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The structure group of an L-algebra is torsion-free

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Abstract. L-algebras arise in algebraic logic, in the theory of one-sided lattice-ordered groups, and in connection with set-theoretic solutions of the quantum Yang–Baxter equation. They apply in several ways to Garside groups. For example, the set of primitive elements, the set of simple elements, and the negative cone of a Garside group are all L-algebras. Picantin's iterated crossed product decomposition of Garside groups can be reformulated and extended in terms of L-algebras. It is proved that the structure group of an L-algebra, introduced in connection with the "logic" of ℓ-groups, is torsion-free. This applies to the left group of fractions of not necessarily noetherian, Garside-like monoids which need not embed into their ambient group.

A topological proof that Artin's braid groups are torsion-free was given in 1962 by Fadell, Fox, and Neuwirth [12, 13]. The first algebraic proof made use of the orderability of braid groups [5, 16]. Later, Dehornoy [6] showed that all Garside groups are torsion-free, and gave a much simpler proof [8] without the noetherian hypothesis. The method of simplifying proofs by dropping redundant hypotheses came to a certain end with a one-line proof that every group with a right invariant lattice order is torsion-free (see [20, Proposition 3]). On the other hand, Dehornoy's slightly longer proof [8] applies to an even more general situation. In the terminology of Section 1, his result shows that the group of fractions of a cancellative *left hoop* (a residuated monoid) is torsion-free.

In this paper, we prove that left cancellability can be dropped. More generally, we replace the left hoop by a unital *cycloid* [18], a set with a single binary operation $\cdot$ which satisfies

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z). \tag{0.1}$$

The relevance of equation (0.1) will be explained in Section 1. We give an example where the natural map from the unital cycloid to its *structure group* – the "group of fractions" – is surjective. So our theorem does not rely on any kind of partial ordering of the group.

1 The structure group of an L-algebra

Equation (0.1) exhibits a close relationship between group theory, algebraic logic, and quantum structures [11]. It first occurred in a logical context [2, 21]. Here $x \cdot y$ stands for the logical implication ("x implies y"). If the left multiplications $y \mapsto x \cdot y$ are bijective, such a set $(X; \cdot)$ is called a *cycle set* [17]. By [17, Proposition 1], cycle sets are in one-to-one correspondence with non-degenerate unitary set-theoretic solutions of the quantum Yang–Baxter equation. Thirdly, equation (0.1) occurs in the theory of *lattice-ordered* groups (*ℓ-groups* for short) [1, 4], the negative cone $G^- := \{x \in G : x \leqslant 1\}$ of an ℓ-group G being a cycloid with respect to

$$x \cdot y := (yx^{-1} \wedge 1).$$

As a fourth instance, every Garside monoid satisfies equation (0.1) (see [7, equation (1.5)]).

In the latter two examples, there is an associative multiplication which should be distinguished from the operation $x \cdot y$. Therefore, we write $x \to y$ instead of $x \cdot y$, which reflects the logical origin of that operation. In the presence of a multiplication, we then have

$$xy \leqslant z \iff x \leqslant y \to z. \tag{1.1}$$

By this equivalence, the unit element of an ℓ-group G satisfies

$$x \to x = x \to 1 = 1, \quad 1 \to x = x \tag{1.2}$$

for all $x \in G^-$. A cycloid $(X; \to)$ with such an element 1 (a *logical unit* [18]) is said to be *unital*. (In propositional logic, 1 stands for the "true" proposition.)

For a unital cycloid $(X; \to)$, the relation

$$x \sim y :\iff x \to y = y \to x = 1$$

is a congruence. If this relation is trivial, that is, $x \sim y$ implies $x = y$, then X is called an *L-algebra* [18]. Thus every unital cycloid X gives rise to an L-algebra $X/{\sim}$. Every L-algebra has a partial order

$$x \leqslant y :\iff x \to y = 1$$

similar to the logical entailment relation.

The negative cone, and every interval $[a, 1]$ of an ℓ-group is an L-algebra in two ways, namely, with respect to each of the operations $x \to y := yx^{-1} \wedge 1$ and $x \rightsquigarrow y := x^{-1}y \wedge 1$. In several respects, L-algebras play a fundamental part in the theory of Garside groups. For example, the primitive elements as well as the simple elements [7, 10] of a Garside group form an L-algebra. Picantin's iterated crossed products of Garside groups [15] admit a simple formulation in terms of L-algebras [19]. More generally, L-algebras arise in connection with *right ℓ-groups*, that is, groups with a right invariant partial order. For example, the structure groups of non-degenerate unitary set-theoretic solutions $X \times X \to X \times X$ of the quantum Yang–Baxter equation are right ℓ-groups with a distributive lattice structure [20]. For finite X, these groups are Garside [3, 9].

