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Beauville structures in p-central quotients

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Abstract. We prove a conjecture of Boston that if $p \geq 5$, all p-central quotients of the free group on two generators and of the free product of two cyclic groups of order p are Beauville groups. In the case of the free product, we also determine Beauville structures in p-central quotients when $p = 3$. As a consequence, we give an infinite family of Beauville 3-groups, which is different from the ones that were known up to date.

1 Introduction

A *Beauville surface* of unmixed type is a compact complex surface isomorphic to $(C_1 \times C_2)/G$, where C_1 and C_2 are algebraic curves of genus at least 2 and G is a finite group acting freely on $C_1 \times C_2$ and faithfully on the factors C_i such that $C_i/G \cong \mathbb{P}_1(\mathbb{C})$ and the covering map $C_i \to C_i/G$ is ramified over three points for $i = 1, 2$. Then the group G is said to be a *Beauville group*.

It is easy to formulate the condition for a finite group G to be a Beauville group in purely group-theoretical terms. For a couple of elements $x, y \in G$, we define

$$\Sigma(x, y) = \bigcup_{g \in G} \left(\langle x \rangle^g \cup \langle y \rangle^g \cup \langle xy \rangle^g \right),$$

that is, the union of all subgroups of G which are conjugate to $\langle x \rangle$, to $\langle y \rangle$ or to $\langle xy \rangle$. Then G is a Beauville group if and only if the following conditions hold:

(i) G is a 2-generator group.

(ii) There exists a pair of generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of G such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1$.

Then $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are said to form a *Beauville structure* for G. We call $\{x_i, y_i, x_i y_i\}$ the *triple* associated to $\{x_i, y_i\}$ for $i = 1, 2$. The *signature* of a triple is the tuple of orders of the elements in the triple.

In 2000, Catanese [6] proved that a finite abelian group is a Beauville group if and only if it is isomorphic to $C_n \times C_n$, where $n > 1$ and $\gcd(n, 6) = 1$. On the

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other hand, all finite quasisimple groups other than A_5 and $\mathrm{SL}_2(5)$ are Beauville groups [8, 9] (see also [11] and [13]).

If p is a prime, Barker, Boston and Fairbairn [1] have shown that the smallest non-abelian Beauville p-groups for $p = 2$ and $p = 3$ are of order 2^7 and 3^5, respectively. They have also proved that there are non-abelian Beauville p-groups of order p^n for every $p \geq 5$ and every $n \geq 3$. The existence of infinitely many Beauville 2-groups and 3-groups has been settled in the affirmative in [2], and in [17] and [12], respectively. In particular, by [17, Theorem 2], there are quotients of the ordinary triangle group $T = \langle x, y \mid x^3 = y^3 = (xy)^9 = 1\rangle$ which are Beauville 3-groups of every order greater than or equal to 3^5. Among them, one can find the 3-central quotients $T/\lambda_n(T)$ for all $n \geq 4$. In all these groups, the signature of one of the triples of the Beauville structure takes the constant value $(3, 3, 9)$. On the other hand, by considering quotients of the Nottingham group over $\mathbb{F}_3$, Fernández-Alcober and Gül [10] have recently given an infinite family of Beauville 3-groups, for all orders at least 3^5, and in this case the signatures of the triples are not bounded.

In [4], Boston conjectured that if $p \geq 5$ and F is either the free group on two generators or the free product of two cyclic groups of order p, then its p-central quotients $F/\lambda_n(F)$ are Beauville groups. The goal of this paper is to prove Boston's conjecture. In fact, in the case of the free product, we extend the result to $p = 3$.

The main results of this paper are as follows.

