J. Group Theory **20** (2017), 289–307
DOI 10.1515/jgth-2016-0026

Subgroup Isomorphism Problem for units of integral group rings

Leo Margolis

Communicated by Evgenii I. Khukhro

Abstract. The Subgroup Isomorphism Problem for Integral Group Rings asks for which finite groups U it is true that if U is isomorphic to a subgroup of $\mathrm{V}(\mathbb{Z}G)$, the group of normalized units of the integral group ring of the finite group G, it must be isomorphic to a subgroup of G. The smallest groups known not to satisfy this property are the counterexamples to the Isomorphism Problem constructed by M. Hertweck. However, the only groups known to satisfy it are cyclic groups of prime power order and elementary-abelian p-groups of rank 2. We give a positive solution to the Subgroup Isomorphism Problem for $C_4 \times C_2$. Moreover, we prove that if the Sylow 2-subgroup of G is a dihedral group, any 2-subgroup of $\mathrm{V}(\mathbb{Z}G)$ is isomorphic to a subgroup of G.

1 Introduction

The first to study the unit group of an integral group ring $\mathbb{Z}G$ of a finite group G was G. Higman in his PhD thesis [20]. There, in Section 5, he states it to be "plausible" that any finite subgroup of units U is isomorphic to a group of trivial units, i.e. units of the form $\pm g$ with $g \in G$. See [30] for details on the thesis. Denote by $\mathrm{V}(\mathbb{Z}G)$ the normalized units of $\mathbb{Z}G$, i.e. the units whose coefficients sum up to 1. In view of the fact that $\pm\mathrm{V}(\mathbb{Z}G)$ are all the units of $\mathbb{Z}G$ it is enough to study normalized units.

After Higman's question remained open for quite a long time, a negative answer to it was given by M. Hertweck's counterexample to the isomorphism problem [16]: There are finite groups G and H such that $\mathbb{Z}G$ is isomorphic to $\mathbb{Z}H$, but G and H are not isomorphic. In particular, $\mathrm{V}(\mathbb{Z}G)$ contains a subgroup isomorphic to H, but G does not, and thus provides a counterexample to the question of Higman. However Hertweck's counterexample is a rather complicated group of or order $2^{21} \cdot 97^{28}$ and derived length 4 and leaves Higman's question totally open for smaller and/or "easier" groups. This leads to the following question.

This research was partly supported by the Sonderforschungsbereich 701 at the University of Bielefeld.

Subgroup Isomorphism Problem (SIP). For which finite groups U does the following statement hold: If $\mathrm{V}(\mathbb{Z}G)$ contains a subgroup isomorphic to U for some finite group G, then G contains a subgroup isomorphic to U?

Not explicitly studied, this was only known to hold for cyclic groups of prime power order, following from a result of Cohn and Livingstone that the exponents of G and $\mathrm{V}(\mathbb{Z}G)$ coincide [9, Corollary 4.1]. Z. Marciniak raised the question for the Klein four group at an ICM Satellite Conference in 2006 and W. Kimmerle immediately gave a positive answer [25], followed by a positive answer for elementary abelian p-groups of rank 2 for odd primes p by M. Hertweck [18]. Since then no further progress has been made on the question. The name "Subgroup Isomorphism Problem" was coined by W. Kimmerle in several talks and is also recorded in [26]. The problem appeared for the first time in the literature in [24, Problem 19]. Denote by C_n the cyclic group of order n. The purpose of this article is the proof of the following result:

Theorem 1.1. *Let G be a finite group. If $\mathrm{V}(\mathbb{Z}G)$ contains a subgroup isomorphic to $C_4 \times C_2$, then so does G, i.e. the Subgroup Isomorphism Problem has a positive solution for $C_4 \times C_2$.*

Doing just a little more we obtain also the following results, being of independent interest. The investigation of the following Theorem was proposed in [18, Comment before Example 7].

Theorem 1.2. *Let G be a finite group possessing a dihedral Sylow 2-subgroup. Then any 2-subgroup of $\mathrm{V}(\mathbb{Z}G)$ is isomorphic to a subgroup of G.*

Moreover, we obtain the following generalisation of [3, Proposition 3.4] and [26, Proposition 4.7].

Proposition 1.3. *Let G be a finite group whose Sylow 2-subgroup has at most order 8 and assume moreover that G is not isomorphic to the alternating group of degree 7. Then any 2-subgroup U of $\mathrm{V}(\mathbb{Z}G)$ is rationally conjugate to a subgroup of G, i.e. there exists a unit x in the rational group algebra $\mathbb{Q}G$ and a subgroup P of G such that $x^{-1}Ux = P$.*

2 Preliminaries

The following lemma from elementary group theory is the key observation to allow the proof of Theorem 1.1.

Lemma 2.1. *Let P be a finite 2-group such that P contains no subgroup isomorphic to $C_4 \times C_2$. Then P is one of the following: Elementary-abelian, cyclic, a generalized quaternion group, a dihedral group or a semidihedral group.*

Remark. Note that semidihedral groups are called "quasidihedral" groups by many authors. A presentation of a semidihedral group of order 2^{n+1} is given by

$$\langle h, a \mid h^{2^n} = a^2 = 1,\ h^a = h^{-1+2^{n-1}} \rangle.$$

For definitions of all groups mentioned in the lemma see [27, Section 5.3].

Proof. If P is abelian, the statement is clear and if P is not abelian and contains only one involution, P is a quaternion group [27, 5.3.7]. So assume P is not abelian and contains at least two involutions. Since P then contains an element of order 4, say g, we have $C_2 \cong Z(P) = \langle g^2 \rangle$. Assume P possesses a non-cyclic abelian normal subgroup M, which must be elementary-abelian, and let a be an element of M which is not central in P. Then $C_P(M) = N$ is also normal, elementary-abelian and contains g^2. If the order of N is 4, we have $M = N = C_P(a) \cong C_2 \times C_2$ and P is a dihedral or semidihedral group by [27, 5.3.10]. So assume $|N| \geq 8$ and let b be an element of N which is not central in P such that ab is also not central in P. Since $a^g a$ commutes with g, we have $a^g = ag^2$ and in the same way $b^g = bg^2$. But then $(ab)^g = ab$, contradicting the existence of N and M. Hence there is no non-cyclic abelian normal subgroup in P and P is dihedral or semidihedral by [27, 5.3.9]. □

Today an important open question concerning torsion units in $\mathbb{Z}G$ is the Zassenhaus Conjecture. It states that for every finite group G and every torsion unit $u \in \mathrm{V}(\mathbb{Z}G)$ there exist a unit x in the rational group algebra $\mathbb{Q}G$ and an element $g \in G$ such that $x^{-1}ux = g$. In this case one says that u and g are rationally conjugate. The main notion to study torsion subgroups of $\mathbb{Z}G$ are the so-called partial augmentations. For a conjugacy class x^G in G and an element $u = \sum_{g \in G} z_g g$ in $\mathbb{Z}G$ the sum

$$\varepsilon_x(u) = \sum_{g \in x^G} z_g$$

is called the partial augmentation of u with respect to x. The connection between partial augmentations and the Zassenhaus Conjecture is established by [29, Theorem 2.5]: A torsion unit $u \in \mathrm{V}(\mathbb{Z}G)$ of order n is rationally conjugate to an element of G if and only if $\varepsilon_x(u^d) \geq 0$ for all $x \in G$ and divisors d of n. If u is a torsion unit and x is an element of G of order not dividing the order of u, then $\varepsilon_x(u) = 0$ (see [17, Theorem 2.3]). Moreover, $\varepsilon_z(u) = 0$ for every central element z of G, unless $u = z$ by the Berman–Higman Theorem [23, Proposition 1.5.1]. These facts will be used in the rest of the paper without further mention.

