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# On asymorphisms of groups

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**Abstract.** Let $G$, $H$ be groups and let $\kappa$ be a cardinal. A bijection $f : G \to H$ is called an asymorphism if, for any $X \in [G]^{<\kappa}$, $Y \in [H]^{<\kappa}$, there exist $X' \in [G]^{<\kappa}$, $Y' \in [H]^{<\kappa}$ such that for all $x \in G$ and $y \in H$, we have $f(Xx) \subseteq Y' f(x)$, $f^{-1}(Yy) \subseteq X' f^{-1}(y)$. For a set $S$, $[S]^{<\kappa}$ denotes the set $\{S' \subseteq S : |S'| < \kappa\}$. Let $\kappa$ and $\gamma$ be cardinals such that $\aleph_0 < \kappa \leq \gamma$. We prove that any two Abelian groups of cardinality $\gamma$ are $\kappa$-asymorphic, but the free group of rank $\gamma$ is not $\kappa$-asymorphic to an Abelian group provided that either $\kappa < \gamma$ or $\kappa = \gamma$ and $\kappa$ is a singular cardinal. It is known [9] that if $\gamma = \kappa$ and $\kappa$ is regular, then any two groups of cardinality $\kappa$ are $\kappa$-asymorphic.

## 1 Introduction

Following [7, 10], we say that a *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where $X$, $P$ are non-empty sets, and for all $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of $X$ which is called a *ball of radius* $\alpha$ around $x$. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set $X$ is called the *support* of $\mathcal{B}$, $P$ is called the *set of radii*.

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we set

$$B^*(x,\alpha) = \{y \in X : x \in B(y,\alpha)\},$$

$$B(A,\alpha) = \bigcup_{a \in A} B(a,\alpha),$$

$$B^*(A,\alpha) = \bigcup_{a \in A} B^*(a,\alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called a *ballean* if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P'$ such that, for every $x \in X$,

$$B(x,\alpha) \subseteq B^*(x,\alpha'),\ B^*(x,\beta) \subseteq B(x,\beta'),$$

- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x,\alpha),\beta) \subseteq B(x,\gamma),$$

- for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x,\alpha)$.

We note that a ballean can be considered as an asymptotic counterpart of a uniform space, and could be defined [11] in terms of the entourages of the diagonal $\Delta_X$ in $X \times X$. In this case a ballean is called a *coarse structure*. For categorical look at the balleans and coarse structures as "two faces of the same coin" see [3].

Let $\mathcal{B} = (X, P, B)$, $\mathcal{B}' = (X', P', B')$ be balleans. A mapping $f : X \to X'$ is called a *$\prec$-mapping* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that, for every $x \in X$, $f(B(x, \alpha)) \subseteq B'(f(x), \alpha')$.

A bijection $f : X \to X'$ is called an *asymorphism* between $\mathcal{B}$ and $\mathcal{B}'$ if $f$ and $f^{-1}$ are $\prec$-mappings. In this case $\mathcal{B}$ and $\mathcal{B}'$ are called *asymorphic*.

Let $\mathcal{B} = (X, P, B)$ be a ballean. Each subset $Y$ of $X$ defines a *subballean* $\mathcal{B}_Y = (Y, P, B_Y)$, where $B_Y(y, \alpha) = Y \cap B(y, \alpha)$. A subset $Y$ of $X$ is called *large* if $X = B(Y, \alpha)$, for $\alpha \in P$. Two balleans $\mathcal{B}$ and $\mathcal{B}'$ with supports $X$ and $X'$ are called *coarsely equivalent* if there exist large subsets $Y \subseteq X$ and $Y' \subseteq X'$ such that the subballeans $\mathcal{B}_Y$ and $\mathcal{B}'_{Y'}$ are asymorphic. In the proof of Theorem 2, we use the following equivalent definition: $\mathcal{B}$ and $\mathcal{B}'$ are coarsely equivalent if there is a $\prec$-mapping $f : X_1 \to X_2$ such that $f(X_1)$ is large and, for every $\alpha' \in P'$ there exists $\alpha \in P$ such that $f^{-1}(B'(f(x), \alpha')) \subseteq \mathcal{B}(x, \alpha)$ for each $x \in X$.

We recall [8] that an ideal $\mathcal{J}$ in the Boolean algebra of all subsets of a group $G$ is a *group ideal* if $AB^{-1} \in \mathcal{J}$ for any $A, B \in \mathcal{J}$. We suppose that $\mathcal{J}$ contains all finite subsets of $G$ and denote by $(G, \mathcal{J})$ the ballean $\mathcal{B} = (G, \mathcal{J}, B)$, where $P(g, A) = (A \cup \{e\})g$, $e$ is the identity of $G$.

For an infinite group $G$ and an infinite cardinal $\kappa$, $\kappa \le |G|$, we denote by $[G]^{<\kappa}$ the group ideal $\{A \subset G : |A| < \kappa\}$.

We say that two groups $G$ and $H$ are *$\kappa$-asymorphic* (*$\kappa$-coarsely equivalent*) if the balleans $(G, [G]^{<\kappa})$ and $(H, [H]^{<\kappa})$ are asymorphic (coarsely equivalent).