Theorem A. *Let $F = \langle x, y\rangle$ be the free group on two generators. Then a p-central quotient $F/\lambda_n(F)$ is a Beauville group if and only if $p \geq 5$ and $n \geq 2$.*

Theorem B. *Let $F = \langle x, y \mid x^p, y^p\rangle$ be the free product of two cyclic groups of order p. Then a p-central quotient $F/\lambda_n(F)$ is a Beauville group if and only if $p \geq 5$ and $n \geq 2$ or $p = 3$ and $n \geq 4$.*

We will see that the signatures of the triples in the Beauville structures arising from Theorem B are unbounded as n goes to infinity. As a corollary, we get examples of Beauville 3-groups of every order greater than or equal to 3^5 which are completely different from the ones given by Stix and Vdovina. We also compare these examples with the Beauville quotients of the Nottingham group over $\mathbb{F}_3$ given in [10], and we show that the two infinite families only coincide at the group of order 3^5, see Theorem 3.7.

Notation. We use standard notation in group theory. If G is a group, then we denote by $\mathrm{Cl}_G(x)$ the conjugacy class of the element $x \in G$. Also, if p is a prime, then we write G^{p^i} for the subgroup generated by all powers g^{p^i} as g runs over G

and $\Omega_i(G)$ for the subgroup generated by the elements of G of order at most p^i. The exponent of G, denoted by $\exp G$, is the maximum of the orders of all elements of G.

2 The free group on two generators

In this section, we give the proof of Theorem A. We begin by recalling the definition of p-central series for the convenience of the reader.

Definition 2.1. For any group G, the normal series

$$G = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \geq \cdots$$

given by $\lambda_n(G) = [\lambda_{n-1}(G), G]\lambda_{n-1}(G)^p$ for $n > 1$ is called the *p-central series* of G.

Then a quotient group $G/\lambda_n(G)$ is said to be a *p-central quotient* of G. To prove the main theorems, we need the following properties of the subgroups $\lambda_n(G)$ (see [15, Definition 1.4 and Theorem 1.8, respectively]): we have

$$\lambda_n(G) = \gamma_1(G)^{p^{n-1}} \gamma_2(G)^{p^{n-2}} \dots \gamma_n(G), \tag{2.1}$$

and any element of $\lambda_n(G)$ can be written in the form

$$a_1^{p^{n-1}} a_2^{p^{n-2}} \dots a_n \quad \text{for some } a_i \in \gamma_i(G). \tag{2.2}$$

Also observe that if $\exp G/G' = p$, then $\lambda_n(G) = \gamma_n(G)$, since for any $i, j \geq 1$ we have $\gamma_i(G)^{p^j} \leq \gamma_{i+j}(G)$.

Lemma 2.2. *Let G be a group and $x, y \in G$. For $n \geq 3$, we have*

$$(xy)^{p^{n-2}} \equiv x^{p^{n-2}} y^{p^{n-2}} \pmod{\lambda_n(G)}.$$

Proof. By the Hall–Petrescu formula (see [15, Lemma 1.1]), we have

$$(xy)^{p^{n-2}} \equiv x^{p^{n-2}} y^{p^{n-2}} \pmod{\gamma_2(G)^{p^{n-2}} \prod_{r=1}^{n-2} \gamma_{p^r}(G)^{p^{n-2-r}}}.$$

Now the result follows, since by (2.1), $\gamma_2(G)^{p^{n-2}} \leq \lambda_n(G)$ and for $1 \leq r \leq n-2$ we have

$$\gamma_{p^r}(G)^{p^{n-2-r}} \leq \lambda_{p^r+n-2-r}(G) \leq \lambda_n(G). \qquad \square$$

Note that if $y \in \lambda_2(G)$ in Lemma 2.2, then

$$(xy)^{p^{n-2}} \equiv x^{p^{n-2}} \pmod{\lambda_n(G)}. \tag{2.3}$$

Before we proceed to prove Theorem A, we will need to introduce a lemma.

Let $F = \langle x, y \rangle$ be the free group on two generators. Notice that for $n \geq 2$, $\Phi(F/\lambda_n(F))$ coincides with $\lambda_2(F)/\lambda_n(F)$, and thus elements outside $\lambda_2(F)$ are potential generators in $F/\lambda_n(F)$. In order to determine Beauville structures in the quotients $F/\lambda_n(F)$, it is fundamental to control p^{n-2}nd powers of elements outside $\lambda_2(F)$ in these quotients groups.