A method to study the torsion units in integral group rings was introduced in [28] for ordinary characters and later generalized in [17] for Brauer characters and has, in recent years, became well known as the HeLP-method. Though developed to study finite cyclic subgroups of $\mathrm{V}(\mathbb{Z}G)$ its idea does also work for non-cyclic groups and has been applied for ordinary characters a few times (namely in [21], [19], [18] and [5]). Since for non-cyclic groups Brauer characters have not been used so far, the following lemma seems necessary. Note that for a prime p and a p-Brauer character φ of a finite group G with corresponding representation D we can extend the domain of φ to the torsion units of $\mathrm{V}(\mathbb{Z}G)$ of order prime to p by linearly extending D to $\mathbb{Z}G$, see [17, Section 3].

Lemma 2.2. *Let G be a finite group, U a finite subgroup of $\mathrm{V}(\mathbb{Z}G)$ of order n and p a prime not dividing n. Let χ be the extension of a p-Brauer character of G to the p-regular elements of $\mathrm{V}(\mathbb{Z}G)$ and let ψ be an ordinary character of U.*

(a) [17, Theorem 3.2] *For every $u \in U$ we have*

$$\chi(u) = \sum_{\substack{x^G \\ x\ p\text{-regular}}} \varepsilon_x(u)\chi(x).$$

(b) $\frac{1}{n}\sum_{u\in U}\chi(u)\psi(u^{-1})$ *is a non-negative integer.*

Proof. Since p does not divide n, there is an ordinary character χ' of U such that $\chi'(u) = \chi(u)$ for every $u \in U$ by [11, Corollary 18.11]. So

$$\frac{1}{n}\sum_{u\in U}\chi(u)\psi(u^{-1}) = \langle \chi', \psi\rangle_U$$

equals the scalar product of two ordinary characters and thus it is a non-negative integer. □

The proof of Theorem 1.1 will be achieved by checking all possible cases for Sylow 2-subgroups given by Lemma 2.1. The knowledge so far is recorded in [26].

Proposition 2.3 ([26, Theorem 4.1, Proposition 4.8]). *Let G be a finite group whose Sylow 2-subgroup is either abelian or a (generalized) quaternion group. Then any 2-subgroup of $\mathrm{V}(\mathbb{Z}G)$ is isomorphic to a subgroup of G.*

Groups possessing dihedral or semidihedral Sylow 2-subgroups were classified in a series of articles [14] (see also [6] for an alternative proof) and [2]. These will be the basis for the rest of the proofs. We will denote by $O_{2'}(G)$ the biggest normal subgroup of odd order of a group G. The following Lemma follows from [12, Lemma 2.1, Theorem 2.2].

Lemma 2.4. *Let G be a finite group, N a normal subgroup of G and U a finite subgroup of* $\mathrm{V}(\mathbb{Z}G)$ *such that the orders of N and U are coprime. Then U is rationally conjugate to a subgroup of G if and only if the image of U in* $\mathrm{V}(\mathbb{Z}(G/N))$ *is rationally conjugate a subgroup of G/N. Moreover,* $\mathrm{V}(\mathbb{Z}(G/N))$ *possesses a subgroup isomorphic to U.*

In the proofs of the main results we will need a lot of information about finite classical groups. We are not going to cite the original literature here of Dickson, Schur, Dieudonné, Brauer, just to mention a few, but literature which is more accessible and easier to understand nowadays. We will also mostly avoid the use of any character tables accessible in the literature, using natural representations of the groups involved instead.

3 Proof of main results

The first lemma concerns a big class of the groups involved in the results.

Lemma 3.1. *Let G be a subgroup of* $\mathrm{P\Gamma L}(2,q) = \mathrm{Aut}(\mathrm{PSL}(2,q))$ *containing* $\mathrm{PSL}(2,q)$ *or* $\mathrm{PGL}(2,q)$ *as a normal subgroup of odd index, where q is an odd prime power. Then units of order* 2 *and* 4 *in* $\mathrm{V}(\mathbb{Z}G)$ *are rationally conjugate to elements of G. In particular, squares of units of order* 4 *from* $\mathrm{V}(\mathbb{Z}G)$ *are rationally conjugate to involutions in* $\mathrm{PSL}(2,q)$.

Proof. The following facts are given in [14, Lemma 3.1]. The Sylow 2-subgroup of G is a dihedral group or an elementary abelian-group of rank 2. The group $\mathrm{PSL}(2,q)$ possesses exactly one conjugacy class of involutions, say $2a$, while the group $\mathrm{PGL}(2,q)$ possesses exactly two conjugacy classes of involutions, the one not lying in $\mathrm{PSL}(2,q)$ will be called $2b$. If $\mathrm{PGL}(2,q)$ is not a normal subgroup of G, the conjugacy class $2b$ can be ignored in the following computations. If G possesses no elements of order 4, then $\mathrm{PGL}(2,q)$ is not a subgroup of G and there exists exactly one conjugacy class of involutions in G. Since for $u \in \mathrm{V}(\mathbb{Z}G)$ of order 2 and $x \in G$ of order different from 2 we have $\varepsilon_x(u) = 0$, this immediately yields the desired result. So assume that G contains elements of order 4 and denote by $4a$ the conjugacy class of these elements. There is exactly one such class, since this is the case in a dihedral group. The class $4a$ may lie in $\mathrm{PSL}(2,q)$ or outside of it and these two cases will be partly separated in the following computations.

If $\mathrm{PGL}(2,q)$ is a subgroup of G, then $G/\mathrm{PSL}(2,q)$ has a Sylow 2-subgroup of order 2. Hence it possesses a normal 2-complement by [27, 7.2.2]. Thus the group G maps onto a cyclic group of order 2 such that $\mathrm{PSL}(2,q)$ is in the kernel of this map and elements of $\mathrm{PGL}(2,q)$ outside of $\mathrm{PSL}(2,q)$ are not. Hence the group G possesses a 1-dimensional representation mapping $\mathrm{PSL}(2,q)$ to 1 and elements of $\mathrm{PGL}(2,q)$ outside of $\mathrm{PSL}(2,q)$ to -1. The character corresponding to this representation will be called χ. If $\mathrm{PGL}(2,q)$ is not a subgroup of G, the character χ can be ignored.

