E(A_0)|$. Thus, the solution space is one-dimensional. Since the system is homogeneous and the entries of its matrix are integers, there exists a rational nonzero solution $\vec{v}$. If $\vec{v}$ is not an integer, then we can multiply $\bar{v}$ by the LCM (least common multiple) of the denominators of its coordinates and obtain a minimal integer solution. If $\vec{v}$ is an integer, then we can divide it by the GCD of its coordinates and obtain a minimal integer solution again. A minimal integer solution is unique up to a sign change because the solution space is one-dimensional.

(2) Necessity follows from Definition 2.11 (3). We now prove sufficiency. We need to complete the set $(\sigma), (\theta)$ defined for T_A. More precisely, we need to find θ_e for $e \in E(A) \setminus E(T_A)$. Using (2.1) and the equations in (2), we can put

$$\theta_e = \theta_{\bar{e}} = \frac{\sigma_{\partial_0 e} \cdot \lambda_e}{\mu_{\phi_e}} = \frac{\sigma_{\partial_1 e} \cdot \lambda_{\bar{e}}}{\mu_{\bar{\phi}_e}} \quad \text{for all } e \in E(A) \setminus E(T_A),$$

which are well defined. At last, we can obtain a minimal set from $(\sigma), (\theta)$ by multiplying or dividing by an appropriate integer if necessary.

(3) The proof is analogous to that of (1).

(4) Take a vertex a of A such that $|\sigma_a| = \min\{|\sigma_c| : c \in V(A)\}$. Take a vertex b of A and denote by p_b some edge path in A connecting a and b. Lemma 3.3 yields $\sigma_b = \sigma_a \cdot R_b$, where $R_b = \Delta(\phi(p_b))/\Delta(p_b)$. The numerator of R_b divides $\prod_{i=1}^{n} \lambda_{e_i} \cdot \mu_{\phi(\bar{e}_i)}$, where $p_b = (e_1, \dots, e_n)$. Thus,

$$\max\{|\sigma_a|, |\theta_e|\} \leqslant (M(\mathbb{A}) \cdot M(\mathbb{B}))^{|V(A)|} \cdot |\sigma_a|.$$

All σ_a and θ_e are integer numbers if and only if the denominators of fractions R_b divide σ_a. The denominator of each R_b divides $\prod_{i=1}^{n} \lambda_{\bar{e}_i} \cdot \mu_{\phi(e_i)}$. Thus,

$$|\sigma_a| \leqslant (M(\mathbb{A}) \cdot M(\mathbb{B}))^{|V(A)|}. \qquad \square$$

Definition 3.5. Given a labeled graph $\mathbb{A}$, a vertex a of A is called *special* if one has $|\mathrm{St}_A\, a| > 2$ or there exists an $e \in \mathrm{St}_A\, a$ such that $\lambda_e \neq 1$. A vertex v is *non-special* if and only if $|\mathrm{St}_A\, v| = 2$ and $\lambda_e = 1$ for all $e \in \mathrm{St}_A\, v$.

A labeled graph $\mathbb{C}$ is said to be an *extension of* $\mathbb{A}$ if there exists a sequence of natural expansions from $\mathbb{A}$ to $\mathbb{C}$.

For an integer n with prime factorization $p_1^{\alpha_1} \cdot p_2^{\alpha_1} \cdots p_k^{\alpha_k}$, write $\log n$ for the sum $\alpha_1 + \alpha_2 + \cdots + \alpha_k$.

For a labeled graph $\mathbb{A}$, put $\log \mathbb{A} = \log \prod_{e \in E(A)} \lambda_e$.

Proposition 3.6. *Given a labeled graph* $\mathbb{A}$*, every extension* $\mathbb{C}$ *of* $\mathbb{A}$ *has at most* $S(\mathbb{A}) = \log \mathbb{A} + (|E(A)| - 2)|V(A)|$ *special vertices.*

Proof. If $|\mathrm{St}_A\, a| = n > 2$, then, using expansions, we can separate this star into at most $n-2$ tridents (see Figure 2). It is easy to see that an expansion move cannot increase $\log \mathbb{A}$. Thus, the number of special vertices such that $\lambda_e \neq 1$ for some edge $e \in \mathrm{St}_A\, a$ cannot exceed $\log \mathbb{A}$. Therefore, we can obtain at most $(|E(A)| - 2) \cdot |V(A)|$ special vertices such that $|\mathrm{St}_A\, v| > 2$. □

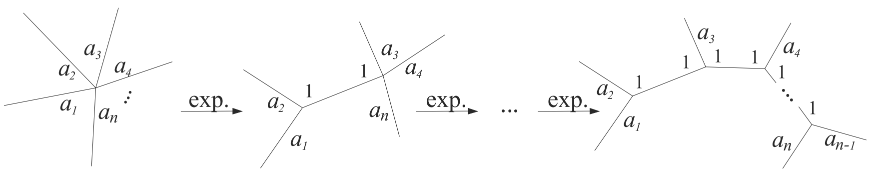

Figure 2. Separation of the star into tridents.

Definition 3.7. Consider a reduced labeled graph $\mathbb{A}$. Denote by $\Omega_N(\mathbb{A})$ the set of labeled graphs such that every $\mathbb{B} \in \Omega_N(\mathbb{A})$ is an extension of $\mathbb{A}$ and every reduced (i.e. without $e, \bar{e}$ inclusions) path $e_1, e_2, \dots, e_{N+1}$ without self-intersections in B contains at least one special vertex.

Proposition 3.8. *We have* $|\Omega_N(\mathbb{A})| \leqslant S(\mathbb{A})^{6 \cdot S(\mathbb{A})+5} \cdot (3 \cdot S(\mathbb{A}) + 2)^{\log \mathbb{A}} \cdot N^{S(\mathbb{A})^2}$.

Proof. Every labeled graph in $\Omega_N(\mathbb{A})$ has k special vertices, where

$$|V(A)| \leqslant k \leqslant S(\mathbb{A}).$$

If $\mathbb{B} \in \Omega_N(\mathbb{A})$ and the number of special vertices of $\mathbb{B}$ is equal to k, then applying collapse moves to $\mathbb{B}$ we can obtain a labeled graph $\mathbb{C}$ such that $|V(C)| = k$, all vertices are special, and $\mathbb{C} \in \Omega_N(\mathbb{A})$.

Given k, we will find an upper estimate for the number of labeled graphs in $\Omega_N(\mathbb{A})$ with k special vertices and without other vertices. It is possible to consider that each new special vertex appears with one new edge during an expansion move. The number of graphs with k vertices and $|E(\mathbb{A})| + (k - |V(\mathbb{A})|) \cdot 2$ edges is, obviously, at most $(C_k^2)^{|E(\mathbb{A})|+(k-|V(\mathbb{A})|)\cdot 2}$, where C_k^2 is equal to the number of 2-combinations from k elements. Let us calculate the number of possible labelings of the edges. Each prime divisor of $\prod_{e \in E(A)} \lambda_e$ (there are $\log \mathbb{A}$ of these) can be placed on any edge ($|E(\mathbb{A})| + (k - |V(\mathbb{A})|) \cdot 2$ elements). Thus, we have at most

$$(C_k^2)^{|E(\mathbb{A})|+(k-|V(\mathbb{A})|)\cdot 2} \cdot (|E(\mathbb{A})| + (k - |V(\mathbb{A})|) \cdot 2)^{\log \mathbb{A}}$$

labeled graphs in $\Omega_N(\mathbb{A})$ with k special vertices and no others.

We can obtain every labeled graph from $\Omega_N(\mathbb{A})$ by adding to one of the labeled graphs described above at most N non-special vertices (by expansion movies) between pairs of special vertices. There are at most $N^{C_k^2}$ possibilities for these extensions. Therefore,

$$|\Omega_N(\mathbb{A})| \leqslant \sum_{k=|V(A)|}^{S(\mathbb{A})} (C_k^2)^{|E(\mathbb{A})|+(k-|V(\mathbb{A})|)\cdot 2} \cdot (|E(\mathbb{A})|+(k-|V(\mathbb{A})|)\cdot 2)^{\log \mathbb{A}} \cdot N^{C_k^2}.$$

Since $C_k^2 \leqslant k^2 \leqslant S(\mathbb{A})^2$ and

$$|E(\mathbb{A})| + (k - |V(\mathbb{A})|) \cdot 2 \leqslant 3 \cdot S(\mathbb{A}) + 2,$$

we can see that each term in the sum is at most

$$S(\mathbb{A})^{6 \cdot S(\mathbb{A})+4} \cdot (3 \cdot S(\mathbb{A}) + 2)^{\log \mathbb{A}} \cdot N^{S(\mathbb{A})^2}. \qquad \square$$

Definition 3.9. Given a labeled graph $\mathbb{B}$, take a path $q = (f_1, f_2, \ldots, f_n)$ in B with $\Delta(q) = \alpha/\beta$, where $\alpha \perp \beta$. Denote by γ_q the minimal positive integer such that the numbers

$$\beta\gamma_q \cdot \frac{1}{\mu_{f_1}}, \beta\gamma_q \cdot \frac{\mu_{\bar{f}_1}}{\mu_{f_1} \cdot \mu_{f_2}}, \ldots, \beta\gamma_q \cdot \frac{\mu_{\bar{f}_1} \cdots \mu_{\bar{f}_i}}{\mu_{f_1} \cdots \mu_{f_{i+1}}}, \ldots, \beta\gamma_q \cdot \frac{\mu_{\bar{f}_1} \cdots \mu_{\bar{f}_{n-1}}}{\mu_{f_1} \cdots \mu_{f_n}}$$

are integers. It is obvious that $\gamma_q \mid \mu_{f_1} \cdots \mu_{f_n}$.