Lemma 2.3. *Let $F = \langle x, y \rangle$ be the free group on two generators. Then $x^{p^{n-2}}$ and $y^{p^{n-2}}$ are linearly independent modulo $\lambda_n(F)$ for $n \geq 2$.*

Proof. We argue by way of contradiction. Suppose that

$$y^{ip^{n-2}} \equiv x^{p^{n-2}} \pmod{\lambda_n(F)}.$$

It follows from (2.2) that

$$x^{-p^{n-2}} y^{ip^{n-2}} = a_1^{p^{n-1}} a_2^{p^{n-2}} \dots a_n \quad \text{for some } a_j \in \gamma_j(F),$$

and then we have

$$y^{-ip^{n-2}} x^{p^{n-2}} a_1^{p^{n-1}} \in \gamma_2(F).$$

Write $a_1 = x^k y^l z$ for some $z \in \gamma_2(F)$ and some $k, l \in \mathbb{Z}$. Then

$$x^{p^{n-2}(1+kp)} y^{p^{n-2}(lp-i)} \in \gamma_2(F).$$

On the other hand, an element of the free group F belongs to $\gamma_2(F)$ if and only if the exponent sum of both generators is zero. Hence we get $p^{n-2}(1 + kp) = 0$, which is a contradiction. □

As a consequence of Lemma 2.3, x and y have order p^{n-1} modulo $\lambda_n(F)$.

By equation (2.3), if we want to know p^{n-2}nd powers of all elements outside $\lambda_2(F)$ in $F/\lambda_n(F)$, it is enough to know the power of each element in the set $\{y, xy^i \mid 0 \leq i \leq p-1\}$. Also, by Lemma 2.2, we have

$$(xy^i)^{p^{n-2}} \equiv x^{p^{n-2}} y^{ip^{n-2}} \pmod{\lambda_n(F)} \quad \text{for } 1 \leq i \leq p-1,$$

and since $x^{p^{n-2}}$ and $y^{p^{n-2}}$ are linearly independent modulo $\lambda_n(F)$ by Lemma 2.3, the following lemma is straightforward.

Lemma 2.4. *If $G = F/\lambda_n(F)$, the power subgroups $M^{p^{n-2}}$ are all different and of order p in $\lambda_{n-1}(F)/\lambda_n(F)$, as M runs over the $p+1$ maximal subgroups of G. In particular, all elements in $M \smallsetminus \Phi(G)$ are of order p^{n-1}.*

After these preliminaries, we can now prove Theorem A.

Theorem 2.5. *A p-central quotient $F/\lambda_n(F)$ is a Beauville group if and only if $p \geq 5$ and $n \geq 2$.*

Proof. For simplicity let us call G the quotient group $F/\lambda_n(F)$. We first show that if $p = 2$ or 3, then G is not a Beauville group. By way of contradiction, suppose that $\{u_1, v_1\}$ and $\{u_2, v_2\}$ form a Beauville structure for G. Since G has $p + 1 \leq 4$ maximal subgroups, we may assume that u_1 and u_2 are in the same maximal subgroup. Then by (2.3), we have

$$\langle u_1^{p^{n-2}} \rangle = \langle u_2^{p^{n-2}} \rangle,$$

which is a contradiction.

Thus we assume that $p \geq 5$. First of all, notice that if $n = 2$, $G \cong C_p \times C_p$ is a Beauville group, by Catanese's criterion. So we will deal with the case $n \geq 3$. Let u and v be the images in G of x and y, respectively. We claim that $\{u, v\}$ and $\{uv^2, uv^4\}$ form a Beauville structure for G. If we have $A = \{u, v, uv\}$ and $B = \{uv^2, uv^4, uv^2uv^4\}$, we need to show that

$$\langle a^g \rangle \cap \langle b^h \rangle = 1 \tag{2.4}$$

for all $a \in A$, $b \in B$, and $g, h \in G$. Observe that a^g and b^h lie in different maximal subgroups of G in every case, since u and v are linearly independent modulo $\Phi(G)$ and $p \geq 5$.