Moreover, $\mathrm{GL}(2,q)$ acts via conjugation on the 2×2-matrices over $\mathbb{F}_q$ with trace 0. The kernel of this action is exactly the centre of $\mathrm{GL}(2,q)$, thus inducing a 3-dimensional representation of $\mathrm{PGL}(2,q)$. The character corresponding to the induced representation on G and its twist with χ will be called ψ_+ and ψ_-. Note that these characters take only integral values on the classes we are interested in. Let N be the normal subgroup of G which is $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$ such that $[G:N]=m$ is odd. The characters described above take the values given in Table 1 and Table 2 on the classes of interest.

	$1a$	$2a$	$4a$	$2b$
χ	1	1	1	-1
ψ_+	$3m$	$-m$	m	$-m$
ψ_-	$3m$	$-m$	m	m

Table 1. Some characters of G, if elements of order 4 are in $\mathrm{PSL}(2,q)$.

	$1a$	$2a$	$4a$	$2b$
χ	1	1	-1	-1
ψ_+	$3m$	$-m$	m	$-m$
ψ_-	$3m$	$-m$	$-m$	m

Table 2. Some characters of G, if elements of order 4 are not in $\mathrm{PSL}(2,q)$.

Let $u\in \mathrm{V}(\mathbb{Z}G)$ be of order 2. We have $\varepsilon_{2a}(u)+\varepsilon_{2b}(u)=1$. Moreover, from χ we obtain

$$\chi(u)=\varepsilon_{2a}(u)-\varepsilon_{2b}(u)\in\{\pm 1\}.$$

Thus $(\varepsilon_{2a}(u),\varepsilon_{2b}(u))\in\{(1,0),(0,1)\}$, proving the claim for units of order 2.

Now let $u\in \mathrm{V}(\mathbb{Z}G)$ be of order 4. Note that since $\chi(u)$ is integral, $\chi(u)\in\{\pm 1\}$ and thus $\chi(u^2)=1$ meaning that u^2 is rationally conjugate to elements in $2a$. We have

$$\varepsilon_{2a}(u)+\varepsilon_{2b}(u)+\varepsilon_{4a}(u)=1. \tag{3.1}$$

Let D be a representation corresponding to ψ_+ or ψ_-. Since $\psi_-(u^2)=\psi_+(u^2)=\psi_+(2a)=-m$ the eigenvalues of $D(u^2)$, with multiplicities, are $2m$ times -1

and m times 1. Thus $2m$ of the eigenvalues of $D(u)$ are i or $-i$, where i denotes a primitive 4th root of unity in $\mathbb{F}_{q^2}$. But since ψ_+ and ψ_- do only take real values, m eigenvalues of $D(u)$ must be i and m must be $-i$. In particular, we have $-m \leq \psi_+(u) \leq m$ and $-m \leq \psi_-(u) \leq m$.

Assume firstly that $4a$ is lies in $\mathrm{PSL}(2,q)$. Then using χ we obtain

$$\chi(u) = \varepsilon_{2a}(u) - \varepsilon_{2b}(u) + \varepsilon_{4a}(u) \in \{\pm 1\}.$$

Subtracting (3.1) this gives $\varepsilon_{2b}(u) \in \{0,1\}$. Using ψ_+ and ψ_- we obtain

$$-m \leq -m\varepsilon_{2a}(u) + m\varepsilon_{4a}(u) - m\varepsilon_{2b}(u) \leq m$$

and

$$-m \leq -m\varepsilon_{2a}(u) + m\varepsilon_{4a}(u) + m\varepsilon_{2b}(u) \leq m$$

respectively. Adding or subtracting (3.1) gives $\varepsilon_{4a}(u) \in \{0,1\}$ and $\varepsilon_{2a}(u) \in \{0,1\}$ respectively. Since we have $\varepsilon_{4a}(u) \not\equiv 0 \bmod 2$ and $\varepsilon_{2a}(u) \equiv \varepsilon_{2b}(u) \equiv 0 \bmod 2$ by [9, Theorem 4.1], this implies the equality $(\varepsilon_{2a}(u), \varepsilon_{2b}(u), \varepsilon_{4a}(u)) = (0,0,1)$.

Now assume $4a$ is not in $\mathrm{PSL}(2,q)$. Doing analogous computations to the above χ gives $\varepsilon_{2a}(u) \in \{0,1\}$, while ψ_+ and ψ_- give $\varepsilon_{2b}(u) \in \{0,1\}$ and $\varepsilon_{4a}(u) \in \{0,1\}$ respectively. By [9, Theorem 4.1] however $\varepsilon_{4a}(u) \not\equiv 0 \bmod 2$ and thus we have $(\varepsilon_{2a}(u), \varepsilon_{2b}(u), \varepsilon_{4a}(u)) = (0,0,1)$, proving that units of order 4 are rationally conjugate to elements of G. □

We are now ready to prove the main results.

Proof of Theorem 1.2. By Lemma 2.4 we may assume that $O_{2'}(G) = 1$. So by [6, Theorem of Gorenstein and Walter] one of the following three cases may be assumed:

(i) G is a subgroup of $\mathrm{P\Gamma L}(2,q)$ containing $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$ as a normal subgroup of odd index m. Here q denotes an odd prime power.

(ii) G is the alternating group of degree 7.

(iii) G is a 2-group.

In case (iii) the result follows from [32, Theorem 1] and case (ii) has been handled in [18, Example 7]. So (i) is the only remaining case and we are exactly in the situation of Lemma 3.1. We may moreover assume that G contains elements of order 4, since otherwise the result follows from Proposition 2.3. We will frequently use Lemma 3.1 and the characters from Tables 1 and 2.

Assume $\mathrm{V}(\mathbb{Z}G)$ contains a subgroup U isomorphic to $C_4 \times C_2$. Note that U contains three involutions and four elements of order 4 and ψ_+ takes the value $-m$

on involutions of U while taking the value m on all elements of order 4 in U. Let $\mathbf{1}$ be the trivial character of U. Then by Lemma 2.2

$$\frac{1}{8}\sum_{u\in U}\psi_+(u)\mathbf{1}(u^{-1}) = \frac{1}{8}(3m - 3m + 4m) = \frac{m}{2}$$

is a non-negative integer, contradicting the existence of U since m is odd.