Proposition 3.10. *Consider labeled graphs $\mathbb{A}$ and $\mathbb{B}$, a morphism $\Phi\colon \mathbb{A} \to \mathbb{B}$ of labeled graphs and the set of integers (σ), (θ) is adjusted to ϕ. Let $\mathbb{A}_1$ be a labeled graph obtained from $\mathbb{A}$ by addition of a path $p = (e_1, \ldots, e_n)$ to the graph A such that $\lambda_{e_i} = \lambda_{\bar{e}_i} = 1$, $i = 1, 2, \ldots, n$. Suppose that the extension of the morphism ϕ to the graph A_1 is given by $\phi(p) = q$, where q is a cycle. Write $\Delta(q) = \alpha/\beta$, where $\alpha \perp \beta$. Then we can find numbers $\theta_{e_i}, \sigma_{\partial_1 e_i}$ for all $i = 1, \ldots, n-1$ and θ_{e_n} using* (2.1) *and these numbers are integers if and only if $\beta \cdot \gamma_q \mid \sigma_{\partial_0 e_1}$.*

Proof. The proof is straightforward. $\square$

Proposition 3.11. *Suppose that $\frac{\alpha}{\beta} \in \langle \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_k}{\beta_k} \rangle \subseteq \mathbb{Q}$, where $\alpha_i \perp \beta_i$, for all $i = 1, \ldots, k$. Denote*

$$\frac{\alpha}{\beta} = \prod_{i=1}^{k} \left(\frac{\alpha_i}{\beta_i} \right)^{s_i},$$

where $\alpha \perp \beta$. Put $S = \sum_{i=1}^{k} s_i$ and $d = \log(\prod_{i=1}^{k} \alpha_i \cdot \beta_i)$. Suppose $\prod_{i=1}^{k} \alpha_i \beta_i$ has exactly P prime divisors $p_1, p_2, \ldots, p_P$. Denote by $\tau_i\colon \mathbb{Q} \to \mathbb{Q}$ the mapping defined by

$$\tau_i\left(\frac{m}{n}\right) = \frac{m}{n} \cdot \frac{\alpha_i}{\beta_i}, \quad i = 1, \ldots, k.$$

Then there exists a sequence $i_1, \dots, i_S \in \{1, \dots, k\}$ *such that*

$$(\tau_{i_1} \cdot \tau_{i_2} \cdots \tau_{i_l})\left(\beta \cdot \prod_{i=1}^{P} p_i^d\right) \in \mathbb{Z} \quad \textit{for all } l = 1, 2, \dots, S, \tag{3.1}$$

and

$$(\tau_{i_1} \cdot \tau_{i_2} \cdots \tau_{i_S})\left(\beta \cdot \prod_{i=1}^{P} p_i^d\right) = \alpha \cdot \prod_{i=1}^{P} p_i^d. \tag{3.2}$$

Proof. Using the Unique Factorization Theorem for rational numbers, define the map $\psi\colon \langle \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_k}{\beta_k} \rangle \to \mathbb{Z}^P$ and put $\vec{v_i} = \psi(\alpha_i/\beta_i)$. Decompose $\vec{v} = \vec{v}^+ + \vec{v}^-$ such that

$$\vec{v}^+, -\vec{v}^- \in \mathbb{Z}_+^P = \{(r_1, \dots, r_P) : r_i \geqslant 0, i = 1, \dots, P\}.$$

Since $\alpha_i \perp \beta_i$, we obtain $\psi(\alpha_i) = \vec{v_i}^+$ and $\psi(\beta_i) = \vec{v_i}^-$. Let $\mathscr{L} = \langle \vec{v_1}, \dots, \vec{v_k} \rangle_{\mathbb{Z}}$ be the integer linear span of $\vec{v_1}, \dots, \vec{v_k}$. The map

$$\psi\colon \left\langle \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_k}{\beta_k} \right\rangle \to \mathscr{L}$$

is a bijection and $\mathscr{L}$ is a lattice in $\mathbb{R}^P$. Put

$$\vec{w} = \psi\left(\frac{\alpha}{\beta}\right) = \sum_{i=1}^{k} s_i \vec{v_i}.$$

Then $\psi(\beta) = \vec{w}^-$ and $\psi(\alpha) = \vec{w}^+$. Let $\mathscr{L}' = \mathscr{L} - \vec{w}^-$ be the lattice $\mathscr{L}$ shifted by the vector $-\vec{w}^-$. Since $\vec{w}^+ + \vec{w}^- \in \mathscr{L}$, it follows that

$$\vec{w}^+ \in \mathscr{L} - \vec{w}^- \quad \text{and} \quad -\vec{w}^- \in \mathscr{L} - \vec{w}^-.$$

Let us say that two lattice points in $\mathscr{L}$ (and in $\mathscr{L}'$) are connected by an edge if and only if one of the corresponding rational numbers can be obtained from another by using τ_i. To prove Proposition 3.11 we find a path (a sequence of lattice points joined by edges) in $\mathscr{L}$ connecting $-w^-$ and w^+ every point of which is contained in

$$\mathbb{Z}^P_{\geqslant -d} = \{(r_1, \dots, r_P) : r_i \geqslant -d\}.$$

Denote by $R = \langle \vec{v_1}, \dots, \vec{v_k} \rangle_{\mathbb{R}}$ the subspace of $\mathbb{R}^k$ generated by $\vec{v_1}, \dots, \vec{v_k}$, and denote $R - \vec{w}^-$ by R'. Then $R \supset \mathscr{L}$ and $R' \supset \mathscr{L}'$.

The points $-\vec{w}^-$ and $\vec{w}^+$ are connected in R' by the segment

$$\mathscr{M} = \{t \cdot \vec{w}^+ + (1-t) \cdot (-\vec{w}^-) : t \in [0, 1]\}.$$

Since R' and $\mathbb{R}^P_+$ are convex, we see that $\mathscr{M} \subset \mathbb{R}^P_+$. Hence

$$t\vec{w}^+ + (1-t)(-\vec{w}^-) \in \mathbb{R}^P_+ \quad \text{for all } t \in [0, 1].$$

The segment $\mathscr{M}$ passes through the basic parallelepipeds of the lattice $\mathscr{L}'$. We will show that every reduced path by the vertices of these parallelepipeds satisfies the conditions. To this end, it suffices to understand that for every vertex of these parallelepipeds there exists a point in $\mathscr{M}$ at distance at most the diameter of this parallelepiped. The diameter of the parallelepiped is at most the length of the path connecting opposite vertices, which is equal to d. □

Cycles $q_1, q_2, \dots, q_k$ with common origin and endpoint a in graph A are called *basic cycles* if any cycle q in A with the same origin and endpoint can be represented as a concatenation of cycles $q_1, q_2, \dots, q_k$ (up to deleting and inserting $e, \bar{e}$ paths) and the number k is minimal (equal to the first Betti number of A).

Proposition 3.12. *Consider a labeled graph $\mathbb{B}$. Take basic cycles $m_1, \dots, m_k$ with $\partial_0 m_i = \partial_1 m_i = a \in V(B)$. Write $\Delta(m_i)$ as $\frac{\alpha_i}{\beta_i}$ with $\alpha_i \perp \beta_i$, and put*

$$d = \log\left(\prod_{i=1}^{k} \alpha_i \beta_i\right).$$

For any integers $s_1, s_2, \dots, s_k$ there exists a cycle q such that $\partial_0 q = \partial_1 q = a$, $\Delta(q) = \prod_{i=1}^{k} (\frac{\alpha_i}{\beta_i})^{s_i}$ and $\gamma_q \mid \mathrm{LCM}(\gamma_{m_1}, \dots, \gamma_{m_k}) \prod_{i=1}^{P} p_i^d$.

Proof. Write

$$\Delta(q) = \prod_{i=1}^{k} \left(\frac{\alpha_i}{\beta_i}\right)^{s_i} = \frac{\alpha}{\beta}$$

and put $S = \sum_{i=1}^{k} s_i$. By Proposition 3.11 there exists a sequence $i_1, \dots, i_S$ satisfying (3.1) and (3.2). To show that

$$q = m_{i_1} \circ m_{i_2} \circ \dots \circ m_{i_S}$$

is an appropriate path, we verify that $\mathrm{LCM}(\gamma_{m_1}, \dots, \gamma_{m_k}) \prod_{i=1}^{P} p_i^d$ satisfies all conditions for γ_q (Definition 3.9) except the minimality. Using (3.1)–(3.2) and an analog of Proposition 3.10, we can see that it suffices to check that

$$\beta_{i_j} \cdot \gamma_{m_{i_j}} \mid \Delta(m_{i_1}) \dots \Delta(m_{i_{j-1}}) \cdot \beta \cdot \mathrm{LCM}(\gamma_{m_1}, \dots, \gamma_{m_k}) \cdot \prod_{i=1}^{P} p_i^d$$

for all $j = 1, \dots, S$, which follows from two facts. First,

$$\gamma_{m_{i_j}} \mid \mathrm{LCM}(\gamma_{m_1}, \dots, \gamma_{m_k}) \quad \text{for all } j = 1, \dots, S.$$

Second, $\Delta(m_{i_1}) \cdots \Delta(m_{i_j}) \beta \prod_{i=1}^{P} p_i^d$ is an integer. Therefore,

$$\beta_{i_j} \mid \Delta(m_{i_1}) \cdots \Delta(m_{i_{j-1}}) \beta \prod_{i=1}^{P} p_i^d \quad \text{for all } j = 1, \dots, S.$$

□

In any metric space, denote by $B_r(x)$ the closed ball of radius r centered at a point x.