Now, all elements $a \in A$ and $b \in B$ are of order p^{n-1}, by Lemma 2.4. If (2.4) does not hold, then

$$\langle (a^g)^{p^{n-2}} \rangle = \langle (b^h)^{p^{n-2}} \rangle,$$

and again by Lemma 2.4, a^g and b^h lie in the same maximal subgroup of G, which is a contradiction. We thus complete the proof that G is a Beauville group. □

3 The free product of two cyclic groups of order p

Now we focus on the free product $F = \langle x, y \mid x^p, y^p \rangle$ of two cyclic groups of order p. Notice that since F/F' has exponent p, we have $\lambda_n(F) = \gamma_n(F)$ for all $n \geq 1$.

We start with an easy lemma whose proof is left to the reader.

Lemma 3.1. *Let $\psi \colon G_1 \to G_2$ be a group homomorphism, let $x_1, y_1 \in G_1$ and $x_2 = \psi(x_1)$, $y_2 = \psi(y_1)$. If $o(x_1) = o(x_2)$, the condition $\langle x_2^{\psi(g)} \rangle \cap \langle y_2^{\psi(h)} \rangle = 1$ implies that $\langle x_1^g \rangle \cap \langle y_1^h \rangle = 1$ for $g, h \in G_1$.*

To prove the main theorem we also need a result of Easterfield [7] regarding the exponent of $\Omega_i(G)$. More precisely, if G is a p-group, then for every $i, k \geq 1$, the condition $\gamma_{k(p-1)+1}(G) = 1$ implies that

$$\exp \Omega_i(G) \le p^{i+k-1}.$$

A key ingredient of the proof of Theorem B will be based on p-groups of maximal class with some specific properties. Let $G = \langle s \rangle \ltimes A$, where s is of order p and $A \cong \mathbb{Z}_p^{p-1}$. The action of s on A is via θ, where θ is defined by the companion matrix of the pth cyclotomic polynomial $x^{p-1} + \cdots + x + 1$. Then G is the only infinite pro-p group of maximal class. Since $s^p = 1$ and $\theta^{p-1} + \cdots + \theta + 1$ annihilates A, this implies that for every $a \in A$,

$$(sa)^p = s^p a^{s^{p-1}+\cdots+s+1} = 1.$$

Thus all elements in $G \setminus A$ are of order p. An alternative construction of G can be given by using the ring of cyclotomic integers (see [16, Example 7.4.14]).

Let P be a finite quotient of G of order p^n for $n \ge 3$. Let us call P_1 the abelian maximal subgroup of P and $P_i = [P_1, P, \overset{i-1}{\ldots}, P] = \gamma_i(P)$ for $i \ge 2$. Then one can easily check that

$$\exp P_i = p^{\lceil \frac{n-i}{p-1} \rceil}$$

and every element in $P_i \setminus P_{i+1}$ is of order $p^{\lceil \frac{n-i}{p-1} \rceil}$.

Now we can begin to determine which p-central quotients of F are Beauville groups. We first assume that $p = 2$. The free product F of two cyclic groups of order 2 is the infinite dihedral group D_∞. Then by [3, Lemma 3.7], no finite quotient of F is a Beauville group. In the remainder, we consider the case where p is an odd prime.

Lemma 3.2. *Let $G = F/\lambda_n(F)$ for $n \ge 2$. If u and v are the images of x and y in G, then for any $i, j \not\equiv 0 \pmod p$ all elements in the coset $u^i v^j \Phi(G)$ have order $p^{\lceil \frac{n-1}{p-1} \rceil}$.*

Proof. Let P be the p-group of maximal class of order p^n which is mentioned above and let $s \in P \setminus P_1$ and $s_1 \in P_1 \setminus P'$. Since all elements in $P \setminus P_1$ are of order p and $\lambda_n(P) = 1$, the map

$$\psi\colon G \to P, \quad u^i \mapsto s^{-1}, \quad v^j \mapsto s s_1,$$

is well-defined and an epimorphism. Set $k = \lceil \frac{n-1}{p-1} \rceil$. Since ψ is an epimorphism, we have $\psi(u^i v^j \Phi(G)) = \psi(u^i v^j)\Phi(P) = s_1 \Phi(P)$, where every element in the coset $s_1 \Phi(P)$ has the same order as s_1, namely p^k. Then for every $g \in u^i v^j \Phi(G)$, we have $o(g) \ge o(s_1) = p^k$. On the other hand, $\gamma_{k(p-1)+1}(G) \le \gamma_n(G) = 1$. Then by (6), we get $\exp G \le p^k$, and consequently $o(g) = p^k$. □

We deal separately with the cases $p \ge 5$ and $p = 3$.