In [18] we associate a *structure group* $G(X)$ to any L-algebra X. To explain this, we have to clarify the rôle of the product in (1.1). Define a *left hoop* [18] to be a monoid H with a binary operation $\to$ satisfying

$$a \to a = 1,$$
$$ab \to c = a \to (b \to c),$$
$$(a \to b)a = (b \to a)b$$

for $a, b, c \in H$. By [18, Propositions 3 and 4], every left hoop is an L-algebra, and any pair $a, b \in H$ has a meet

$$a \wedge b = (a \to b)a. \tag{1.3}$$

Therefore, the partial order of H is given by

$$a \leqslant b \iff \exists c \in H\colon a = cb. \tag{1.4}$$

A left hoop H is *self-similar* if it is right cancellative. By [18, Proposition 5], this can be expressed in terms of equations:

Proposition 1. *For a left hoop H, the following are equivalent.*

(a) *H is self-similar, that is, $ac = bc \Longrightarrow a = b$ holds for $a, b, c \in H$.*

(b) *$a \to ba = b$ for all $a, b \in H$.*

(c) *$a \to bc = ((c \to a) \to b)(a \to c)$ for all $a, b, c \in H$.*

Since $a \to a = 1$ follows by (b), a self-similar left hoop is given by the axioms

$$a \to ba = b,$$
$$ab \to c = a \to (b \to c),$$
$$(a \to b)a = (b \to a)b.$$

Note that by equation (1.1), self-similarity is a property of L-algebras. In fact, an L-algebra X is self-similar if and only if for any $x \in X$, the map $y \mapsto (x \to y)$ gives a bijection $\{y \in X : y \leqslant x\} \xrightarrow{\sim} X$ (see [18, Definition 2]). Moreover, self-similarity implies that X admits a unique multiplication which makes X into a left hoop.

Now let H be a self-similar L-algebra with an *L-subalgebra* X, which means that the operation $\to$ of X is induced by that of H. As remarked above, H can be regarded as a left hoop. If the monoid H is generated by X, we call H a *self-similar closure* of X. By [18, Theorem 3], a self-similar closure of any L-algebra X exists and is unique, up to isomorphism. We denote it by $S(X)$. In particular, every self-similar left hoop H satisfies

$$S(H) = H.$$

Since any self-similar left hoop satisfies the left Ore condition, the self-similar closure $S(X)$ of an L-algebra X admits a left group of fractions $G(X)$ with a natural map

$$q\colon X \hookrightarrow S(X) \to G(X). \tag{1.5}$$

Example 1. Classical logic is built upon the two-element L-algebra $X_a := \{1, a\}$, which is completely determined by the property of 1 being a logical unit. Thus $a < 1$, where a and 1 stand for the two possible truth values. The self-similar closure $S(X_a)$ is the negative cone of the ℓ-group $G(X_a) \cong \mathbb{Z}$, the additive group of integers. Thus $S(X_a) := G(X_a)^-$ consists of the powers $1 > a > a^2 > a^3 > \cdots$ with $a^m \to a^n = a^{n-m}$ for $m \leqslant n$. As the two-element L-algebra X_a generates the group $G(X_a)$, this shows that logic creates arithmetic, the theory of the ring $\mathbb{Z} \cong \mathrm{End}(G(X_a))$.

Example 2. If the pair of truth values is replaced by the interval $[-1, 0] \subset \mathbb{R}$, we obtain an L-algebra with $x \to y := \min\{x, y\} - x$, which is fundamental for measure and integration theory [14]. Its self-similar closure is the negative cone of $G([-1, 0]) \cong \mathbb{R}$, the additive group of the reals.

Example 3. Let V be an $(n+1)$-dimensional Euclidean space, and let $\mathscr{E}(V)$ be the corresponding elliptic real n-space, that is, the projective space $\mathscr{P}(V)$ of one-dimensional subspaces of V. The scalar product of V defines an elliptic polarity $\mathscr{P}(V) \xrightarrow{\sim} \mathscr{P}(V^*)$ which associates a hyperplane $P^\perp$ to any point $P \in \mathscr{P}(V)$. "Elliptic" means that $P \notin P^\perp$ for all $P \in \mathscr{P}(V)$. Let $\mathscr{L}(V)$ denote the lattice of $\mathbb{R}$-linear subspaces of V. Thus $U \mapsto \mathscr{P}(U)$ gives a lattice isomorphism between the $U \in \mathscr{L}(V)$ and the projective subspaces of $\mathscr{E}(V)$. Define

$$H \to L := (H \wedge L) \vee H^\perp$$

for $H, L \in \mathscr{L}(V)$. Then $1 := \mathscr{E}(V)$ is a logical unit, and

$$\begin{aligned} H \subset L &\Longrightarrow (H \wedge L) \vee H^\perp = 1 \\ &\Longrightarrow H = ((H \wedge L) \vee H^\perp) \wedge H = (H \wedge L) \vee (H^\perp \wedge H) = H \wedge L, \end{aligned}$$