Proposition 3.13. *Consider linearly independent vectors $\vec{a}_1, \dots, \vec{a}_l \in \mathbb{R}^n$ and additional vectors $\vec{a}_{l+1}, \dots, \vec{a}_m \in \mathbb{R}^n$. If the system of equations and inequalities*

$$\begin{cases} \vec{a}_i \cdot \vec{x} = b_i, & i = 1, \dots, l, \\ \vec{a}_i \cdot \vec{x} \leqslant b_i, & i = l+1, \dots, m, \end{cases} \tag{3.3}$$

is solvable in $\mathbb{R}^n$, then there exists a subset $I \subseteq \{l+1, \dots, m\}$ such that the system of equations

$$\begin{cases} \vec{a}_i \cdot \vec{x} = b_i, & i = 1, \dots, l, \\ \vec{a}_i \cdot \vec{x} = b_i, & i \in I, \end{cases} \tag{3.4}$$

is solvable and the solutions to (3.4) *form a subset of solutions to* (3.3).

Proof. Denote by P the set of solutions to (3.3) and by P_i, $i = l+1, \dots, m$, the set of solutions to the equation $\vec{a}_i \cdot \vec{x} = b_i$. Take a solution x_0 to (3.3) and denote by r_1 the minimal real number r such that there exists k with $l+1 \leqslant k \leqslant m$ and $B_r(x_0) \cap P \cap P_k \neq \emptyset$. Take l_1 with $l+1 \leqslant l_1 \leqslant m$ and $B_{r_1}(x_0) \cap P \cap P_{l_1} \neq \emptyset$. Then, the system of equations and inequalities

$$\begin{cases} \vec{a}_i \cdot \vec{x} = b_i, & i \in \{1, \dots, l, l_1\}, \\ \vec{a}_i \cdot \vec{x} \leqslant b_i, & i \in \{l+1, \dots, m\} \setminus \{l_1\}, \end{cases} \tag{3.5}$$

is solvable. Excluding those inequalities of (3.5) which are identically true for all x satisfying $\vec{a}_i \cdot \vec{x} = b_i$, for $i \in \{1, \dots, l, l_1\}$, we obtain an equivalent system. The number of inequalities in (3.5) is less than in (3.3) and the set of solutions to (3.5) contains in the set of solutions of (3.3). Continuing this argument, we obtain a required system after at most $m - l$ steps. □

Lemma 3.14. *Given an $m \times n$ matrix A over $\mathbb{Z}$ and $b \in \mathbb{R}^m$, if the system of Diophantine equations $A \cdot \vec{x} = b$ is solvable over $\mathbb{R}$, then there exists a solution $\vec{x}_0 \in \mathbb{R}^m$ satisfying*

$$|\vec{x}_0| \leqslant k^2 \cdot a^{k-1} \cdot (k-1)! \cdot |b|,$$

where $k = \mathrm{rk}(A)$ and $a = \max\{|a_{ij}| : i = 1, \dots, m,\ j = 1, \dots, n\}$.

Proof. Since $A \cdot \vec{x} = b$ is solvable, we have $\mathrm{rk}(A) = \mathrm{rk}(A \mid b) = k$. Without loss of generality we may assume that the first k rows and columns define a maximal minor in A (denote it by B) with nonzero value. Then we can choose the solution so that

$$x = (y_1, y_2, \dots, y_k, 0, 0, \dots, 0), \quad B \cdot (y_1, y_2, \dots, y_k)^T = (b_1, b_2, \dots, b_k)^T.$$

Put $\vec{y} = (y_1, y_2, \dots, y_k)^T$ and $\vec{c} = (b_1, b_2, \dots, b_k)^T$. Since $B \in \mathrm{GL}_k(\mathbb{Z})$, we infer that

$$\vec{y} = B^{-1} \cdot \vec{c}.$$

Find an upper bound for $\|B^{-1}\|$ using the formula $B^{-1} = C/\det(B)$, where C is the adjoint matrix (thus we have $\|C\| \leqslant k^2 \cdot a^{k-1} \cdot (k-1)!$). Since $B \in \mathrm{GL}_k(\mathbb{Z})$, we have $|\det(B)| \geqslant 1$. Thus $1/|\det(B)| \leqslant 1$ and the required estimate follows. □

Lemma 3.15. *Given an $m \times n$ matrix A over $\mathbb{Z}$ with* $\mathrm{rk}(A) = k$*, denote by P the nullspace of A. If $z \in P$, then $B_R(z)$ contains an integer solution to $A \cdot \vec{x} = 0$, where $R = (n-k) \cdot 2k \cdot a^{2k-1} \cdot k!^2$.*

Proof. As above, we may assume that the first k rows and columns define a maximal nonzero minor in A (denote it by B). After excluding linearly dependent equations we have $m = k$. Denote column $k+i$ of A by $\vec{a}_i$ for $i = 1, 2, \dots, n-k$. Then $\vec{a}_i$ belongs to the linear span of the first k columns of A. Therefore, the systems $B \cdot \vec{x}_i = \vec{a}_i$ are solvable for all $i = 1, 2, \dots, n-k$. Since $B \in \mathrm{GL}_k(\mathbb{Z})$ and $\vec{x}_i = B^{-1} \cdot \vec{a}_i$, where $\vec{a}_i \in \mathbb{Z}^m$, we infer that $\det(B) \cdot \vec{x}_i$ is an integer vector. Moreover, the set of vectors $\vec{y}_i = (\vec{x}_i^T, 0, \dots, 0, -1, 0, \dots, 0)$ for $i = 1, \dots, n-k$ (the -1 appears in slot $(k+i)$) is a basis for P. Using the upper bound of $\|B^{-1}\|$, we can prove that

$$\begin{aligned} |\det(B) \cdot y_i| &\leqslant \|C\|(|a_i| + |\det(B)|) \\ &\leqslant k^2 \cdot a^{k-1} \cdot (k-1)! \cdot (k \cdot a + k! \cdot a^k) \\ &\leqslant 2k \cdot a^{2k-1} \cdot k!^2. \end{aligned}$$

The vectors $\vec{z}_i = \det(B) \cdot \vec{y}_i$ for $i = 1, \dots, n-k$ constitute an integer basis for P. The basis $\{\vec{z}_i\}$ generates a lattice in P with the elements

$$\alpha_1 \cdot \vec{z}_1 + \alpha_2 \cdot \vec{z}_2 + \cdots + \alpha_{n-k} \cdot \vec{z}_{n-k},$$

where $\alpha_1, \alpha_2, \dots, \alpha_{n-k} \in \mathbb{Z}$. The diameter of the basic parallelepiped is at most

$$|\det(B)| \cdot (|\vec{y}_1| + |\vec{y}_2| + \cdots + |\vec{y}_{n-k}|) \leqslant (n-k) \cdot 2k \cdot a^{2k-1} \cdot k!^2 = R.$$

Therefore, $B_R(\vec{z})$ contains at least one basic parallelepiped for all $\vec{z} \in P$. □

Proposition 3.16. *Given an $m \times n$ matrix A over $\mathbb{Z}$ and $b \in \mathbb{Z}^n$. If the system $A \cdot \vec{x} = \vec{b}$ is solvable, then there exists an integer solution $\vec{x}_0$ satisfying*

$$|\vec{x}_0| \leqslant m^2 \cdot a^{m-1} \cdot (m-1)! \cdot |\vec{b}| + (n-m) \cdot 2m \cdot a^{2m-1} \cdot m!^2,$$

where $m =$ $\mathrm{rk}(A)$ and $a = \max\{|a_{ij}| : i = 1, \dots, m,\ j = 1, \dots, n\}$.

Proof. Lemma 3.14 shows that there exists a solution $\vec{z} \in \mathbb{R}^n$ to $A \cdot \vec{x} = \vec{b}$ such that $|\vec{z}| \leqslant m^2 \cdot a^{m-1} \cdot (m-1)! \cdot |\vec{b}|$. Then $A \cdot (\vec{x} - \vec{z}) = 0$ satisfies the hypotheses of Lemma 3.15. Therefore, for every solution $\vec{z}$ to $A \cdot \vec{x} = \vec{b}$ the ball $B_R(\vec{z})$ contains at least one basic parallelepiped. This implies the required estimate. □

Lemma 3.17. *Consider labeled graphs $\mathbb{A}$ and $\mathbb{B}$ with A being a tree, as well as a graph morphism $\phi\colon A \to B$. We can compute the minimal set $(\sigma), (\theta)$ adjusted to ϕ by using at most*

$$\begin{aligned} N(\mathbb{A}, \mathbb{B}) &= |E(B)| \cdot |E(A)| \cdot |V(A)| + 4 \cdot |E(A)| + \log_2(|E(A)| + 1) \cdot 5 \\ &\quad \cdot \log_{10}(M(A)^{2\cdot|E(A)|} \cdot M(B)^{|E(A)|+|E(B)|\cdot|V(A)|}) + |E(A)| + 1 \\ &= O(|E(B)| \cdot |E(A)| \cdot |V(A)| \cdot \ln(M(A) \cdot M(B)) \cdot \ln|E(A)|) \end{aligned}$$

division and multiplication operations.