Now in view of Lemma 2.1 and the fact that the exponents of G and $\mathrm{V}(\mathbb{Z}G)$ coincide, it only remains to show that there are no elementary-abelian groups of rank bigger than 2, quaternion or semidihedral groups in $\mathrm{V}(\mathbb{Z}G)$. Since every quaternion and semidihedral group contains a quaternion group of order 8, it will be enough to prove that $\mathrm{V}(\mathbb{Z}G)$ does not contain neither a subgroup isomorphic to $C_2 \times C_2 \times C_2$ nor Q_8. Let U be an elementary-abelian group of rank 3. Then $\psi_+(u) = -m$ for every $u \neq 1$ in U and by Lemma 2.2

$$\frac{1}{8}\sum_{u\in U}\psi_+(u)\mathbf{1}(u^{-1}) = \frac{1}{8}(3m - 7m) = \frac{-m}{2}$$

is a non-negative integer, contradicting the existence of U.

Finally, let $U \leq \mathrm{V}(\mathbb{Z}G)$ be a quaternion group of order 8. Note that U contains exactly one involution and six elements of order 4. Since the involution in U is a square of an element of order 4, it is rationally conjugate to elements in $2a$. We claim that $4a$ is in $\mathrm{PSL}(2,q)$. Indeed, otherwise

$$\frac{1}{8}\sum_{u\in U}\chi(u)\mathbf{1}(u^{-1}) = \frac{1}{8}(2-6) = \frac{-1}{2},$$

contradicting Lemma 2.2. So assume that $4a$ lies in $\mathrm{PSL}(2,q)$. Note that since the 2-Sylow subgroup of $\mathrm{PSL}(2,q)$ is dihedral and $|\mathrm{PSL}(2,q)| = \frac{(q-1)q(q+1)}{2}$, this is equivalent to $q \equiv \pm 1 \bmod 8$. Any irreducible ordinary character of degree $q \pm 1$ of $\mathrm{PSL}(2,q)$ may be extended to a character of $\mathrm{PGL}(2,q)$ by [33, Lemma 4.5], meaning it has the same values on the elements lying in $\mathrm{PSL}(2,q)$. If $\mathrm{PGL}(2,q)$ is not a normal subgroup of G, this fact may just be ignored. In any case every ordinary irreducible character η' of degree $q \pm 1$ of $\mathrm{PSL}(2,q)$ can be induced to a character η of G such that $\eta(1a) = m \cdot \eta'(1a)$, $\eta(2a) = m \cdot \eta'(2a)$ and $\eta(4a) = m \cdot \eta'(4a)$. Since in $\mathrm{PGL}(2,q)$ every element is conjugate to its inverse, the extension of η' to $\mathrm{PGL}(2,q)$ is real-valued. Moreover, the Schur index of this extension is 1 by [15, Theorem 2 (a)] and hence it is affordable by a real representation and then so is η.

We will apply the same arguments as in the last paragraph of the proof of [19, Theorem 2.1]. Let $\varepsilon \in \{\pm 1\}$ such that $q \equiv \varepsilon \bmod 4$. The ordinary character table of $\mathrm{PSL}(2,q)$ was first computed independently by Schur and Jordan and is also given in [19, Table 1]. Let η' be an irreducible ordinary character of $\mathrm{PSL}(2,q)$ of degree $q+\varepsilon$ such that $\eta'(2a) = -2\varepsilon$ and $\eta'(4a) = 0$. Let η be the character of G obtained from η' the way described in the preceding paragraph. Then $\eta(1a) = m(q+\varepsilon)$, $\eta(2a) = -2m\varepsilon$ and $\eta(4a) = 0$. Let λ be the 2-dimensional irreducible character of U, so λ takes the value -2 on the involution in U and the value 0 on elements of order 4. Taking the scalar product of η and λ with respect to U we obtain

$$\langle \lambda, \eta \rangle_U = \frac{1}{8}(2m(q+\varepsilon) + 4m\varepsilon) = \frac{m(q-\varepsilon)}{4} + m\varepsilon.$$

Since $q \equiv \pm 1 \bmod 8$, the number $\frac{q-\varepsilon}{4}$ is even, thus $\langle \lambda, \eta \rangle_U$ is odd. By the preceding paragraph η is affordable by a real representation. Linearly extending this representation we obtain a real representation of U with character $\eta|_U$. Since the Frobenius–Schur indicator of λ is -1, it is not affordable by a real representation [22, Chapter XI, Theorem 8.3]. So by [22, Chapter XI, Theorem 8.9] the integer $\langle \lambda, \eta \rangle_U$ must be even, a contradiction. Thus $\mathrm{V}(\mathbb{Z}G)$ does not contain a quaternion group of order 8 and this finishes the proof of the theorem. □

Proof of Proposition 1.3. By [26, Proposition 4.7] we may assume that the Sylow 2-subgroup of G is not abelian, thus it is a dihedral or quaternion group of order 8. First assume that the Sylow 2-subgroup is dihedral. By Lemma 2.4 we may assume that $O_{2'}(G) = 1$ and so again by [6, Theorem of Gorenstein and Walter] one of the three cases given in the beginning of the proof of Theorem 1.2 may be assumed. If G is a 2-group, the result is well known and follows also from [32, Theorem 1]. Since by assumption G is not isomorphic to an alternating group of degree 7, it is a subgroup of $\mathrm{P\Gamma L}(2,q)$ containing $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$ as a normal subgroup of odd index with some odd prime power q. Let U be a 2-subgroup of $\mathrm{V}(\mathbb{Z}G)$. Then U is isomorphic to a subgroup of G by Theorem 1.2. Let the conjugacy classes of elements of order 2 and 4 be named as in the proof of Lemma 3.1. If $\mathrm{PGL}(2,q)$ is not a subgroup of G, Lemma 3.1 implies that any isomorphism between U and a subgroup S of G preserves values of all ordinary irreducible characters of G and then U and S are rationally conjugate by [31, Lemma 4].

So assume that $\mathrm{PGL}(2,q)$ is a subgroup of G. Note that in this case elements of order 4 in G do not lie in $\mathrm{PSL}(2,q)$ since otherwise the Sylow 2-subgroup of G would have more then eight elements. We thus may assume that G has characters as given in Table 2. If U is cyclic, then it is rationally conjugate to a subgroup of G by Lemma 3.1. If U is elementary-abelian of rank 2, either all elements of U are rationally conjugate to elements in $2a$ or exactly two elements are rationally conjugate to elements of $2b$, since otherwise $\frac{1}{4}\sum_{u \in U} \chi(u)$ is not a non-negative integer, contradicting Lemma 2.2. Thus U is rationally conjugate to a subgroup of G by [31, Lemma 4], since in both cases we can find a subgroup S of G and an isomorphism between U and S preserving values on all ordinary characters of G.

So finally assume that U is isomorphic to a dihedral group of order 8. The central involution of U is rationally conjugate to elements in $2a$ by Lemma 3.1 since it is a square of an unit of order 4. Since $\frac{1}{8}\sum_{u \in U} \chi(u)$ is an integer, one of the non-central involutions in U is rationally conjugate to elements of $2b$ and the character values of χ then determine for every involution in U, whether it is rationally conjugate to an element of $2a$ or $2b$. This allows again to construct an isomorphism between U and a Sylow 2-subgroup of G preserving character values and thus U is rationally conjugate to Sylow 2-subgroups of G by [31, Lemma 4].