which gives

$$H \subset L \iff H \to L = 1.$$

Furthermore, $((H \wedge L) \vee H^{\perp}) \wedge H = (H \wedge L) \vee (H^{\perp} \wedge H) = H \wedge L$, which yields

$$(H \to L) \wedge H = H \wedge L.$$

Note that $H^{\perp} \subset H \to L$. For $H, L, M \in \mathscr{L}(V)$, this implies that

$$\begin{aligned}(H \to L) \wedge (H \to M) &= (H^{\perp} \vee (H \wedge L)) \wedge (H \to M)\\ &= H^{\perp} \vee (H \wedge L \wedge (H \to M))\\ &= (H \wedge L \wedge M) \vee H^{\perp}.\end{aligned}$$

Hence

$$\begin{aligned}(H \to L) \to (H \to M) &= ((H \to L) \wedge (H \to M)) \vee (H \to L)^{\perp}\\ &= (H \wedge L \wedge M) \vee H^{\perp} \vee (H \wedge (H \wedge L)^{\perp})\\ &= (H \wedge L \wedge M) \vee ((H^{\perp} \vee H) \wedge (H \wedge L)^{\perp})\\ &= (H \wedge L \wedge M) \vee (H \wedge L)^{\perp} = (H \wedge L) \to M,\end{aligned}$$

which shows that $\mathscr{L}(V)$ is an L-algebra. Assume that H and L are *orthogonal* ($H \perp L$) in the sense that $H^{\perp} \subset L$ or equivalently, $L^{\perp} \subset H$. Then

$$H \to L = H^{\perp} \vee (H \wedge L) = (H^{\perp} \vee H) \wedge L = L.$$

So $L \to (H \to M) = (H \to L) \to (H \to M) = (H \wedge L) \to M$, which yields

$$H \wedge L \subset M \iff L \subset H \to M.$$

Thus $H \wedge L$ coincides with the product $HL \in S(\mathscr{L}(V))$ whenever $H \perp L$. Let $X(V) \subset \mathscr{L}(V)$ be the L-subalgebra of subspaces $H \subset V$ of codimension $\leqslant 1$. Since every $H \in \mathscr{L}(V)$ is a finite meet of pairwise orthogonal hyperplanes, it follows that $S(\mathscr{L}(V)) = S(X(V))$. By [20, Proposition 19], $S(X(V))$ is a modular lattice.

Furthermore, it can be shown that the equation

$$(H \to L) \to (L \to H) = H$$

holds for distinct $H, L \in X(V)$. By [20, Theorem 4], this implies that $G(X(V))$ is a Garside group. The example gives an L-algebraic interpretation of the well-known Gram–Schmidt process in the Euclidean space V^*.

For a unital cycloid X, we define the *structure group* $G(X)$ to be $G(X/{\sim})$. Note that in general, the map (1.5) need not be injective. Indeed, two elements x, y of an L-algebra X satisfy $q(x) = q(y)$ if and only if $cx = cy$ holds for some $c \in S(X)$. Thus q is injective if and only if $S(X)$ is left cancellative. Two elements $x, y \in X$ with $q(x) = q(y)$ are called *equipollent* [18]. If X has a smallest element 0 (= "false"), equipollence of x and y is equivalent to $x \to 0 = y \to 0$. Our aim is to show that the structure group of any L-algebra X is torsion-free. Thus, by (1.5), we can assume, without loss of generality, that X is a self-similar left hoop.

Let H be a self-similar left hoop. For a pair of elements $a, b \in H$, we define the *derived pair* $(a', b') \in H^2$ by

$$a' := b \to a, \quad b' := a \to b. \tag{1.6}$$

The n-th derived pair will be denoted by $(a^{(n)}, b^{(n)})$ for $n \in \mathbb{N}$. In particular, $a^{(0)} := a$ and $b^{(0)} := b$. The notation does not express the dependency of $a^{(n)}$ on both a and b, which should always be clear from the context. Furthermore, we write

$$a_n := a^{(n-1)} \cdots a'a, \quad b_n := b^{(n-1)} \cdots b'b \tag{1.7}$$

for $n \in \mathbb{N}$. Thus $a_0 = b_0 = 1$. More generally, we abbreviate

$$a_{m,n} := a^{(m-1)} \cdots a^{(n)}, \quad b_{m,n} := b^{(m-1)} \cdots b^{(n)} \tag{1.8}$$

for $m \geqslant n$ in $\mathbb{N}$. The concept of derived pair (1.6) is justified by

Proposition 2. *Let H be a self-similar left hoop. Two elements $a, b \in H$ are equipollent if and only if a' and b' are equipollent.*

Proof. By Proposition 1, we have

$$a \to cb = ((b \to a) \to c)(a \to b) = (a' \to c)b'.$$

Hence

$$ca \to cb = c \to (a' \to c)b' = ((b' \to c) \to (a' \to c))(c \to b').$$

So $ca \leqslant cb$ is equivalent to the conjunction of $b' \to c \leqslant a' \to c$ and $c \leqslant b'$. Thus a and b are equipollent if and only if $a' \to c = b' \to c$ for some $c \leqslant a', b'$. By [18, Proposition 12], the latter condition is equivalent to $q(a') = q(b')$. □

The product expressions (1.7) satisfy the following.