Proof. To compute some set $(\sigma), (\theta)$ adjusted to ϕ, put

$$\sigma_a = \Big(\prod_{f \in E(B)} \mu_f\Big)^{|V(A)|} \prod_{e \in E(A)} \lambda_e$$

for some $a \in V(A)$. Using Definition 2.11 and (2.1), we can compute $\theta_e = \theta_{\bar{e}}$ for all $e \in E(A)$ such that $\partial_o(e) = a$. Then we can compute σ_b for all $b \in E(A)$ such that $b = \partial_1(e)$ for all computed θ_e, and so on. Sooner or later we will compute $(\sigma), (\theta)$ adjusted to ϕ (the choice of the first σ_a ensures that the set $(\sigma), (\theta)$ is integer). Let us calculate the number of operations. It takes $|E(B)| \cdot |E(A)| \cdot |V(A)|$ multiplications to find first σ_a. It takes $2 \cdot |\mathrm{St}_A(a)|$ operations to find θ_e for all edges $e \in E(A)$ such that $\partial_0(e) = a$. Therefore, to determine σ_b for all $b \in V(A)$ connected with a by an edge, we need $4 \cdot |\mathrm{St}_A(a)|$ operations. Thus we need $|E(B)| \cdot |E(A)| \cdot |V(A)| + 4 \cdot |E(A)|$ operations to determine some set adjusted to ϕ.

To find the minimal set adjusted to ϕ, we need to divide the set just found by $D = \mathrm{GCD}(\sigma_a, a \in V(A), \theta_e, e \in E(A))$. Use Euclid's algorithm to compute D. First of all, we need to see that the maximal element of $(\sigma), (\theta)$ adjusted to ϕ is at most

$$\begin{aligned} &M(B)^{|V(A)|\cdot|E(B)|} \cdot M(A)^{|E(A)|} \cdot (M(A) \cdot M(B))^{|E(A)|} \\ &\quad = M(A)^{2\cdot|E(A)|} \cdot M(B)^{|E(A)|+|E(B)|\cdot|V(A)|)}. \end{aligned}$$

Lame's theorem asserts that the number of modulo operations required to compute $\mathrm{GCD}(a, b)$ using Euclid's algorithm is at most $5 \cdot \log_{10}(\max(a, b))$. Thus, it takes

$$\log_2(|E(A)| + 1) \cdot 5 \cdot \log_{10}(M(A)^{2\cdot|E(A)|} \cdot M(B)^{|E(A)|+|E(B)||V(A)|})$$

operations to compute D.

Finally, to find the minimal set adjusted to ϕ we need $|E(A)| + 1$ divisions. □

Proposition 3.18. *Given labeled graphs* $\mathbb{A}$ *and* $\mathbb{B}$, *it takes at most*

$$\begin{aligned} K(\mathbb{A},\mathbb{B}) = |E(B)|^{|E(A)|} \cdot |V(B)|^{|V(A)|} \cdot \big(&|E(A)|/2 - |V(A)| + 1 + N(\mathbb{A},\mathbb{B}) \\ &+ |V(A)| \cdot |E(B)| \cdot M(\mathbb{B})^{|E(B)|} \cdot 5 \\ &\qquad \cdot \log_{10}(M(\mathbb{A})^{2\cdot|V(A)|} \cdot M(\mathbb{B})^{2|V(A)|+E(B)}) \\ &+ 2 \cdot |E(B)| \cdot |V(A)|\big) \\ = O\big(|E(B)|^{|E(A)|+1} \cdot |V(B)|^{|V(A)|} &\cdot |V(A)|^2 \cdot |E(A)|^2 \cdot M(\mathbb{B})^{|E(B)|} \\ &\cdot \ln(M(\mathbb{A}) \cdot M(\mathbb{B}))\big) \end{aligned}$$

operations to check whether an immersion $\mathbb{A} \to \mathbb{B}$ *exists.*

Proof. To find an immersion, we check all morphisms $\mathbb{A} \to \mathbb{B}$. To determine a morphism $\mathbb{A} \to \mathbb{B}$, we need to find a graph morphism ϕ and a set adjusted to ϕ. It is not difficult to see that there are at most $|E(B)|^{|E(A)|} \cdot |V(B)|^{V(A)}$ graph morphisms.

Fix a graph morphism $\phi: A \to B$ and find a minimal set adjusted to ϕ. Then Lemma 3.17 yields a minimal set adjusted to $\phi|_{T_A}$. By Lemma 3.4 (2), we need to check $|E(A)|/2 - |V(A)| + 1$ equalities to whether a morphism $\mathbb{A} \to \mathbb{B}$ with the graph morphism ϕ exists. If $(\sigma_a), (\theta_e)$ is a minimal set adjusted to ϕ, then every set adjusted to ϕ is equal to $(l \cdot \sigma_a), (l \cdot \theta_e)$ for some $l \in \mathbb{Z}$.

The morphism ϕ and the set $(l \cdot \sigma_a), (l \cdot \theta_e)$ adjusted to ϕ define a morphism Φ_l of labeled graphs. Using Remarks 2.13 and 2.14, we see that Φ_l is an immersion if and only if for all valid pairs (a, f) condition (2.2) holds. It is easy to see that if for all $l = 1, 2, \dots, \mathrm{LCM}(\mu_f, f \in E(B))$ some condition fails, the morphism $\mathbb{A} \to \mathbb{B}$ with the graph morphism ϕ cannot be an immersion. To check the above conditions, we need to calculate $\mathrm{GCD}(l \cdot \sigma_a, \mu_f)$ at most $|V(A)| \cdot |E(B)| \cdot \mathrm{LCM}(\mu_f)$ times. Finally, to obtain the required estimate for the total number of operations, we need to consider all upper estimates, calculations, and information from the proof of Lemma 3.17. □

Lemma 3.19. *Given positive integers* a_i *and* b_i *for all* $i = 1, \dots, n$. *Suppose that* $\mathrm{GCD}(b_1, \dots, b_n) = 1$. *Then there exist* $\alpha_1, \dots, \alpha_n \in \mathbb{N}$ *such that*

$$\alpha = \mathrm{GCD}(a_1 \cdot b_1, a_2 \cdot b_2, \dots, a_n \cdot b_n) = \alpha_1 \cdot \alpha_2 \cdots \alpha_n$$

and α_i *divides* a_i *for all* $i = 1, \dots, n$.

Proof. Let $p_1, \dots, p_k$ be distinct prime numbers such that

$$\alpha = p_1^{\delta_1} \cdot p_2^{\delta_2} \cdots p_k^{\delta_k}.$$

Then $p_i^{\delta_i}$ divides $a_j \cdot b_j$ for all $i = 1, \dots, k$ and $j = 1, \dots, n$. Therefore,

$$\delta_i = m_{ji} + n_{ji}$$

with $m_{ji} \geqslant 0$, $n_{ji} \geqslant 0$, $p_i^{m_{ji}} \mid a_j$, and $p_i^{n_{ji}} \mid b_j$. Thus,

$$p_i^{\min_j n_{ji}} \mid \mathrm{GCD}(b_1, b_2, \dots, b_n) = 1.$$

It follows that for every i there exists a j such that $n_{ji} = 0$. Hence, $p_i^{\delta_i} \mid a_j$. We can define a map $\rho\colon \{1, 2, \dots, k\} \to \{1, 2, \dots, n\}$ such that $p_i^{\delta_i} \mid a_{\rho(i)}$. Then it is easy to see that

$$\alpha_j = \prod_{i \in \rho^{-1}(j)} p_i^{\delta_i}. \qquad \square$$

4 The main theorem

Consider a labeled graph $\mathbb{A}$ such that $R(\mathbb{A}) = \{\mathbb{A}_1, \dots, \mathbb{A}_R\}$. Put

$$\begin{aligned} \log_{\mathbb{A}} &= \max_i \log \mathbb{A}_i, & V_{\mathbb{A}} &= \max_i |V(\mathbb{A}_i)|, \\ E_{\mathbb{A}} &= \max_i |E(\mathbb{A}_i)|, & M_{\mathbb{A}} &= \max_i M(\mathbb{A}_i), \\ S_{\mathbb{A}} &= \max_i S(\mathbb{A}_i). \end{aligned}$$

Theorem 1.1. *Consider labeled graphs $\mathbb{A}$ and $\mathbb{B}$ such that there exists an immersion $\mathbb{A} \to \mathbb{B}$. Suppose that $R(\mathbb{A}) = \{\mathbb{A}_1, \dots, \mathbb{A}_R\}$. Then there exists a computable natural number $N = N(R(\mathbb{A}), \mathbb{B})$ such that some $\mathbb{C} \in \bigcup_{i=1}^{R} \Omega_N(\mathbb{A}_i)$ can be immersed into $\mathbb{B}$.*

Proof. Denote by $\mathrm{Path}(\mathbb{A})$ the set of reduced paths in A up to inversion whose endpoints are special vertices and other vertices are non-special by $\mathrm{Path}(\mathbb{A})$. Then

$$|\mathrm{Path}(\mathbb{A})| \leqslant 3 \cdot C^2_{S(\mathbb{A})}.$$

For $\mathbb{A} \in \bigcup_{i=1}^{R} \Omega_{|V(B)|}(\mathbb{A}_i)$ there is nothing to prove, since otherwise the lengths of some paths in $\mathrm{Path}(\mathbb{A})$, which we denote by p_i for $i = 1, \dots, r$, is greater than $|V(B)|$.