Theorem 3.3. *If $p \geq 5$, then the p-central quotient $F/\lambda_n(F)$ is a Beauville group for every $n \geq 2$.*

Proof. For simplicity let us call G the quotient group $F/\lambda_n(F)$. Observe that $\Omega_1(G) = G$.

If $n = 2$, then $G \cong C_p \times C_p$ is a Beauville group, by Catanese's criterion. Thus we assume that $n \geq 3$. Let u and v be the images of x and y in G, respectively. We claim that $\{u, v\}$ and $\{uv^2, uv^4\}$ form a Beauville structure for G. Let $A = \{u, v, uv\}$ and $B = \{uv^2, uv^4, uv^2uv^4\}$. Assume first that $a = u$ or v, which are elements of order p, and $b \in B$. If $\langle a^g \rangle \cap \langle b^h \rangle \neq 1$ for some $g, h \in G$, then $\langle a^g \rangle \subseteq \langle b^h \rangle$, and hence $\langle a\Phi(G) \rangle = \langle b\Phi(G) \rangle$, which is a contradiction since $p \geq 5$. Next we assume that $a = uv$. Since $p \geq 5$, for every $b \in B$ we have $\psi(b) \in P \smallsetminus P_1$, which is of order p. Thus for all $g, h \in G$ we have

$$\langle s_1^{\psi(g)} \rangle \cap \langle \psi(b)^{\psi(h)} \rangle = 1.$$

Since $o(uv) = o(s_1)$, it then follows from Lemma 3.1 that $\langle a^g \rangle \cap \langle b^h \rangle = 1$. This completes the proof. □

In order to deal with the prime 3, we need the following lemmas.

Lemma 3.4. *Let G be a p-group which is not of maximal class such that $d(G) = 2$. Then for every $x \in G$ there exists $t \in \Phi(G) \smallsetminus \{[x, g] \mid g \in G\}$.*

Proof. Note that a p-group has maximal class if and only if it has an element with centralizer of order p^2 (see [14, III.14.23]). Thus for every $x \in G$ we have $|C_G(x)| \geq p^3$, and hence

$$|\{[x, g] \mid g \in G\}| = |\operatorname{Cl}_G(x)| = |G : C_G(x)| \leq p^{n-3}.$$

Since $|\Phi(G)| = p^{n-2}$, there exists $t \in \Phi(G)$ such that $t \notin \{[x, g] \mid g \in G\}$. □

Lemma 3.5 ([10, Lemma 3.8]). *Let G be a finite p-group and let $x \in G \smallsetminus \Phi(G)$ be an element of order p. If $t \in \Phi(G) \smallsetminus \{[x, g] \mid g \in G\}$, then*

$$\Big(\bigcup_{g \in G} \langle x \rangle^g\Big) \bigcap \Big(\bigcup_{g \in G} \langle xt \rangle^g\Big) = 1.$$

Theorem 3.6. *Let $p = 3$. Then the following hold:*

(i) *The p-central quotient $F/\lambda_n(F)$ is a Beauville group if and only if $n \geq 4$.*

(ii) *The series $\{\lambda_n(F)\}_{n \geq 4}$ can be refined to a normal series of F such that two consecutive terms of the series have index p and for every term N of the series F/N is a Beauville group.*

Proof. Since the smallest Beauville 3-group is of order 3^5, the quotient $F/\lambda_n(F)$ can only be a Beauville group if $n \geq 4$. We first assume that $n = 4$. Now consider the group

$$H = \langle a, b, c, d, e \mid a^3 = b^3 = c^3 = d^3 = e^3 = 1, \\ [b,a] = c,\ [c,a] = d,\ [c,b] = e \rangle,$$

where we have omitted all commutators between generators which are trivial. This is the smallest Beauville 3-group. Since $\lambda_4(H) = 1$, $F/\lambda_4(F)$ maps onto H. On the other hand, it is clear that $|F/\lambda_4(F)| \leq 3^5$ and so $F/\lambda_4(F) \cong H$. Consequently, $F/\lambda_4(F)$ is a Beauville group. Thus we assume that $n \geq 5$.