Lemma 1. *Let H be a self-similar left hoop. For $a, b \in H$ and $m, n \in \mathbb{N}$,*

$$a^{(m+n)} = (a_m \to b_n) \to a^{(m)}, \quad b^{(m+n)} = (b_n \to a_m) \to b^{(n)}. \tag{1.9}$$

Proof. We proceed by induction over the sum $s := m + n$. For $s = 0$ or $n = 0$, the equations are obvious. Assume that the equations hold for a fixed sum $s \in \mathbb{N}$. For $s + 1 = m + (n + 1)$ with $m, n \in \mathbb{N}$, using Proposition 1, this gives

$$\begin{aligned}
a^{(m+n+1)} &= b^{(m+n)} \to a^{(m+n)} \\
&= ((b_n \to a_m) \to b^{(n)}) \to ((a_m \to b_n) \to a^{(m)}) \\
&= ((b_n \to a_m) \to b^{(n)})(a_m \to b_n) \to a^{(m)} \\
&= (a_m \to b^{(n)} b_n) \to a^{(m)} \\
&= (a_m \to b_{n+1}) \to a^{(m)}.
\end{aligned}$$

By symmetry, the lemma is proved. □

The expressions in equation (1.8) admit the following interpretation.

Proposition 3. *Let H be a self-similar left hoop. For $a, b \in H$ and $m, n \in \mathbb{N}$,*

$$a_m \to b_n = b_{m+n,m}, \quad b_n \to a_m = a_{m+n,n}.$$

Proof. We prove the first equation by induction on n. For $n = 0$, the equation is trivial. Thus, assume that the equation holds for a fixed n. Then Lemma 1 gives

$$\begin{aligned}
a_m \to b_{n+1} &= a_m \to b^{(n)} b_n \\
&= ((b_n \to a_m) \to b^{(n)})(a_m \to b_n) \\
&= b^{(m+n)} b_{m+n,m} = b_{m+n+1,m},
\end{aligned}$$

which completes the inductive step.

The second equation is proved analogously. □

The following corollary is essentially due to Dehornoy [8].

Corollary 1. *Let H be a self-similar left hoop. For $a, b \in H$ and $m, n \in \mathbb{N}$,*

$$a_m \wedge b_n = a_{m+n,n} b_n = b_{m+n,m} a_m. \tag{1.10}$$

Proof. This follows immediately by equation (1.3). □

Corollary 2. *Let H be a self-similar left hoop. For $a, b \in H$ and $n \in \mathbb{N}$, we have*

$$(q(a)^{-1} q(b))^n = q(a_n)^{-1} q(b_n). \tag{1.11}$$

Proof. Equation (1.11) can be rewritten as

$$q(a_n)(q(a)^{-1}q(b))^n = q(b_n).$$

For $n = 0$, this is obvious. Assume the equation holds for some n. Then equation (1.10) gives

$$\begin{aligned} q(a_{n+1})(q(a)^{-1}q(b))^{n+1} &= q(a_{n+1,1})q(b_1)(q(a)^{-1}q(b))^n \\ &= q(b_{n+1,n})q(a_n)(q(a)^{-1}q(b))^n \\ &= q(b_{n+1,n})q(b_n) \\ &= q(b_{n+1}). \end{aligned}$$

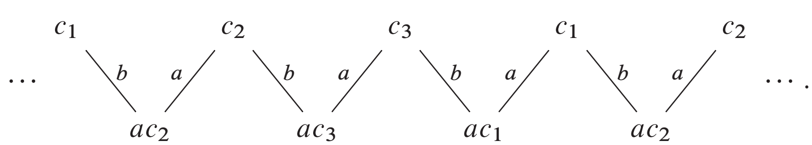
So the corollary follows by induction. □

Definition 1. Let H be a self-similar left hoop. For a pair $a, b \in H$, we define a *weak* (a, b)*-cycle* to be a non-empty sequence $(c_1, \ldots, c_n) \in H^n$ with

$$bc_i = ac_{i+1} \quad \text{for } 1 \leqslant i < n \qquad \text{and} \qquad q(ac_1) = q(bc_n).$$

If $ac_1 = bc_n$, we speak of an (a, b)*-cycle*.