Take an immersion $\Phi\colon \mathbb{A} \to \mathbb{B}$ with graph morphism $\phi\colon A \to B$. From the fact that $|p_i| > |V(B)|$, we see that p_i passes through some vertices $v_i, w_i \in V(B)$ such that $\phi(v_i) = \phi(w_i)$. Collapsing a part of the path p_i connecting the vertices v_i and w_i, we obtain a graph A_1. It is easy to see that ϕ induces a well-defined graph morphism $\phi|_{A_1}\colon A_1 \to B$. Let us choose some non-special vertices $a_i \in p_i$ for $i = 1, \dots, r$ and make the transformations as above (where $v_i = a_i$ or $w_i = a_i$)

while possible. This yields a labeled graph $\mathbb{F} \in \bigcup_{i=1}^{R} \Omega_{|V(B)|}(\mathbb{A}_i)$ with a well-defined morphism $\phi|_{\mathbb{F}}$, which we denote by ϕ for simplicity. It is not hard to check that $\mathbb{A}$ is obtained from $\mathbb{F}$ by replacing the vertices a_i with some paths q_i, with the labels of all edges in q_i equal to 1. Moreover, $\phi(q_i)$ is a cycle in B for all $i = 1, \dots, r$.

Using Proposition 3.18 and Remark 2.14, we can check whether an immersion $\mathbb{F} \to \mathbb{B}$ exists. If the answer is "yes", then the theorem is proved. Otherwise we use Lemma 3.17 and Lemma 3.4 to find a minimal set $(\sigma), (\theta)$ adjusted to $\phi|_{T_F}$ for some maximal subtree T_F.

Below we prove the theorem separately for $r = 1$ and in the general case.

4.1 Case $r = 1$

Assume that $r = 1$. In this case we obtain $\mathbb{A}$ from $\mathbb{F}$ by replacing a vertex a with a path q. All labels on q are equal to 1 and $\phi(q) = q'$ is a cycle in B. Fix basic cycles $m_1, m_2, \dots, m_k$ in B with endpoints in $\phi(a)$ such that $\Delta(m_i) = \alpha_i/\beta_i$ with $\alpha_i \perp \beta_i$ for $i = 1, \dots, k$. Then,

$$\Delta(q') = \prod_{i=1}^{k} \left(\frac{\alpha_i}{\beta_i}\right)^{x_i} = \left(\frac{\alpha}{\beta}\right)$$

with $\alpha \perp \beta$. Our goal is to establish bounds for the integers x_i, $i = 1, \dots, k$, in terms of the parameters of $\mathbb{B}$ and $R(\mathbb{A})$ such that the corresponding extension of $\mathbb{F}$ immerses into $\mathbb{B}$.

The vertex a divides T_F into T_+ and T_- such that

$$T = T_- \cup T_+ \quad \text{and} \quad T_- \cap T_+ = \{a\}.$$

Denote by $\mathbb{D}$ the labeled graph obtained from $\mathbb{F}$ by replacing a vertex a with a path q as above such that $\partial_0(q) = a_1$ and $\partial_1(q) = a_2$. We describe how the minimal set $(\sigma), (\theta)$ determined for T_F changes when we insert the path q. By Proposition 3.10, the numbers σ_a for all $a \in V(q)$ and θ_e for all $e \in E(q)$ are integers if and only if $\beta \cdot \gamma_q \mid \sigma_{a_1}$ if and only if $\alpha \cdot \gamma_{\bar{q}} \mid \sigma_{a_2}$. By Proposition 3.12, it is possible to define ϕ on q such that γ_q divides $\mathrm{LCM}(\gamma_{m_i}) \cdot \prod_{i=1}^{P} p_i^d = H$ and H is independent of x_i. Thus, we can extend the morphism ϕ to D.

Therefore, the minimal set $(\sigma^+), (\theta^+)$ adjusted to $\phi \colon D \to B$ divides element by element a set $(\sigma'), (\theta')$ where

$$\sigma'_c = \begin{cases} \sigma_c \cdot \beta \cdot H, & c \in V(T_+), \\ \sigma_c \cdot \alpha \cdot H, & c \in V(T_-), \end{cases}$$

and we can obtain the other elements of $(\sigma'), (\theta')$ from Definition 2.11 (3) using Lemmas 3.17 and 3.4.

Denote the vertices of q by $v_1 = a_1, v_2, \dots, v_{|q|} = a_2$. Put

$$\Sigma_\pm = \mathrm{GCD}(\sigma_v, v \in T_\pm).$$

To determine the minimal set $(\sigma), (\theta)$ adjusted to $\phi\colon D \to B$, we can find a set $(\sigma''), (\theta'')$ such that

$$\sigma''_c = \begin{cases} \dfrac{\sigma_c \cdot \beta}{\mathrm{GCD}(\Sigma_+ \cdot \beta, \Sigma_- \cdot \alpha, \sigma_{v_i}, i = 1, \dots, |q|)}, & c \in V(T_+), \\ \dfrac{\sigma_c \cdot \alpha}{\mathrm{GCD}(\Sigma_+ \cdot \beta, \Sigma_- \cdot \alpha, \sigma_{v_i}, i = 1, \dots, |q|)}, & c \in V(T_-), \end{cases}$$

where we can find σ_{v_i} from Definition 2.11 (3) using Lemmas 3.17 and 3.4. It is possible that some numbers in $(\sigma''), (\theta'')$ are not integers, but there exists an integer l such that $(l \cdot \sigma''), (l \cdot \theta'')$ is a minimal set adjusted to ϕ. Let us define a fourth set $(\sigma'''), (\theta''')$ such that

$$\sigma'''_c = \begin{cases} \dfrac{\sigma_c \cdot \beta}{\mathrm{GCD}(\Sigma_+ \cdot \beta, \Sigma_- \cdot \alpha)}, & c \in V(T_+), \\ \dfrac{\sigma_c \cdot \alpha}{\mathrm{GCD}(\Sigma_+ \cdot \beta, \Sigma_- \cdot \alpha)}, & c \in V(T_-), \end{cases}$$

and we can find σ_{v_i} as above. It is not hard to see that there exists an integer κ such that $(\kappa \cdot \sigma'''), (\kappa \cdot \theta''')$ coincides with $(\sigma''), (\theta'')$.

To sum up, we can see that there exists some integer κ such that

$$\kappa \mid H \cdot \mathrm{GCD}(\Sigma_+ \cdot \beta, \Sigma_- \cdot \alpha)$$

and

$$\sigma^+_c = \begin{cases} \dfrac{\sigma_c \cdot \beta \cdot \kappa}{\mathrm{GCD}(\Sigma_+ \cdot \beta, \Sigma_- \cdot \alpha)}, & c \in V(T_+), \\ \dfrac{\sigma_c \cdot \alpha \cdot \kappa}{\mathrm{GCD}(\Sigma_+ \cdot \beta, \Sigma_- \cdot \alpha)}, & c \in V(T_-), \end{cases}$$

defines a minimal set $(\sigma^+), (\theta^+)$ adjusted to ϕ.

Every set $(\sigma^*), (\theta^*)$ adjusted to ϕ is equal to $(\tau \cdot \sigma^+), (\tau \cdot \theta^+)$ for some $\tau \in \mathbb{Z}_+$. By Remarks 2.13 and 2.14 as well as Lemma 3.4 (2), there exists an immersion $\mathbb{D} \to \mathbb{B}$ with the morphism ϕ and adjusted set $(\sigma^*), (\theta^*)$ if and only if Lemma 3.4 (2) and (2.2) hold for $(\sigma^*), (\theta^*)$ and T_F.

Let us show that it suffices to verify (2.2) only for the valid pairs (a, f), where $a \in V(A)$ is special. If the vertex a is non-special, then $\mathrm{St}_A(a) = \{e_1, e_2\}$ and $\lambda_{e_1} = \lambda_{e_2} = 1$. Therefore, condition (2.2) is given by $\mu_f = (\sigma^*_a, \mu_f)$, which is equal to $\mu_f \mid \sigma^*_a$. Recall that $(\sigma^*), (\theta^*)$ is adjusted to ϕ; hence, $\theta^*_e = \sigma^*_a / \mu_f$ is an integer.

We will show that Lemma 3.4 (2) and (2.2) are equivalent to some system of linear equations in the variables $x_1, \dots, x_k$.