It remains to consider the case where the Sylow 2-subgroup of G is a quaternion group of order 8. By [26, Proposition 4.5] it is enough to show that units of order 2 and 4 are rationally conjugate to elements of G. By the famous theorem of Brauer and Suzuki [8], G contains a central involution, say g, and by the Berman–Higman Theorem, g is the only involution in $\mathrm{V}(\mathbb{Z}G)$. So assume $u \in \mathrm{V}(\mathbb{Z}G)$ has order 4. Then $\varepsilon_g(u) = 0$ by the Berman–Higman Theorem. If G contains only one conjugacy class of elements of order 4, this implies that u is rationally conjugate to an element of G. So assume that G has more then one conjugacy class of elements of order 4 and that u is not rationally conjugate to an element of G. Different conjugacy classes of elements of order 4 in G map to different conjugacy classes of involutions in $G/\langle g \rangle$. Thus, if u has non-trivial partial augmentations, the image of u in $\mathrm{V}(\mathbb{Z}(G/\langle g \rangle))$ is an involution with non-trivial partial augmentations. Such an involution does not exist by [3, Proposition 3.4] and hence u is rationally conjugate to an element of G and the proposition is proved. □

Remark. As already remarked in [18, Example 7] it is not known whether units of order 4 in $\mathrm{V}(\mathbb{Z}A_7)$ are rationally conjugate to elements of the group base. If one could prove that this is the case, then the exception of A_7 from Proposition 1.3 would not be necessary.

Proof of Theorem 1.1. Let G be a group not containing a subgroup isomorphic to $C_4 \times C_2$ such that $\mathrm{V}(\mathbb{Z}G)$ contains a subgroup $U = \langle t \rangle \times \langle s \rangle \cong C_4 \times C_2$. By Lemma 2.1 we may assume that the Sylow 2-subgroup P of G is abelian, (generalized) quaternion, dihedral or semidihedral. By Proposition 2.3 and Theorem 1.2 only the case where P is semidihedral remains and by Lemma 2.4 we may assume $O_{2'}(G) = 1$. The groups of interest are classified in [2], but are not given in a single theorem there. By [2, Chapter II, Section 1, Proposition 1] four different cases may appear. These cases get their names in [2, Chapter II, Section 2, Definition 1].

So G is one of the following:

(i) A QD-group. G possesses exactly one conjugacy class of involutions and one conjugacy class of elements of order 4.

(ii) A Q-group. G possesses exactly two conjugacy classes of involutions and one conjugacy class of elements of order 4.

(iii) A D-group. G possesses exactly one conjugacy class of involutions and two conjugacy classes of elements of order 4.

(iv) A 2-group.

In case (iv) [32, Theorem 1] gives the result, so we have to consider the first three cases.

Case (i): G is a QD-group. Denote by $2a$ the conjugacy class of involutions in G and by $4a$ the class of elements of order 4.

By [2, Chapter II, Section 2, Proposition 2], G possesses a simple normal subgroup N of odd index m and by [2, Third Main Theorem] the normal subgroup N is either $\mathrm{PSL}(3,q)$ with $q \equiv -1 \bmod 4$ or $\mathrm{PSU}(3,q)$ with $q \equiv 1 \bmod 4$, for some prime power q, or the Mathieu group M_{11}. Note that $\mathrm{PSU}(3,q)$ is actually defined over $\mathbb{F}_{q^2}$, the field with q^2 elements. The groups $\mathrm{SL}(3,q)$ and $\mathrm{SU}(3,q)$ act on the homogeneous polynomials of degree 3 in three commuting variables x, y, z over the field $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$ respectively. See e.g. [1, pp. 14–16] for a discussion of the analogous action of $\mathrm{SL}(2,q)$. The centres of the groups are in the kernel of the action and thus this action supplies a 10-dimensional representation of $\mathrm{PSL}(3,q)$ and $\mathrm{PSU}(3,q)$ respectively. Let χ' be the character of this representation. Elements of $\mathrm{SL}(3,q)$ and $\mathrm{SU}(3,q)$ projecting to elements in the classes $2a$ and $4a$ in G are given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

where the Gram matrix of the underlying unitary form is taken to be the identity matrix. I.e. we understand $\mathrm{SU}(3,q)$ to be those matrices of $\mathrm{GL}(3,q^2)$ having determinant 1 and leaving the unitary form

$$\alpha : \mathbb{F}_{q^2}^3 \times \mathbb{F}_{q^2}^3 \to \mathbb{F}_{q^2}, \quad (x,y) \mapsto \sigma(x)^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} y$$

invariant. Here σ denotes the automorphism of $\mathbb{F}_{q^2}$ of order 2.

The eigenspace of A for the eigenvalue 1 is spanned by x^3, xy^2, xz^2, xyz while the eigenspace for the eigenvalue -1 is spanned by $y^3, z^3, x^2y, x^2z, y^2z, yz^2$. Denote by i a primitive 4th root of unity in $\mathbb{F}_{q^2}$. The eigenspaces of B are spanned by:

- for the eigenvalue i by $y^3 + iz^3, yz^2 + iy^2z, x^2y + ix^2z$,
- for the eigenvalue $-i$ by $y^3 - iz^3, yz^2 - iy^2z, x^2y - ix^2z$,
- for the eigenvalue -1 by $xy^2 - xz^2, xyz$,
- for the eigenvalue 1 by $x^3, xy^2 + xz^2$.

One thus obtains $\chi'(1) = 10$, $\chi'(2a) = -2$ and $\chi'(4a) = 0$. The Mathieu group M_{11} does also possess an (ordinary) character with the same values, see e.g. [10]. Inducing this character of N to G we get the character given in Table 3.

	$1a$	$2a$	$4a$
χ	$10m$	$-2m$	0

Table 3. A character of a QD-group G.