An (a, b)-cycle can be visualized by a periodic diagram, e. g., for $n = 3$:



So we can assume that (c_i) is an n-periodic sequence defined for all $i \in \mathbb{Z}$, and there is a single condition $bc_i = ac_{i+1}$ for all $i \in \mathbb{Z}$.

Proposition 4. *Let H be a self-similar left hoop. If the sequence $(c_1, \ldots, c_n) \in H^n$ is an (a, b)-cycle, then $(ac_1, \ldots, ac_n)$ is an (a', b')-cycle.*

Proof. For $1 \leqslant i < n$, equation (1.10) yields $b'ac_i = a'bc_i = a'ac_{i+1}$. Furthermore, $a'ac_1 = a'bc_n = b'ac_n$, which proves the claim. □

Lemma 2. *If a self-similar left hoop H admits an (a, b)-cycle, then $q(a) = q(b)$.*

Proof. Let $(c_1, \ldots, c_n) \in H^n$ be an (a, b)-cycle. We show that

$$b_i c_j = a_i c_{i+j} \tag{1.12}$$

holds for $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. For $i = 0$, this is trivial. Assume that equation (1.12) is valid for a given $i \in \mathbb{N}$. Then equation (1.10) implies that

$$\begin{aligned} b_{i+1}c_j &= b^{(i)}b_i c_j \\ &= b_{i+1,i}a_i c_{i+j} \\ &= a_{i+1,1}b_1 c_{i+j} \\ &= a_{i+1,1}ac_{i+j+1} \\ &= a_{i+1}c_{i+1+j}, \end{aligned}$$

which proves equation (1.12). In particular, $b_n c_j = a_n c_j$. Hence

$$b_n = a_1 \wedge b_n = a_{n+1,n}b_n,$$

which yields $a^{(n)} = a_{n+1,n} = 1$. Similarly, $a_n = a_n \wedge b_1 = b_{n+1,n}a_n$ implies that $b^{(n)} = 1$. By Proposition 2, this proves that a and b are equipollent. □

Proposition 5. *If a self-similar left hoop H admits a weak (a,b)-cycle, then $q(a)$ is equal to $q(b)$.*

Proof. Let $(c_1,\dots,c_n) \in H^n$ be a weak (a,b)-cycle. So there is an element $c \in H$ with $cac_1 = cbc_n$. Choose $d \leqslant cbc_n \wedge bc_i$ for all $i \in \{1,\dots,n\}$. Since $d \leqslant bc_i$, there exist elements $d_i \in H$ with $d = d_i bc_i$ for all $i \in \{1,\dots,n\}$. In particular, $d_n bc_n = d \leqslant cbc_n$ shows that $d_n \leqslant c$. Hence we have $d_n ac_1 = d_n bc_n$. Furthermore, $d_{i+1}bc_{i+1} = d = d_i bc_i = d_i ac_{i+1}$ gives $d_i a = d_{i+1}b$ for $1 \leqslant i < n$, and $d_1 bc_1 = d = d_n bc_n = d_n ac_1$ yields $d_n a = d_1 b$. So the d_i extend to an n-periodic sequence (d_i) with $d_0 ac_1 = d_0 bc_n$ and

$$d_i a = d_{i+1}b$$

for all $i \in \mathbb{Z}$. Hence

$$\begin{aligned} (d_{i-1} \to d_i)d_i b &= (d_{i-1} \to d_i)d_{i-1}a \\ &= (d_i \to d_{i-1})d_i a \\ &= (d_i \to d_{i-1})d_{i+1}b, \end{aligned}$$

which yields

$$(d_{i-1} \to d_i)d_i = (d_i \to d_{i-1})d_{i+1} \tag{1.13}$$

for all $i \in \mathbb{Z}$. Similarly,

$$\begin{aligned} (d_{i+1} \to d_i)d_i a &= (d_{i+1} \to d_i)d_{i+1}b \\ &= (d_i \to d_{i+1})d_i b \\ &= (d_i \to d_{i+1})d_{i-1}a \end{aligned}$$

gives

$$(d_{i+1} \to d_i)d_i = (d_i \to d_{i+1})d_{i-1} \tag{1.14}$$

for all $i \in \mathbb{Z}$. Now equations (1.13) and (1.14) give

$$\begin{aligned} d_{i-1} \to d_i &= d_i \to (d_i \to d_{i-1})d_{i+1} \\ &= ((d_{i+1} \to d_i) \to (d_i \to d_{i-1}))(d_i \to d_{i+1}) \\ &= ((d_{i+1} \to d_i)d_i \to d_{i-1})(d_i \to d_{i+1}) \\ &= ((d_i \to d_{i+1})d_{i-1} \to d_{i-1})(d_i \to d_{i+1}) \\ &= d_i \to d_{i+1}, \end{aligned}$$

which shows that the element

$$p := d_i \to d_{i+1}$$

does not depend on i. Similarly,

$$\begin{aligned} d_{i+1} \to d_i &= d_i \to (d_i \to d_{i+1})d_{i-1} \\ &= ((d_{i-1} \to d_i) \to (d_i \to d_{i+1}))(d_i \to d_{i-1}) \\ &= ((d_{i-1} \to d_i)d_i \to d_{i+1})(d_i \to d_{i-1}) \\ &= ((d_i \to d_{i-1})d_{i+1} \to d_{i+1})(d_i \to d_{i-1}) \\ &= d_i \to d_{i-1}. \end{aligned}$$