Denote by $p_1, p_2, \dots, p_P$ the set of all prime divisors of the labels of $\mathbb{A}$ and $\mathbb{B}$. If $\frac{m}{n} \in \langle p_1, p_2, \dots, p_P \rangle_{\mathbb{Q}} \subseteq \mathbb{Q}$, then $\frac{m}{n} = p_1^{r_1} \cdot p_2^{r_2} \cdots p_P^{r_P}$ and we put

$$\psi\left(\frac{m}{n}\right) = (r_1, r_2, \dots, r_P) \in \mathbb{Z}^P.$$

Then we have

$$\psi\left(\frac{\alpha_i}{\beta_i}\right) = \vec{\gamma}_i, \quad \psi\left(\left(\frac{\alpha_i}{\beta_i}\right)^{x_i}\right) = x_i \cdot \vec{\gamma}_i, \quad \psi\left(\frac{\alpha}{\beta}\right) = \sum_{i=1}^{k} x_i \cdot \vec{\gamma}_i.$$

Denote the i-th coordinate of $\vec{v} \in \mathbb{Z}^P$ by $[\vec{v}]_i$. If $\vec{v} \in \mathbb{Z}^P$, then denote the nonnegative part of $\vec{v}$ by $\vec{v}^+$ and the nonpositive part of $\vec{v}$ by $\vec{v}^-$, so that $\vec{v} = \vec{v}^+ + \vec{v}^-$ and $\vec{v}^+, -\vec{v}^- \in \mathbb{Z}^P_{\geqslant 0}$. Since $\alpha \perp \beta$, it follows that

$$\psi(\alpha) = \left(\psi\left(\frac{\alpha}{\beta}\right)\right)^+ \quad \text{and} \quad \psi(\beta) = -\left(\psi\left(\frac{\alpha}{\beta}\right)\right)^-.$$

The values $\psi(\alpha)$ and $\psi(\beta)$ depend on x_i, $i = 1, \dots, k$, but we know that at least one of the 2^P systems

$$\left[\sum_{i=1}^{k} x_i \cdot \vec{\gamma}_i\right]_j \geqslant 0 \text{ (or } \leqslant 0), \quad j = 1, 2, \dots, P, \tag{4.1}$$

holds and determines the values of $\psi(\alpha)$ and $\psi(\beta)$.

Since $\Sigma_+ \perp \Sigma_-$ and $\alpha \perp \beta$, it is easy to see that

$$(\Sigma_+ \cdot \beta, \Sigma_- \cdot \alpha) = (\Sigma_+, \alpha) \cdot (\Sigma_-, \beta).$$

The values of Σ_+ and Σ_- can be effectively computed and depend only on $R(\mathbb{A})$ and $\mathbb{B}$. Lemma 3.4 (4) implies

$$\Sigma_+, \Sigma_- \leqslant (M(\mathbb{F}) \cdot M(\mathbb{B}))^{2|V(F)|},$$

where

$$|V(F)| \leqslant 3 \cdot |V(B)| \cdot C^2_{S_{\mathbb{A}}} \quad \text{and} \quad M(\mathbb{F}) \leqslant M_{\mathbb{A}}.$$

The values of $(\Sigma_+, \alpha), (\Sigma_-, \beta)$ depend on x_i for $i = 1, \dots, k$ and, as above, are determined by one of the 2^{2P} systems

$$\begin{aligned} &[\psi(\alpha)]_j \geqslant [\psi(\Sigma_+)]_j \text{ (or } \leqslant [\Sigma_+]_j), \quad j = 1, 2, \dots, P, \\ &[\psi(\beta)]_j \geqslant [\psi(\Sigma_-)]_j \text{ (or } \leqslant [\Sigma_-]_j), \quad j = 1, 2, \dots, P. \end{aligned} \tag{4.2}$$

The integer κ divides H, which is independent of x_i, $i = 1, \dots, k$, and depends only on $R(\mathbb{A})$ and $\mathbb{B}$. Therefore, the number of possibilities for κ is bounded in terms of $R(\mathbb{A})$ and $\mathbb{B}$.

In the set of equations (2) of Lemma 3.4 we need to replace the symbols σ_v by the fractions

$$\frac{\sigma_v \cdot \beta \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)} \quad \text{or} \quad \frac{\sigma_v \cdot \alpha \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)}$$

depending on $v = \partial_0(e) \in V(T_+)$ or $v = \partial_0(e) \in V(T_-)$. Counting the edges outside T_F, we obtain from Lemma 3.4 (2) at most $P \cdot (E_{\mathbb{A}}/2 - V_{\mathbb{A}} + 1)$ linear equations. Therefore, the following equations hold for all $e \in E(F) \setminus E(T_F)$: if $\partial_0(e) \in V(T_+)$ and $\partial_1(e) \in V(T_-)$, then

$$\begin{aligned} &\left[\psi\left(\frac{\sigma_{\partial_0(e)} \cdot \beta \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)} \cdot \lambda_e \cdot \mu_{\overline{\phi(e)}}\right)\right]_i \\ &\qquad = \left[\psi\left(\frac{\sigma_{\partial_1(e)} \cdot \alpha \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)} \cdot \lambda_{\bar{e}} \cdot \mu_{\phi(e)}\right)\right]_i; \end{aligned} \tag{4.3}$$

if $\partial_1(e) \in V(T_+)$ and $\partial_0(e) \in V(T_-)$, then

$$\begin{aligned} &\left[\psi\left(\frac{\sigma_{\partial_0(e)} \cdot \alpha \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)} \cdot \lambda_e \cdot \mu_{\overline{\phi(e)}}\right)\right]_i \\ &\qquad = \left[\psi\left(\frac{\sigma_{\partial_1(e)} \cdot \beta \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)} \cdot \lambda_{\bar{e}} \cdot \mu_{\phi(e)}\right)\right]_i. \end{aligned} \tag{4.4}$$

Finally, for all special vertices a and edges $f \in \mathrm{St}_B(\phi(a))$ we need to take into consideration the equations and inequalities resulting from (2.2):

$$\left[\psi\left(\lambda_{e_i} \cdot \left(\frac{\sigma_a \cdot \beta \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)}, \mu_f\right)\right)\right]_j = [\psi(\mu_f)], \quad e_i \in \phi_{(a)}^{-1}(f), \tag{4.5}$$

$$k(a, f) \leqslant \left(\frac{\sigma_a \cdot \beta \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)}, \mu_f\right) = \frac{\mu_f}{\lambda_{e_i}}. \tag{4.6}$$

It is easy to see that (4.6) is independent of x_i, $i = 1, \dots, k$, and depends only on ϕ. We can determine the value of

$$\left(\frac{\sigma_a \cdot \beta \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)}, \mu_f\right)$$

in (4.5) in one of 2^P ways as in (4.2) or (4.1):

$$\left[\psi\left(\frac{\sigma_a \cdot \beta \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)}\right)\right]_j \geqslant [\psi(\mu_f)]_j \ \left(\text{or} \leqslant [\psi(\mu_f)]_j\right) \tag{4.7}$$

for $j = 1, 2, \dots, P$.

Therefore, we obtain P linear Diophantine inequalities that determine

$$\left(\frac{\sigma_a \cdot \beta \cdot \kappa \cdot \tau}{(\Sigma_+, \alpha) \cdot (\Sigma_-, \beta)}, \mu_f\right)$$

for each valid pair (a, f). Thus, we have at most $P \cdot S_{\mathbb{A}} \cdot |E(B)|$ such inequalities.

It is easy to see that (4.3) and (4.4) are independent of τ and κ, but (4.5) depends on τ and κ. Nevertheless, it is not hard to see that if (4.5) holds for some τ, then we can choose τ in the set of divisors of $\mathrm{LCM}(\mu_f, f \in E(B))$. Therefore, we need to check only finitely many possibilities for τ whose number is independent of x_i for $i = 1, \dots, k$ and depends only on $\mathbb{B}$.

As a result, there are some systems (their number depends only on $\mathbb{A}$ and $\mathbb{B}$) of equations and inequalities (4.1)–(4.7). Each system consists of

- $P \cdot (3 + 2 \cdot S_{\mathbb{A}} \cdot |E(B)|)$ inequalities,
- $P \cdot S_{\mathbb{A}} \cdot |E(B)| + P \cdot (E_{\mathbb{A}}/2 - V_{\mathbb{A}} + 1)$ equations.

If an extension $\mathbb{D}$ of some $\mathbb{A}_i$ as above exists, then one of the systems is solvable. Conversely, if some system is solvable, then we can use Propositions 3.12 and 3.10 as well as Remark 2.14 to choose the path q and $\mathbb{D}$ so that $\mathbb{D}$ immerses into $\mathbb{B}$. We verify now that the parameters of the system depend only on $\mathbb{A}$ and $\mathbb{B}$ and find its upper estimation.

Let us consider one of these systems. Denote its matrix by Q and its right hand side by $\vec{b}$. Then $\mathrm{rk}(Q) \leqslant k \leqslant |E(B)|/2 - |V(B)| + 1$ because k is equal to the number of columns in Q.

Denote by M the maximal absolute value the entries of Q. We find an upper bound on M. The coefficients of x_i in (4.1) and (4.2) are at most

$$\left[\sum_{i=1}^{k} \gamma_i\right]_j = \sum_{i=1}^{k} [\gamma_i]_j \leqslant \log \mathbb{B} \cdot k. \tag{4.8}$$

The coefficients of x_i in (4.3), (4.4), (4.5), and (4.7) appear only from α and β. Therefore, (4.8) holds.