Now let us call G the quotient group $F/\lambda_n(F)$. Consider the map $\psi\colon G \to P$ defined in the proof of Theorem 3.3. Since ψ is an epimorphism, we have

$$\psi(\lambda_{n-1}(G)) = \lambda_{n-1}(P).$$

Observe that the subgroup $\operatorname{Ker}\psi \cap \lambda_{n-1}(G)$ has index 3 in $\lambda_{n-1}(G)$, since the subgroup $\lambda_{n-1}(P)$ is of order 3. Choose a normal subgroup N of F such that $\lambda_n(F) \leq N < \lambda_{n-1}(F)$ and $N/\lambda_n(F) \leq \operatorname{Ker}\psi$. Then ψ induces an epimorphism $\overline{\psi}$ from F/N to P.

We will see that $L = F/N$ is a Beauville group, which simultaneously proves (i) and (ii). Let u and v be the images of x and y in L, respectively. Set $k = \lceil \frac{n-1}{2} \rceil$. Then $o(uv) \leq o(xy\lambda_n(F)) = 3^k$. On the other hand, since $\overline{\psi}(uv) = s_1$, we have $o(uv) \geq o(s_1) = 3^k$, and consequently we get $o(uv) = 3^k$ in L. Since the subgroup $F/\lambda_4(F) \cong H$ is not of maximal class, L is not of maximal class. Thus, by Lemma 3.4, there exist elements $z, t \in \Phi(L)$ such that $z \notin \{[u,l] \mid l \in L\}$ and $t \notin \{[v,l] \mid l \in L\}$. We claim that $\{u, v\}$ and $\{(uz)^{-1}, vt\}$ form a Beauville structure for L. Let $A = \{u, v, uv\}$ and $B = \{(uz)^{-1}, vt, (uz)^{-1}vt\}$.

If $a = u$, which is of order 3, and $b = vt$ or $(uz)^{-1}vt$, we get $\langle a^g \rangle \cap \langle b^h \rangle = 1$ for every $g, h \in L$, as in the proof of Theorem 3.3. When $a = v$ and $b = (uz)^{-1}$ or $(uz)^{-1}vt$, the same argument applies. If we are in one of the following cases: $a = u$ and $b = (uz)^{-1}$, or $a = v$ and $b = vt$, then the condition $\langle a^g \rangle \cap \langle b^h \rangle = 1$ follows from Lemma 3.5.

It remains to check the case when $a = uv$ and $b \in B$. For every element $b \in B$, we have $\overline{\psi}(b) \in P \smallsetminus P_1$, which has order 3. Since $o(uv) = o(s_1)$, the condition $\langle a^g \rangle \cap \langle b^h \rangle = 1$ follows from Lemma 3.1, as in the proof of Theorem 3.3. This completes the proof. □

Thus the quotients in Theorem 3.6 constitute an infinite family of Beauville 3-groups of order 3^n for all $n \geq 5$.

Observe that as a consequence of Lemma 3.2, the signatures of the triples in the Beauville structures arising from Theorems 3.3 and 3.6 are unbounded as n

goes to infinity. Consequently, these examples are different from those of Stix and Vdovina, since in their examples the signatures of one of the triples of the Beauville structures take the constant value.

We next compare the infinite family of Beauville 3-groups in Theorem 3.6 with the ones given in [10], by considering quotients of the Nottingham group over $\mathbb{F}_3$. We will show that these two infinite families of Beauville 3-groups only coincide at the group of order 3^5.