So the element

$$q := d_i \to d_{i-1}$$

does not depend on i. Hence we obtain

$$pd_i = (d_i \to d_{i+1})d_i = (d_{i+1} \to d_i)d_{i+1} = qd_{i+1}$$

for all $i \in \mathbb{Z}$. Thus $(d_1, \dots, d_n)$ is a (q, p)-cycle. By Lemma 2, this shows that p and q are equipollent. In particular, $d_0 \to d_1$ and $d_1 \to d_0$ are equipollent. So Proposition 2 implies that d_0 and d_1 are equipollent. Since $d_0a = d_1b \leqslant a \wedge b$, there is an element $e \in H$ with $d_0a = e(a \wedge b)$. Hence we have $d_0 = e(a \to b)$ and $d_1 = e(b \to a)$. Thus $a \to b$ and $b \to a$ are equipollent. Again by Proposition 2, this yields $q(a) = q(b)$. □

Now we are ready to prove our main result.

Theorem. *The structure group $G(X)$ of an L-algebra X is torsion-free.*

Proof. Every element of $G(X)$ is of the form $q(a)^{-1}q(b)$ with $a, b \in S(X)$. Assume that $(q(a)^{-1}q(b))^n = 1$ for some $n \geqslant 2$. Then equation (1.11) gives

$$q(a_n) = q(b_n).$$

Define $c_i := b_{n-1,n-i}a_{n-i}$ for $i \in \{1, \dots, n\}$. Then equation (1.10) gives

$$\begin{aligned} b^{(n-1)}c_i &= b_{n,n-i}a_{n-i} \\ &= a_{n,i}b_i \\ &= a^{(n-1)}a_{n-1,i}b_i \\ &= a^{(n-1)}b_{n-1,n-i-1}a_{n-i-1} \\ &= a^{(n-1)}c_{i+1} \end{aligned}$$

for $1 \leqslant i < n$. Furthermore,

$$\begin{aligned} q(a^{(n-1)}c_1) &= q(a^{(n-1)}b_{n-1,n-1}a_{n-1}) \\ &= q(a_n) \\ &= q(b_n) \\ &= q(b^{(n-1)}b_{n-1,0}a_0) \\ &= q(b^{(n-1)}c_n). \end{aligned}$$

This shows that $(c_1, \dots, c_n)$ is a weak $(a^{(n-1)}, b^{(n-1)})$-cycle in $S(X)$. Thus $a^{(n-1)}$ and $b^{(n-1)}$ are equipollent by Proposition 5. So Proposition 2 implies that $q(a) = q(b)$. Whence $q(a)^{-1}q(b) = 1$. □

Since every self-similar left hoop H is an L-algebra and satisfies $S(H) = H$, we have:

Corollary. *The left group of fractions $G(H)$ of a self-similar left hoop H is torsion-free.*

2 Examples and constructions

We start with two new constructions of self-similar left hoops. Let M be a self-similar left hoop. By $\vec{S}(M)$ we denote the group of bijections

$$\alpha\colon M \smallsetminus \{1\} \to M \smallsetminus \{1\}$$

with $(xy)^\alpha = x(y^\alpha)$ for all $x, y \in M \smallsetminus \{1\}$. The equivalence (1.4) shows that $\vec{S}(M)$ consists of order automorphisms, that is,

$$x \leqslant y \iff x^\alpha \leqslant y^\alpha$$

holds for $x, y \in M \smallsetminus \{1\}$ and $\alpha \in \vec{S}(M)$. Note that the equation $(xy)^\alpha = x(y^\alpha)$ is valid also for $x = 1$, while for $y = 1$, it turns into $x^\alpha = x(1^\alpha)$. So the $\alpha \in \vec{S}(M)$ can be conceived as right multiplications with an "external" element 1^α.

Definition 2. Let H and M be self-similar left hoops, and let $\sigma\colon H \to \vec{S}(M)$ be a monoid homomorphism. We write $x^a := \sigma(a)(x)$. Define the *lexicographic semidirect product* $H \vec{\times} M$ to be the set of all formal products ax with $a \in H$ and $x \in M$ such that

$$ax \cdot by := \begin{cases} (ab)y & \text{for } x = 1, \\ a(x^b y) & \text{for } x \neq 1, \end{cases} \tag{2.1}$$

for $a, b \in H$ and $x, y \in M$.