Let us estimate $|\vec{b}|$. There are no constants in (4.1). Since we already have an upper bound for Σ_+ and Σ_-, it follows that the constants in (4.2) are at most

$$\max\{[\psi(\Sigma_+)]_j, [\psi(\Sigma_-)]_j\} \leqslant (6 \cdot |V(B)| \cdot C^2_{S_{\mathbb{A}}}) \cdot \log_2(M_{\mathbb{A}} \cdot M(\mathbb{B})) = \Sigma.$$

The constants in (4.3) and (4.4) are at most

$$\max_j [\psi(\max_v \sigma_v^2 \cdot \max_e \lambda_e^2 \cdot \max_f \mu_f^2)]_j \leqslant 2\Sigma + 2 \cdot \log \mathbb{B} + 2 \log_{\mathbb{A}}.$$

The constants in (4.5) and (4.7) are at most

$$\max_j[\psi(\max_f \mu_f^2 \cdot \max_e \lambda_e \cdot \max_v \sigma_v \cdot \kappa \cdot \tau \cdot \Sigma_+ \cdot \Sigma_-)]_j$$
$$\leqslant 5\Sigma + (3 \cdot k + 3) \cdot \log \mathbb{B} + \log_{\mathbb{A}},$$

because $\tau \mid \mathrm{LCM}(\mu_f, f \in E(B))$ and $\kappa \mid P(\mathbb{B})^k \prod_{i=1}^k p_i^d \cdot \Sigma_+ \cdot \Sigma_-$, where

$$d = \log \prod_{i=1}^{k} \alpha_i \cdot \beta_i \leqslant 2 \cdot \log \mathbb{B} \cdot k.$$

Therefore, we have

$$|\vec{b}| \leqslant 2 \cdot P \cdot \Sigma + P \cdot (E_{\mathbb{A}}/2 - V_{\mathbb{A}} + 1) \cdot 2 \cdot (\Sigma + \log \mathbb{B} + \log_{\mathbb{A}})$$
$$+ 2 \cdot P \cdot S_{\mathbb{A}} \cdot |E(B)| \cdot (5 \cdot \Sigma + (3 \cdot k + 3) \cdot \log \mathbb{B} + \log_{\mathbb{A}}).$$

Using Proposition 3.16, we easily obtain the required estimate for N.

4.2 The general case

Recall that

$$r \leqslant |\,\mathrm{Path}(\mathbb{A})| \leqslant 3 \cdot C_{S_{\mathbb{A}}}^2.$$

Let us sketch the proof in the general case. Its main idea is the same as in $r = 1$. The labeled graph $\mathbb{A}$ is obtained from $\mathbb{F}$ by replacing vertices a_i with paths q_i with labels 1, for $i = 1, 2, \ldots, r$, such that $\phi\colon q_i \to q_i'$ and q_i' are loops in B. Consider basic loops $m_{1i}, m_{2i}, \ldots, m_{ki}$ be a basic loops in B with origins $\phi(a_i)$. Then

$$\Delta(m_{ji}) = \frac{\alpha_j}{\beta_j}, \quad \text{where } \alpha_j \perp \beta_j,$$

is independent of i. Put $\gamma_{m_j} = \mathrm{LCM}(\gamma_{m_{j1}}, \gamma_{m_{j2}}, \ldots, \gamma_{m_{jr}})$. Since q_j' is a loop in B, it follows that

$$\Delta(q_j') = \prod_{i=1}^{k} \left(\frac{\alpha_i}{\beta_i}\right)^{x_{ij}}.$$

Our goal is to find integers x_{ij} for $i = 1, \ldots, k$ and $j = 1, 2, \ldots, r$ bounded in terms of the parameters of $\mathbb{B}$ and $R(\mathbb{A})$ such that the corresponding extension $\mathbb{D}$ is immersible into $\mathbb{B}$ (such an extension will be defined further using Propositions 3.10 and 3.12).

The vertex a_i divides the tree T_F into T_i^+ and T_i^-, $i = 1, 2, \ldots, r$. Suppose that $(\sigma), (\theta)$ is a minimal set adjusted to ϕ for T_F and write

$$\Delta(q_j') = \prod_{i=1}^{k} \left(\frac{\alpha_i}{\beta_i}\right)^{x_{ij}} = \frac{\omega_j}{\zeta_j},$$

where $\omega_j \perp \zeta_j$ for all $j = 1, 2, \ldots, r$.

The vertices a_i divide the tree T_F into subtrees, which we describe as follows. Given a vertex v of T_F, consider the functions

$$\chi_v(j) \begin{cases} 1 & \text{if } v \in T_j^+, \\ 0 & \text{if } v \in T_j^-, \end{cases} \quad j = 1, 2, \ldots, r.$$

Then $\chi_v = (\chi_v(1), \chi_v(2), \ldots, \chi_v(r))$ defines the position of v relative to a_i for $i = 1, 2, \ldots, r$. Put $V_\chi = \{v : \chi_v = \chi\}$ for all $\chi \in \{0,1\}^r$. It is easy to see that

$$\bigsqcup_{\chi \in \{0,1\}^r} V_\chi = V(T_F).$$

Put $\Sigma_\chi = \mathrm{GCD}(\sigma_v, v \in V_\chi)$. Since $(\sigma), (\theta)$ is a minimal set adjusted to ϕ, we see that $\mathrm{GCD}(\Sigma_\chi, \chi \in \{0,1\}^r) = 1$. Put

$$\Theta = \mathrm{GCD}\left(\prod_{j=1}^{r} \omega_j^{\chi(j)} \cdot \zeta_j^{1-\chi(j)}, \chi \in \{0,1\}^r \right),$$

which depends on x_{ij}.

Applying Lemma 3.19 with

$$a_\chi = \Sigma_\chi \quad \text{and} \quad b_\chi = \frac{1}{\Theta} \prod_{j=1}^{r} \omega_j^{\chi(j)} \cdot \zeta_j^{1-\chi(j)},$$

we see that

$$\mathrm{GCD}\left(\Sigma_\chi \cdot \prod_{j=1}^{r} \omega_j^{\chi(j)} \cdot \zeta_j^{1-\chi(j)}, \chi \in \{0,1\}^r \right) = \Theta \cdot \prod_{\chi \in \{0,1\}^r} \eta_\chi,$$

where $\eta_\chi \mid \Sigma\chi$.

As in the case $r = 1$, a minimal set adjusted to ϕ on $\mathbb{D}$ is given by

$$\frac{\sigma_v \cdot \kappa \cdot \prod_{j=1}^{r} \omega_j^{\chi_v(j)} \cdot \zeta_j^{1-\chi_v(j)}}{\mathrm{GCD}(\Sigma_\chi \cdot \prod_{j=1}^{r} \omega_j^{\chi(j)} \cdot \zeta_j^{1-\chi(j)}, \chi \in \{0,1\}^r)} = \frac{\sigma_v \cdot \kappa \cdot \frac{\prod_{j=1}^{r} \omega_j^{\chi_v(j)} \cdot \zeta_j^{1-\chi_v(j)}}{\Theta}}{\prod_{\chi \in \{0,1\}^r} \eta_\chi},$$

where

$$\kappa \mid \mathrm{LCM}(\gamma_{m_1}, \gamma_{m_2}, \ldots, \gamma_{m_k}) \cdot \prod_{i=1}^{P} p_i^d \prod_{\chi \in \{0,1\}^r} \eta_\chi.$$

Let us now find a system of linear equalities and inequations in variables x_{ij} for $i = 1, \ldots, k$ and $j = 1, \ldots, r$.

List all prime divisors of the labels of the graphs $\mathbb{A}$ and $\mathbb{B}$ as $p_1, p_2, \ldots, p_P$. If $\frac{m}{n} \in \langle p_1, p_2, \ldots, p_P \rangle \subseteq \mathbb{Q}$, then $\frac{m}{n} = p_1^{r_1} \cdot p_2^{r_2} \cdots p_P^{r_P}$ and we put

$$\psi\left(\frac{m}{n}\right) = (r_1, r_2, \ldots, r_P) \in \mathbb{Z}^P$$

as above. Then

$$\psi\left(\frac{\alpha_i}{\beta_i}\right) = \vec{\gamma}_i, \quad \psi\left(\left(\frac{\alpha_i}{\beta_i}\right)^{x_{ij}}\right) = x_{ij} \cdot \vec{\gamma}_i, \quad \psi\left(\frac{\omega_j}{\zeta_j}\right) = \sum_{i=1}^{k} x_{ij} \cdot \vec{\gamma}_i.$$

For $\vec{v} \in \mathbb{Z}^P$ denote the nonnegative part of $\vec{v}$ by $\vec{v}^+$ and the nonpositive part of $\vec{v}$ by $\vec{v}^-$ such that $\vec{v} = \vec{v}^+ + \vec{v}^-$ and $\vec{v}^+, -\vec{v}^- \in \mathbb{Z}^P_{\geqslant 0}$. Since $\omega_j \perp \zeta_j$, it follows that

$$\psi(\omega_j) = \left(\psi\left(\frac{\omega_j}{\zeta_j}\right)\right)^+ \quad \text{and} \quad \psi(\zeta_j) = -\left(\psi\left(\frac{\omega_j}{\zeta_j}\right)\right)^- \quad \text{for } j = 1, \ldots, r.$$

The values of $\psi(\omega_j)$ and $\psi(\zeta_j)$ depend on $x_1, x_2, \ldots, x_k$, but we know that at least one of the $2^{P \cdot r}$ systems

$$\left[\sum_{i=1}^{k} x_{ij} \cdot \gamma_i\right]_s \geqslant 0 \text{ (or } \leqslant 0), \quad s = 1, 2, \ldots, P, \ j = 1, \ldots, r, \tag{4.1$'$}$$

is satisfied. Therefore, the values of $\psi(\omega_j)$ and $\psi(\zeta_j)$ for all $j = 1, \ldots, r$ can be determined by a system of $P \cdot r$ inequalities.