Before proceeding we recall the definition of the Nottingham group and some of its properties. The *Nottingham group* $\mathcal{N}$ over the field $\mathbb{F}_p$, for odd p, is the (topological) group of normalised automorphisms of the ring $\mathbb{F}_p[[t]]$ of formal power series. For any positive integer k, the automorphisms $f \in \mathcal{N}$ such that $f(t) = t + \sum_{i \geq k+1} a_i t^i$ form an open normal subgroup $\mathcal{N}_k$ of $\mathcal{N}$ of index p^{k-1}. The lower central series of $\mathcal{N}$ is given by

$$\gamma_i(\mathcal{N}) = \mathcal{N}_{r(i)}, \quad \text{where } r(i) = i + 1 + \left\lfloor \frac{i-2}{p-1} \right\rfloor, \tag{3.1}$$

and

$$\mathcal{N}_k^p = \mathcal{N}_{kp+r}, \quad \text{where } 0 \leq r \leq p-1 \text{ is the residue of } k \text{ modulo } p \tag{3.2}$$

(see [5, Remark 1 and Theorem 6, respectively]).

Also, each non-trivial normal subgroup of $\mathcal{N}$ lies between some $\gamma_i(\mathcal{N})$ and $\gamma_{i+1}(\mathcal{N})$ (see [5, Remark 1 and Proposition 2]).

By [10, Theorem 3.10], if $p = 3$, a quotient $\mathcal{N}/\mathcal{N}_k$ is a Beauville group if and only if $k \geq 6$ and $k \neq z_m$ for all $m \geq 1$, where $z_m = p^m + p^{m-1} + \cdots + p + 2$. Furthermore, by [10, Theorem 3.11], for $i \geq 1$ there exists a normal subgroup $\mathcal{W}$ between $\mathcal{N}_{ip+3}$ and $\mathcal{N}_{ip+1}$ such that $\mathcal{N}/\mathcal{W}$ is a Beauville group. This gives quotients of $\mathcal{N}$ which are Beauville groups of every order 3^n with $n \geq 5$.

Theorem 3.7. *Let $N \neq \gamma_4(F)$ be a normal subgroup of F such that F/N is a Beauville group. Then F/N is not isomorphic to any quotient of $\mathcal{N}$ which is a Beauville group. On the other hand, $F/\gamma_4(F)$ is isomorphic to $\mathcal{N}/\gamma_4(\mathcal{N})$.*

Proof. Since there is only one Beauville group of order 3^5 [1], it follows that $F/\gamma_4(F)$ is isomorphic to $\mathcal{N}/\gamma_4(\mathcal{N})$. Now suppose that $F/N \cong \mathcal{N}/\mathcal{W}$, where $\gamma_n(F) \leq N < \gamma_{n-1}(F)$ for $n \geq 5$ and F/N is a Beauville group. Since F/N is of class $n-1$ and $\mathcal{W}$ lies between two consecutive terms of the lower central series, we have $\gamma_n(\mathcal{N}) \leq \mathcal{W} < \gamma_{n-1}(\mathcal{N})$. Note that if $n = 5$, then

$$\mathcal{N}_7 = \gamma_5(\mathcal{N}) \leq \mathcal{W} < \gamma_4(\mathcal{N}) = \mathcal{N}_6$$

and so $\mathcal{W} = \gamma_5(\mathcal{N})$. If $n > 5$, then $\mathcal{W} \leq \gamma_5(\mathcal{N})$. Consequently, the isomorphism $F/N \cong \mathcal{N}/\mathcal{W}$ implies that $F/\gamma_5(F)N \cong \mathcal{N}/\gamma_5(\mathcal{N})$. We next show that this is not possible.

Note that by (3.1), we have $\gamma_2(\mathcal{N}) = \mathcal{N}_3$ and by (3.2), $\mathcal{N}_3^3 = \mathcal{N}_9$. Thus the exponent of $\gamma_2(\mathcal{N}/\gamma_5(\mathcal{N}))$ is 3. On the other hand, as in the proof of Theorem 3.6, there is an epimorphism from $F/\gamma_5(F)N$ to a p-group of maximal class P of order 3^5 with $\exp P' = 3^2$. It follows that $\mathcal{N}/\gamma_5(\mathcal{N})$ cannot be isomorphic to $F/\gamma_5(F)N$. □

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