Lemma 3. *Let H be a monoid with a partial order* (1.4) *and a binary operation $a \to b$ which satisfies $ab \leqslant c \iff a \leqslant b \to c$ for all $a, b, c \in H$. Then H is a left hoop.*

Proof. The inequality $1 \leqslant a \to a$ is trivial. For all $d \in H$, we have

$$d \leqslant ab \to c \iff dab \leqslant c \iff da \leqslant b \to c \iff d \leqslant a \to (b \to c).$$

Hence we have $ab \to c = a \to (b \to c)$. To verify equation (1.3), we note first that $(a \to b)a \leqslant a, b$. Conversely, assume that $c \leqslant a, b$. Then $c = da \leqslant b$ for some $d \in H$. Hence $d \leqslant a \to b$, and thus $d = e(a \to b)$ for some $e \in H$. So we obtain $c = da = e(a \to b)a \leqslant (a \to b)a$. □

Proposition 6. *Let H and M be left hoops with M self-similar. The lexicographic semidirect product $H \vec{\times} M$ is a left hoop which satisfies*

$$ax \leqslant by \iff x < y \ \ or \ \ (x = y \ and \ a \leqslant b). \tag{2.2}$$

Moreover, $H \vec{\times} M$ is self-similar if and only if H is self-similar.

Proof. We show first that the multiplication (2.1) is associative. So we have to verify

$$(ax \cdot by) \cdot cz = ax \cdot (by \cdot cz) \tag{2.3}$$

for $a, b, c \in H$ and $x, y, z \in M$. For $x = 1$, this follows by the associativity of H. Assume that $x \neq 1$. Since $x^b y \neq 1$, the left-hand side of equation (2.3) becomes $a(x^b y) \cdot cz = a((x^b y)^c z)$. If $y = 1$, the right-hand side is $ax \cdot (bc)z = a(x^{bc} z)$. So the equation follows since $x^{bc} = (x^b)^c$. Similarly, for $x, y \neq 1$, equation (2.3) follows by $(x^b y)^c = (x^b)(y^c)$. Furthermore, the multiplication (2.1) admits a neutral element. Thus $H \vec{\times} M$ is a monoid.

With respect to the natural ordering (1.4), $ax \leqslant by$ means that $ax = cz \cdot by$ holds for some $c \in H$ and $z \in M$. Here $z = 1$ yields $x = y$ and $a \leqslant b$, while $z \neq 1$ gives $x < y$. This proves (2.2), which shows, in particular, that (1.4) endows $H \vec{\times} M$ with a partial order.

In what follows, let $x \mapsto (a \to x)$ denote the inverse of the action $x \mapsto x^a$, that is,

$$a \to x^a = x = (a \to x)^a$$

for $a \in H$ and $x \in M \smallsetminus \{1\}$. Now we define a second operation on $H \vec{\times} M$ by

$$ax \to by := \begin{cases} a \to (x \to y) & \text{for } y \nleqslant x \nleqslant y, \\ b(a \to (x \to y)) & \text{for } y < x, \\ a \to b & \text{for } y = x, \\ 1 & \text{for } x < y. \end{cases} \tag{2.4}$$

Next we verify the equivalence

$$ax \cdot by \leqslant cz \iff ax \leqslant by \to cz \tag{2.5}$$

in $H \vec{\times} M$. Assume that $x = 1$. Then the left-hand side states that $(ab)y \leqslant cz$, that is, $y < z$ or ($y = z$ and $ab \leqslant c$). For $y < z$, both sides of (2.5) are true, while for $y = z$, both sides of (2.5) are equivalent to $a \leqslant b \to c$. So we can assume that $x \neq 1$. The left-hand side of (2.5) is then equivalent to $a(x^b y) \leqslant cz$, that is,

$$x^b y < z \quad \text{or} \quad (x^b y = z \text{ and } a \leqslant c). \tag{2.6}$$

If y and z are incomparable, $by \to cz = b \to (y \to z)$. So the right-hand side of (2.5) becomes $x \leqslant b \to (y \to z)$, that is, $x^b \leqslant y \to z$. Since $z \nleqslant y$, this is equivalent to $x^b y < z$. For $z < y$, the right-hand side of (2.5) is $ax \leqslant c(b \to (y \to z))$, which gives either $x^b < y \to z$ or ($x^b = y \to z$ and $a \leqslant c$). Since M is self-similar, $x^b = y \to z$ is equivalent to $x^b y = z$. So the right-hand side of (2.5) can be transformed into (2.6). Finally, since $x \neq 1$, both sides of (2.5) are true if $y \leqslant z$. This completes the proof of (2.5).