The value of Θ depends on the variables x_{ij}. To determine Θ we introduce $[\Theta]_i$ for $i = 1, \ldots, P$. The value of $[\Theta]_i$ can be determined by one of the 2^r systems

$$\left[\psi\left(\prod_{j=1}^{r} \omega_j^{\chi_i(j)} \cdot \zeta_j^{1-\chi_i(j)}\right)\right]_i \leqslant \left[\psi\left(\prod_{j=1}^{r} \omega_j^{\chi(j)} \cdot \zeta_j^{1-\chi(j)}\right)\right]_i, \quad \chi \neq \chi_i. \tag{4.2$'$}$$

Therefore, Θ can be defined by system of $(2^r - 1) \cdot P$ inequalities.

The integer κ divides H, which is independent of x_{ij} and depends only on $R(\mathbb{A})$ and $\mathbb{B}$. Therefore, the number of possibilities for κ has an upper bound in terms of the parameters of $R(\mathbb{A})$ and $\mathbb{B}$.

The numbers Σ_χ are independent of x_{ij} and depend only on $R(\mathbb{A})$ and $\mathbb{B}$. Therefore, the number of possibilities for η_χ has an upper bound in terms of the parameters of $R(\mathbb{A})$ and $\mathbb{B}$.

Denote the analogue of equalities and inequations (4.3)–(4.7) with $r \neq 1$ by (4.3)′–(4.7)′. As a result, there are some systems (their number depends only on $R(\mathbb{A})$ and $\mathbb{B}$) of equations and inequalities (4.1)′–(4.7)′. Each system consists of

- $P \cdot (r + 2^r - 1 + S_{\mathbb{A}} \cdot |E(B)|)$ inequalities,
- at most $P \cdot (E_{\mathbb{A}}/2 - V_{\mathbb{A}} + 1 + S_{\mathbb{A}} \cdot |E(B)|)$ equations.

If an extension $\mathbb{F}$ of some $\mathbb{A}_i$ as above exists, then one of the systems is solvable. Conversely, if some system is solvable, then we can use Propositions 3.12 and 3.10 to choose the path q and an extension $\mathbb{D}$ of some $\mathbb{A}_i$ immersible into $\mathbb{B}$. We verify that the parameters of the system depend only on $\mathbb{A}$ and $\mathbb{B}$ and obtain its upper bound.

Consider one of these systems. Denote its matrix by Q and its right hand side by $\vec{b}$. Then $\mathrm{rk}(Q) \leqslant k \cdot r \leqslant (|E(B)|/2 - |V(B)| + 1) \cdot 3 \cdot C^2_{S_{\mathbb{A}}}$ because k is equal to the number of columns in Q.

Denote by M the maximal absolute value the entries of Q. We find an upper bound on M. The coefficients of x_i in $(4.1)'$ and $(4.2)'$ are at most

$$4 \cdot r \cdot \left[\sum_{i=1}^{k} \gamma_i\right]_j = 4 \cdot r \cdot \sum_{i=1}^{k} [\gamma_i]_j \leqslant 4 \cdot \log \mathbb{B} \cdot k \cdot r. \qquad (4.8)'$$

The coefficients of x_{ij} in $(4.3)'$, $(4.4)'$, $(4.5)'$, and $(4.7)'$ appear only from ω_j, ζ_j, and Θ. Therefore, $(4.8)'$ holds.

Let us estimate $|\vec{b}|$. There are no constants in $(4.1)'$–$(4.2)'$. By Lemma 3.4 (4),

$$\max_{\chi} [\psi(\Sigma_\chi)]_j \leqslant (6 \cdot |V(B)| \cdot C^2_{S_{\mathbb{A}}}) \cdot \log_2(M_{\mathbb{A}} \cdot M(\mathbb{B})) = \Sigma.$$

The constants in $(4.3)'$ and $(4.4)'$ are at most

$$\max_j [\psi(\max_v \sigma_v^2 \cdot \max_e \lambda_e^2 \cdot \max_f \mu_f^2)]_j \leqslant 2 \cdot \Sigma + 2 \cdot \log \mathbb{B} + 2 \log_{\mathbb{A}}.$$

The constants in $(4.5)'$ and $(4.7)'$ are at most

$$\max_j [\psi(\max_f \mu_f^2 \cdot \max_e \lambda_e \cdot \max_v \sigma_v \cdot \kappa \cdot \tau \cdot (\max_\chi \Sigma_\chi)^{2^r}]_j$$
$$\leqslant (2 \cdot 2^r + 1)\Sigma + (3 \cdot k + 3) \cdot \log \mathbb{B} + \log_{\mathbb{A}},$$

because $\tau \mid \mathrm{LCM}(\mu_f, f \in E(B))$ and $\kappa \mid P(\mathbb{B})^k \prod_{i=1}^{k} p_i^d \cdot \prod_{\chi \in \{0,1\}^r} \eta_\chi$, where

$$d = \log \prod_{i=1}^{k} \omega_i \zeta_i \leqslant 2 \cdot \log \mathbb{B} \cdot k.$$

Therefore, we have

$$|\vec{b}| \leqslant P \cdot (E_{\mathbb{A}}/2 - V_{\mathbb{A}} + 1) \cdot 2 \cdot (\Sigma + \log \mathbb{B} + \log_{\mathbb{A}})$$
$$+ 2 \cdot P \cdot S_{\mathbb{A}} \cdot |E(B)| \cdot ((2^{r+1} + 1) \cdot \Sigma + (3 \cdot k + 3) \cdot \log \mathbb{B} + \log_{\mathbb{A}}).$$

Using Proposition 3.16, we easily obtain the required estimate. □

5 The algorithm

Now we are ready to prove Theorem 1.2. Suppose we are given labeled graphs $\mathbb{A}$ and $\mathbb{B}$. In order to check whether $\pi_1(\mathbb{A})$ embeds into $\pi_1(\mathbb{B})$ it is suffices to execute the following steps.

- **Step 0.** Reduce $\mathbb{B}$ by elementary moves.
- **Step 1.** Check whether $|R(\mathbb{A})|$ is finite using the algorithm from [3]. If the answer is positive, then go to Step 2, otherwise stop, giving no answer.
- **Step 2.** Find $R(\mathbb{A})$ using standard transformations.
- **Step 3.** Compute N using Theorem 1.1. Then find $\Omega_N(\mathbb{A}_i)$ for all $\mathbb{A}_i$ for $i = 1, \dots, |R(\mathbb{A})|$. By Proposition 3.8 all these sets are finite and $|\Omega_N(\mathbb{A}_i)|$ depends only on $\mathbb{A}$ and $\mathbb{B}$.
- **Step 4.** For all $\mathbb{C}$ in $\bigcup_i \Omega_N(\mathbb{A}_i)$ check whether an immersion $\mathbb{C} \to \mathbb{B}$ exists. Proposition 3.18 shows that it takes finitely many operations. Once a positive answer occurs for some $\mathbb{C}$ in $\bigcup_i \Omega_N(\mathbb{A}_i)$, the algorithm says "yes" and stops. Otherwise, by Theorem 1.1 the algorithm says "no" and stops.

Acknowledgments. The author is grateful to V. A. Churkin for valuable comments and advice.

Bibliography

[1] H. Bass, Covering theory for graphs of groups, *J. Pure Appl. Algebra* **89** (1993), no. 1, 3–47.

[2] M. Clay, Deformation spaces of G-trees and automorphisms of Baumslag–Solitar groups, *Groups Geom. Dyn.* **3** (2009), 39–69.

[3] M. Clay and M. Forester, On the isomorphism problem for generalized Baumslag–Solitar groups, *Algebr. Geom. Topol.* **8** (2008), 2289–2322.

[4] M. Clay and M. Forester, Whitehead moves for G-trees, *Bull. Lond. Math. Soc.* **41** (2009), no. 2, 205–212.

[5] F. A. Dudkin, Subgroups of Baumslag–Solitar groups, *Algebra Logic* **48** (2009), no. 1, 1–19.

[6] F. A. Dudkin, Subgroups of finite index in Baumslag–Solitar groups, *Algebra Logic* **49** (2010), no. 3, 221–232.

[7] F. A. Dudkin, Embedding of Baumslag–Solitar groups into the generalized Baumslag–Solitar groups, *Sib. Math. J.* **55** (2014), no. 1, 72–77.

[8] M. Forester, Deformation and rigidity of simplicial group actions on trees, *Geom. Topol.* **6** (2002), 219–267.

[9] M. Forester, On uniquenes of JSJ decomposition of finitely generated groups, *Comment. Math. Helv.* **78** (2003), 740–751.

[10] M. Forester, Splittings of generalized Baumslag–Solitar groups, *Geom. Dedicata* **121** (2006), no. 1, 43–59.

[11] G. Levitt, On the automorphism group of generalized Baumslag–Solitar groups, *Geom. Topol.* **11** (2007) 473–515.

[12] G. Levitt, Quotients and subgroups of Baumslag–Solitar groups, preprint (2013), `http://arxiv.org/abs/1308.5122`.

[13] J. P. Serre, *Trees*, Springer, Berlin, 1980.

Received April 13, 2014; revised November 18, 2014.

Author information

Fedor A. Dudkin, Sobolev Institute of Mathematics, Novosibirsk State University, Novosibirsk, Russia.
E-mail: `dudkinf@ngs.ru`