Unlike in Lemma 3.1 it is not possible to show using only the HeLP-method that units of order 4 in $\mathrm{V}(\mathbb{Z}G)$ are rationally conjugate to elements of G. This may be checked e.g. using a GAP-package implementing the method [4]. See also [7] concerning M_{11}. Let $u \in \mathrm{V}(\mathbb{Z}G)$ be of order 4. If D is a representation affording χ, then $\chi(u)$ is integral and the eigenvalues of $D(u)$ are 4th roots of unity and thus $\chi(u) = \chi(u^3)$. Recall that $C_4 \times C_2 \cong U = \langle t \rangle \times \langle s \rangle$ is the group we want to study. There are three involutions in U taking the value $-2m$ for χ. Moreover, the elements of order 4 in U are t, t^3, st, st^3 and for them we have $\chi(t) = \chi(t^3)$ and $\chi(st) = \chi(st^3)$. So by Lemma 2.2

$$\frac{1}{8}\sum_{u \in U} \chi(u) = \frac{1}{8}(10m - 6m + 2\chi(t) + 2\chi(st))$$

is a non-negative integer. Note that since $\chi(4a) = 0$, we have $\chi(t) = -2m\varepsilon_{2a}(t)$ and $\chi(st) = -2m\varepsilon_{2a}(st)$. As $\varepsilon_{2a}(t) \equiv \varepsilon_{2a}(st) \equiv 0 \bmod 2$ by [9, Theorem 4.1], this implies $\chi(t) \equiv \chi(st) \equiv 0 \bmod 4$. Thus $2\chi(t) \equiv 2\chi(st) \equiv 0 \bmod 8$, implying $8 \mid 4m$, a contradiction.

Case (ii): G is a Q-group. Denote by $2a$ and $2b$ the conjugacy classes of involutions in G and by $4a$ the class of elements of order 4.

By [2, Chapter II, Section 3, Proposition 2], G possesses a normal subgroup isomorphic to $\mathrm{SL}(2,q)$ for some odd prime power q and hence by [2, Chapter II, Section 3, Proposition 3], G possesses a normal subgroup N isomorphic to a group denoted by Alperin, Brauer and Gorenstein as $\mathrm{SL}_k(2,q)$ or $\mathrm{SU}_k(2,q)$ and $[G:N]=m$ is odd. The groups appearing for N are defined in [2, p. 17, above Lemma 1]. The only relevant fact for us will be that these are subgroups of $\mathrm{GL}(2,q)$ and $\mathrm{GU}(2,q)$ respectively and that they contain all matrices of these groups having determinant 1 or -1. Representatives of the conjugacy classes of interest in N are given by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ where again the Gram matrix of the underlying unitary form is taken to be the identity matrix. Inducing the character of N given by the determinant map to a character of G turns out to be already enough. The values of this character χ are given in Table 4.

	$1a$	$2a$	$2b$	$4a$
χ	m	m	$-m$	m

Table 4. A character of a Q-group G.

Claim. *Elements of order* 2 *and* 4 *in* $\mathrm{V}(\mathbb{Z}G)$ *are rationally conjugate to elements of* G. *Hence if* $u \in \mathrm{V}(\mathbb{Z}G)$ *is of order* 4, *then* u^2 *is rationally conjugate to elements in* $2a$.

Let first $u \in \mathrm{V}(\mathbb{Z}G)$ be of order 2. If u is not in $2a$, then $\varepsilon_{2a}(u)=0$ by the Berman-Higman Theorem, since $2a$ consists of a single element which is central in G. Thus u is rationally conjugate to an element of $2b$ in this case. If u is of order 4, again $\varepsilon_{2a}(u)=0$. So from $\varepsilon_{2b}(u)+\varepsilon_{4a}(u)=1$ and the inequalities

$$-m \le -m\varepsilon_{2b}(u)+m\varepsilon_{4a}(u) \le m$$

coming from χ together with the fact that $\varepsilon_{4a} \not\equiv 0 \bmod 2$ (see [9, Theorem 4.1]) we conclude $(\varepsilon_{2a}(u),\varepsilon_{2b}(u),\varepsilon_{4a}(u))=(0,0,1)$.

Now let again $C_4 \times C_2 \cong U = \langle t\rangle \times \langle s\rangle$ be a subgroup of $\mathrm{V}(\mathbb{Z}G)$. As t^2 is a square of an element of order 4, it must be the element of $2a$ by the above claim. The other involutions in U must thus be rational conjugates of elements in $2b$ since the element of $2a$, being central, has no conjugates. Then

$$\frac{1}{8}\sum_{u\in U}\chi(u)=\frac{1}{8}(6m-2m)=\frac{m}{2}$$

is a non-negative integer by Lemma 2.2, a contradiction.

Case (iii): *G* is a *D*-group. Denote by $2a$ the conjugacy class of involutions in G and by $4a$ and $4b$ the classes of elements of order 4.

By [2, Chapter II, Section 3, Proposition 4] we may assume that G is a $\mathrm{PGL}_n(2,3)$, defined in [2, p. 20, before Lemma 5], or A_7 or it is a subgroup of $\mathrm{P\Gamma L}(2,q) = \mathrm{Aut}(\mathrm{PSL}(2,q))$, for some prime power q, and contains one of the groups $\mathrm{PSL}(2,q)$, $\mathrm{PGL}(2,q)$ or $\mathrm{PGL}^*(2,q)$ as a normal subgroup N such that $[G:N] = m$ and q are both odd. In case G is a $\mathrm{PGL}_n(2,3)$ its Sylow 2-subgroup is not a semidihedral group by [2, Chapter II, Section 2, Lemma 5]. In case G is A_7 or N is $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$, the Sylow 2-subgroup is also not semidihedral by [2, Chapter II, Section 2, Lemma 3]. So only the case $N = \mathrm{PGL}^*(2,q)$ remains, this group is defined in [2, p. 20, above Lemma 4], and we are going to describe it now.

The group $N = \mathrm{PGL}^*(2,q)$ is a non-split extension of $\mathrm{PSL}(2,q)$ of degree 2 and does only exist, if q is a square, say $q = r^2$. The groups $\mathrm{PGL}^*(2,q)$ are Zassenhaus groups and were introduced, to my knowledge, in [34] where they are denoted by M_q. They are also discussed in [22, Chapter XI, Example 1.3 c)]. Let σ be the unique automorphism of $\mathbb{F}_q$ of order 2. So in particular, the elements of $\mathbb{F}_r$ are fixed by σ. By acting entrywise on $\mathrm{GL}(2,q)$ the automorphism σ induces an automorphism, also called σ, of $\mathrm{GL}(2,q)$ and after projection an automorphism $\bar{\sigma}$ of $\mathrm{PSL}(2,q)$. Denote by $\bar{}$ the natural homomorphism from $\mathrm{GL}(2,q)$ to $\mathrm{PGL}(2,q)$. Then via conjugation every $A \in \mathrm{GL}(2,q)$ induces an automorphism $\bar{A}$ of $\mathrm{PSL}(2,q)$. We understand $\mathrm{PSL}(2,q)$ to be its own group of inner automorphisms. Let α be an element of maximal 2-power order in $\mathbb{F}_q$ and i a primitive 4th root of unity in $\mathbb{F}_q$. Set

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \qquad \text{if } r \equiv -1 \bmod 4,$$

$$g = \begin{pmatrix} 0 & \alpha \\ -1 & 0 \end{pmatrix} \qquad \text{if } r \equiv 1 \bmod 4.$$

Note that $\sigma(\alpha) = -\alpha^{-1}$, if $r \equiv -1 \bmod 4$, and $\sigma(\alpha) = -\alpha$, if $r \equiv 1 \bmod 4$. Then $\bar{g}\bar{\sigma}$ is an automorphism of order 4 of $\mathrm{PSL}(2,q)$ such that $(\bar{g}\bar{\sigma})^2 = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ is an inner automorphism of $\mathrm{PSL}(2,q)$. We set $\mathrm{PGL}^*(2,q) = \langle \mathrm{PSL}(2,q), \bar{g}\bar{\sigma} \rangle$, understanding it as a subgroup of the automorphism group of $\mathrm{PSL}(2,q)$. The concrete choice of g is inspired by the proof of [13, Lemma 2.3] where the structure of the Sylow 2-subgroup of $\mathrm{PGL}^*(2,q)$ is analysed.