By Lemma 3, $H \vec{\times} M$ is a left hoop. Using Proposition 1, it remains to verify that H is self-similar if and only if

$$ax \to by \cdot ax = by \tag{2.7}$$

holds in $H \vec{\times} M$. For $y = 1$, we have $ax \to by \cdot ax = ax \to (ba)x = a \to ba$. So equation (2.7) becomes $a \to ba = b$. If $y \neq 1$, then

$$\begin{aligned} ax \to by \cdot ax &= ax \to b(y^a x) \\ &= b(a \to (x \to (y^a)x)) \\ &= b(a \to y^a) = by. \end{aligned}$$

□

To any totally ordered abelian group G we associate a self-similar left hoop $\mathrm{Lex}(G)$ as follows. Let $M(G)$ be the free monoid over G. To distinguish the multiplication of $M(G)$ with that of G, we write the elements of $M(G)$ as $a_1.a_2.\cdots.a_n$.

As G is an additive group, there is no confusion between the neutral element 0 of G and the unit element of $M(G)$, the empty word 1. We define $\mathrm{Lex}(G)$ to be the free monoid over G with the relation

$$a.b = a + b \quad \text{for } a < 0. \tag{2.8}$$

Proposition 7. *Let G be a totally ordered abelian group. Then* $\mathrm{Lex}(G)$ *is a self-similar left hoop.*

Proof. Using equation (2.8), the elements of $\mathrm{Lex}(G)$ can be put into the normal form $a_1.a_2.\cdots.a_n$ with $a_1, \dots, a_{n-1} \geqslant 0$. The existence of such a form is trivial. Uniqueness follows by considering the maximal intervals $[i, k]$ with $a_i + \cdots + a_j < 0$ for all $j \in \{i, \dots, k\}$. Thus, from now on, we represent the elements of $\mathrm{Lex}(G)$ by their normal form. Multiplying $a_1.a_2.\cdots.a_n$ from the left by some $a \in G^+$, we get $a.a_1.a_2.\cdots.a_n$. If $n \geqslant 2$, multiplying by $-a_1 - a$ for some $a > 0$ yields $(a_2 - a).a_3.\cdots.a_n$. So the natural ordering (1.4) provides $\mathrm{Lex}(G)$ with a linear, lexicographic order. Note, however, that the map $s\colon \mathrm{Lex}(G) \to \mathbb{Z}$ with

$$s(a_1.a_2.\cdots.a_n) := a_1 + \cdots + a_n$$

is neither increasing nor decreasing.

For $u, v \in \mathrm{Lex}(G)$ and $a \in G$, it is easily checked that $ua = va$ implies $u = v$. Hence $\mathrm{Lex}(G)$ is right cancellative. It follows that, if $u \geqslant v$, there is a unique $u \to v \in \mathrm{Lex}(G)$ with $v = (u \to v)u$. If $u \leqslant v$, we set $u \to v := 1$. Hence we have $(u \to v)u = u \wedge v$. In particular, $u \to vu = v$ holds for all $u, v \in \mathrm{Lex}(G)$. Since $u \leqslant v \iff uw \leqslant vw$, it follows that $uw \wedge vw = (u \wedge v)w$, that is, we have $(uw \to vw)uw = (u \to v)uw$. Hence $uw \to vw = u \to v$. Thus

$$\begin{aligned}(uv \to w)uv &= uv \wedge w \\ &= uv \wedge v \wedge w \\ &= (uv \to (v \to w)v)uv \\ &= (u \to (v \to w))uv,\end{aligned}$$

which yields $uv \to w = u \to (v \to w)$. □

Example 4. Let X_a be the L-algebra of Example 1. Consider the monoid H generated by a, b with a single relation $b^2a = b$. It is easily checked that the elements of H can be put into the normal form

$$u = a^m ba^{n_1+1} ba^{n_2+1} \cdots ba^{n_r+1} b^n$$

with $r, m, n, n_1, \dots, n_r \in \mathbb{N}$. The elements with $m = 0$ form a submonoid M. To any $u \in M$, we associate the sequence $(n_1, \dots, n_r)$ if $n = 0$, and $(n_1, \dots, n_r, -n)$ if $n > 0$. Then a straightforward calculation shows that M is isomorphic to $\mathrm{Lex}(\mathbb{Z})$, and

$$H \cong S(X_a) \vec{\times} \mathrm{Lex}(\mathbb{Z}),$$

a self-similar left hoop with structure group $\mathbb{Z}$. The canonical map $q\colon H \to \mathbb{Z}$ is given by

$$u \mapsto m + n_1 + \cdots + n_r - n.$$

In general, the image of the map $q\colon H \to G(H)$ is a submonoid of $G(H)$, but $q(H) \cap q(H)^{-1}$ need not be trivial. So the partial order of H need not induce a partial order of $G(H)$. In the present example, $q(H) = q(H)^{-1} = G(H)$.

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