Since $G/\mathrm{PSL}(2,q)$ has a Sylow 2-subgroup of order 2, it contains a normal 2-complement by [27, 7.2.2]. Thus G maps onto a cyclic group of order 2 and $\mathrm{PSL}(2,q)$ is in the kernel of this map while elements of N outside of $\mathrm{PSL}(2,q)$

are not. Hence G has a 1-dimensional representation containing $\mathrm{PSL}(2, q)$ in its kernel and mapping elements of N outside of $\mathrm{PSL}(2, q)$ to -1. Call the corresponding character χ. Moreover, $\mathrm{SL}(2, q)$ acts via conjugation on the 2×2-matrices over $\mathbb{F}_q$ having trace 0 giving a 3-dimensional representation of $\mathrm{PSL}(2, q)$. This then induces a $6m$-dimensional representation of G. Call the corresponding character ψ.

Furthermore $\mathrm{GL}(2, q)$ acts on the 4-dimensional $\mathbb{F}_r$-vector space

$$\mathcal{H} = \left\{ \begin{pmatrix} a & c \\ \sigma(c) & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{F}_r,\ c \in \mathbb{F}_q \right\}$$

by $A * H = \sigma(A)^t HA$ for $A \in \mathrm{GL}(2, q)$ and $H \in \mathcal{H}$, where X^t denotes the transpose of a matrix X. The kernel of this operation is $\{(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}) \mid x \in \mathbb{F}_q,\ x\sigma(x) = 1\}$. Since the centre of $\mathrm{SL}(2, q)$ is contained in this kernel, we obtain a 4-dimensional $\mathbb{F}_r$-representation of $\mathrm{PSL}(2, q)$ which we want to extend to a representation of N. Also σ acts on $\mathcal{H}$ by entry-wise application. Let $H = (\begin{smallmatrix} a & c \\ \sigma(c) & b \end{smallmatrix})$ be an arbitrary element of $\mathcal{H}$. In case $r \equiv -1 \bmod 4$ the element $g\sigma$, as an element of the semilinearities $\Gamma\mathrm{L}(2, q)$, has order 4 and the action of $\langle g\sigma \rangle$ on $\mathcal{H}$ has trivial kernel. Thus setting

$$\bar{g}\bar{\sigma} * H = \begin{pmatrix} a & \alpha\sigma(c) \\ \sigma(\alpha)c & -b \end{pmatrix}$$

extends the action of $\mathrm{PSL}(2, q)$ to an action of N providing a 4-dimensional $\mathbb{F}_r$-representation of N. In case $r \equiv 1 \bmod 4$ the order of the semilinearity $g\sigma$ is twice the order of α while the order of $\bar{g}\bar{\sigma}$ is 4. We have

$$g\sigma * H = \begin{pmatrix} b & -\alpha\sigma(c) \\ -\sigma(\alpha)c & -\alpha^2 a \end{pmatrix},$$

so $(g\sigma)^4 * H = \alpha^4 H$. Hence viewing a basis of $\mathcal{H}$ as a basis of the 4-dimensional $\mathbb{F}_q$-vector space $\mathbb{F}_q \otimes_{\mathbb{F}_r} \mathcal{H}$ and setting

$$\bar{g}\bar{\sigma} * H = \alpha^{-1} \begin{pmatrix} b & -\alpha\sigma(c) \\ -\sigma(\alpha)c & -\alpha^2 a \end{pmatrix} = \begin{pmatrix} \alpha^{-1} b & -\sigma(c) \\ c & -\alpha a \end{pmatrix},$$

we obtain a 4-dimensional $\mathbb{F}_q$-representation of N. Let η be the character corresponding to the induced representation of G. Computing the eigenvalues of these actions one obtains the character values given in Table 5.

Claim. *Elements of order* 2 *and* 4 *in* $\mathrm{V}(\mathbb{Z}G)$ *are rationally conjugate to elements of* G. *Hence especially an element of order* 4 *is rationally conjugate to its inverse.*

	$1a$	$2a$	$4a$	$4b$
χ	1	1	1	-1
ψ	$6m$	$-2m$	$2m$	0
η	$4m$	0	$-2m$	0

Table 5. Some characters of a D-group G.

This is clear for elements of order 2, so let $u \in \mathrm{V}(\mathbb{Z}G)$ be of order 4. From χ we obtain the equation

$$\varepsilon_{2a}(u) + \varepsilon_{4a}(u) - \varepsilon_{4b}(u) \in \{\pm 1\},$$

and adding $\varepsilon_{2a}(u) + \varepsilon_{4a}(u) + \varepsilon_{4b}(u) = 1$ onto this gives $\varepsilon_{4b}(u) \in \{0, 1\}$. Since u^2 is rationally conjugate to an element in $2a$ under a representation affording η, the unit u^2 has $2m$ times the eigenvalue 1 and $2m$ times the eigenvalue -1. This implies

$$-2m \leq \eta(u) = -2m\varepsilon_{4a}(u) \leq 2m$$

and thus $\varepsilon_{4a}(u) \in \{-1, 0, 1\}$. Under a representation affording ψ the involution u^2 has $4m$ times the eigenvalue -1 and $2m$ times the eigenvalue 1. Thus

$$-2m \leq \psi(u) = -2m\varepsilon_{2a}(u) + 2m\varepsilon_{4a}(u) \leq 2m,$$

implying

$$-\varepsilon_{2a}(u) + \varepsilon_{4a}(u) \in \{-1, 0, 1\}.$$

Together with the fact that $\varepsilon_{2a}(u) \equiv 0 \bmod 2$ and $\varepsilon_{4a}(u) + \varepsilon_{4b}(u) \not\equiv 0 \bmod 2$ by [9, Theorem 4.1] these equations prove the claim. The fact that u is rationally conjugate to its inverse follows from the character values of χ.

So let $C_4 \times C_2 \cong U = \langle t \rangle \times \langle s \rangle$. Then by Lemma 2.2

$$\frac{1}{8}\sum_{u \in U} \chi(u) = \frac{1}{8}(1 + 3 + 2\chi(t) + 2\chi(st))$$

is a non-negative integer. Since $\chi(t), \chi(st) \in \{\pm 1\}$, this implies $\chi(t) = \chi(st)$ and thus t and st are rationally conjugate. Now

$$\frac{1}{8}\sum_{u \in U} \eta(u) = \frac{1}{8}(4m + 3 \cdot 0 + 4\eta(t))$$

is also a non-negative integer. Since η vanishes on the conjugacy classes $2a$ and $4b$, we obtain that $\eta(t) = \eta(4a)\varepsilon_{4a}(t) = -2m\varepsilon_{4a}(t)$. So $4\eta(t) \equiv 0 \bmod 8$ meaning that $\frac{m}{2}$ has to be an integer. This contradicts the existence of U and finalizes the proof of the theorem. □

Acknowledgments. I am thankful to Markus Stroppel for many interesting conversations about actions of classical groups. I am moreover thankful to the referee for his corrections and suggestions improving the readability of the paper.

Bibliography

[1] J. L. Alperin, *Local Representation Theory. Modular Representations as an Introduction to the Local Representation Theory of Finite Groups*, Cambridge Stud. Adv. Math. 11, Cambridge University Press, Cambridge, 1986.

[2] J. L. Alperin, R. Brauer and D. Gorenstein, Finite groups with quasi-dihedral and wreathed Sylow 2-subgroups, *Trans. Amer. Math. Soc.* **151** (1970), 1–261.

[3] A. Bächle and W. Kimmerle, On torsion subgroups in integral group rings of finite groups, *J. Algebra* **326** (2011), 34–46.

[4] A. Bächle and L. Margolis, HeLP – A GAP-package for torsion units in integral group rings, preprint (2015), `http://arxiv.org/abs/1507.08174`.

[5] A. Bächle and L. Margolis, Torsion subgroups in the units of the integral group ring of $\mathrm{PSL}(2, p^3)$, *Arch. Math. (Basel)* **105** (2015), 1–11.

[6] H. Bender, Finite groups with dihedral Sylow 2-subgroups, *J. Algebra* **70** (1981), 216–228.

[7] V. Bovdi and A. Konovalov, Integral group ring of the first Mathieu simple group, in: *Groups St. Andrews 2005. Vol. 1* (St. Andrews 2005), London Math. Soc. Lecture Note Ser. 339, Cambridge University Press, Cambridge (2007), 237–245.

[8] R. Brauer and M. Suzuki, On finite groups of even order whose 2-Sylow group is a quaternion group, *Proc. Natl. Acad. Sci. USA* **45** (1959), 1757–1759.

[9] J. A. Cohn and D. Livingstone, On the structure of group algebras I, *Canad. J. Math.* **17** (1965), 583–593.

[10] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups*, Clarendon Press, Oxford, 1985.

[11] C. W. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders. Volume 1*, John Wiley & Sons, New York, 1990.

[12] M. A. Dokuchaev and S. O. Juriaans, Finite subgroups in integral group rings, *Canad. J. Math.* **48** (1996), 1170–1179.

[13] D. Gorenstein, Finite groups the centralizers of whose involutions have normal 2-complements, *Canad. J. Math.* **21** (1969), 335–357.

[14] D. Gorenstein and J. H. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups. I, *J. Algebra* **2** (1965), 85–151.

[15] R. Gow, Schur indices of some groups of Lie type, *J. Algebra* **42** (1976), 102–120.

[16] M. Hertweck, A counterexample to the isomorphism problem for integral group rings, *Ann. of Math. (2)* **154** (2001), 115–138.

[17] M. Hertweck, Partial augmentations and Brauer character values of torsion units in group rings, preprint (2007), `http://arxiv.org/abs/math/0612429v2`.

[18] M. Hertweck, Unit groups of integral finite group rings with no noncyclic Abelian finite p-subgroups, *Comm. Algebra* **36** (2008), 3224–3229.

[19] M. Hertweck, C. R. Höfert and W. Kimmerle, Finite groups of units and their composition factors in the integral group rings of the group $\mathrm{PSL}(2,q)$, *J. Group Theory* **12** (2009), 873–882.

[20] G. Higman, *Units in group rings*, Ph.D. thesis, Oxford University, Oxford, 1940.

[21] C. R. Höfert, *Bestimmung von Kompositionsfaktoren endlicher Gruppen aus Burnsideringen und ganzzahligen Gruppenringen*, Ph.D. thesis, Universität Stuttgart, Stuttgart, 2008.

[22] B. Huppert and N. Blackburn, *Finite Groups. III*, Grundlehren Math. Wiss. 243, Springer, Berlin, 1982.

[23] E. Jespers and Á. del Río, *Group Ring Groups. Volume 1: Orders and Generic Constructions of Units*, De Gruyter, Berlin, 2016.

[24] E. Jespers, Z. Marciniak, G. Nebe and W. Kimmerle, Mini-Workshop: Arithmetic of group rings, *Oberwolfach Rep.* **4** (2007), no. 4, 3209–3239.

[25] W. Kimmerle, Torsion units in integral group rings of finite insoluble groups, *Oberwolfach Rep.* **4** (2007), 3229–3230.

[26] W. Kimmerle, Sylow like theorems for $\mathrm{V}(\mathbb{Z}G)$, *Int. J. Group Theory* **4** (2015), 49–59.

[27] H. Kurzweil and B. Stellmacher, *The Theory of Finite Groups. An Introduction*, Universitext, Springer, New York, 2004.

[28] I. S. Luthar and I. B. S. Passi, Zassenhaus conjecture for A_5, *Proc. Indian Acad. Sci. Math. Sci.* **99** (1989), 1–5.

[29] Z. Marciniak, J. Ritter, S. K. Sehgal and A. Weiss, Torsion units in integral group rings of some metabelian groups II, *J. Number Theory* **25** (1987), 340–352.

[30] R. Sandling, Graham Higman's thesis "Units in group rings", in: *Integral Representations and Applications* (Oberwolfach 1980), Lecture Notes in Math. 882, Springer, Berlin (1981), 93–116.

[31] A. Valenti, Torsion units in integral group rings, *Proc. Amer. Math. Soc.* **120** (1994), 1–4.

[32] A. Weiss, Rigidity of p-adic p-torsion, *Ann. of Math. (2)* **127** (1988), 317–332.

[33] D. L. White, Character degrees of extensions of $\mathrm{PSL}_2(q)$ and $\mathrm{SL}_2(q)$, *J. Group Theory* **16** (2013), 1–33.

[34] H. Zassenhaus, Kennzeichnung endlicher linearer Gruppen als Permutationsgruppen, *Abh. Math. Semin. Univ. Hambg.* **11** (1935), 17–40.

Received December 20, 2015; revised May 13, 2016.

Author information

Leo Margolis, Departamento de Matematicas, Facultad de Matematicas,
Universidad de Murcia, Murcia 30100, Spain.
E-mail: `leo.margolis@um.es`