In the case $\kappa = \aleph_0$, we say that $G$ and $H$ are *finitarily asymorphic* and *finitarily coarsely equivalent* respectively. We note that finitely generated groups are finitarily coarsely equivalent if and only if $G$ and $H$ are quasi-isometric [5, Chapter 4].

A classification of countable locally finite groups (each finite subset generates finite subgroup) up to finitary asymorphisms is obtained in [6] (cf. [7, p. 103]).

*Two countable locally finite groups $G_1$ and $G_2$ are finitarily asymorphic if and only if the following conditions hold:*

(i) *for every finite subgroup $F \subset G_1$, there exists a finite subgroup $H$ of $G_2$ such that $|F|$ is a divisor of $|H|$,*

(ii) *for every finite subgroup $H$ of $G_2$, there exists a finite subgroup $F$ of $G_1$ such that $|H|$ is a divisor of $|F|$.*

It follows that there are continuum many distinct types of countable locally finite groups and each group is finitarily asymorphic to some direct sum of finite cyclic groups.

The following coarse classification of countable Abelian groups is obtained in [2].

*Two countable Abelian groups are finitarily coarsely equivalent if and only if the torsion-free ranks of* $G$ *and* $H$ *coincide and* $G$ *and* $H$ *are either both finitely generated or infinitely generated.*

In particular, any two countable torsion Abelian groups are finitely coarsely equivalent.

The referee suggested to mention [1, 4, 12] in connection to the coarse classification of Abelian groups, as well as the asymptotic dimension and its connection to the Hirsh length.

This note is motivated by the following result [9, Theorem 3]. Let $G$, $H$ be groups of cardinality $\gamma$, $\gamma > \aleph_0$. If $\gamma$ is regular, then $G$ and $H$ are $\gamma$-asymorphic.

What happens if $\gamma$ is singular? Are $G$ and $H$ $\kappa$-asymorphic for uncountable $\kappa < \gamma$?

## 2 Theorems

**Theorem 1.** *Let* $G$ *be an Abelian group of cardinality* $\gamma$*,* $\gamma > \aleph_0$ *and let* $\kappa$ *be a cardinal such that* $\aleph_0 < \kappa \leq \gamma$*. Then* $G$ *is* $\kappa$*-asymorphic to the free Abelian group* $A_\gamma$ *of rank* $\gamma$*.*

**Theorem 2.** *Let* $G$ *be an Abelian group of cardinality* $\gamma$*,* $\gamma > \aleph_0$*. Then* $G$ *is not* $\kappa$*-coarsely equivalent to the free group* $F_\gamma$ *of rank* $\gamma$ *provided that either* $\aleph_0 < \kappa < \gamma$ *or* $\kappa = \gamma$ *and* $\gamma$ *is a singular cardinal. In particular,* $G$ *and* $F_\gamma$ *are not* $\kappa$*-asymorphic.*

The referee asks: "The 'huge jump' from Abelian to free (non-Abelian) groups in Theorem 2 gives rise to the natural question of what one can do for groups close to being Abelian (as done in [1] in the case of finitary coarse equivalence)."

In this line, we conjecture that Theorem 1 remains true if $G$ satisfies the following condition: $|\{x^{-1}gx : x \in G\}| < \kappa$ for each $g \in G$.

## 3 Proofs

*Proof of Theorem* 1. We choose inductively an increasing chain of subgroups $\{G_\alpha : \alpha < \gamma\}$ of $G$ such that

(1) $G_0 = \{e\}$, $G = \bigcup_{\alpha<\gamma} G_\alpha$,

(2) $G_\alpha \subset G_\beta$ for all $\alpha < \beta < \gamma$,

(3) $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ for every limit ordinal $\beta < \gamma$,

(4) $|G_{\alpha+1} : G_\alpha| = \aleph_0$ for every $\alpha < \gamma$.

For each $\alpha < \gamma$, we fix some system $X_\alpha$ of representatives of cosets of $G_{\alpha+1} \setminus G_\alpha$ by $G_\alpha$ so $G_{\alpha+1} \setminus G_\alpha = X_\alpha G_\alpha$.

We take an arbitrary element $g \in G \setminus \{e\}$ and choose the smallest subgroup $G_\alpha$ such that $g \in G_\alpha$. By (3), $\alpha = \alpha_0 + 1$ for some $\alpha_0 < \gamma$. Then $g = g_0 x_0$, $g_0 \in G_{\alpha_0}$, $x_0 \in X_{\alpha_0}$. If $g_0 \neq e$, we repeat the argument for $g_0$: choose $\alpha_1$ such that $g_0 \in G_{\alpha_1+1} \setminus G_{\alpha_1}$ and write $g_0 = g_1 x_1$, where $g_1 \in G_{\alpha_1}$, $x_1 \in X_{\alpha_1}$ and so on. Since the set of ordinals less than $\kappa$ is well ordered, after finite number of steps, we get

$$g = x_{s(g)} \dots x_1 x_0, \quad x_i \in X_{\alpha_i}, \, i \in \{0, \dots, s(g)\}, \quad \alpha_0 > \alpha_1 > \dots > \alpha_{s(g)}.$$

By the construction, the representation is unique, so we can denote

$$\operatorname{supt}(g) = \{\alpha_{s(g)}, \dots, \alpha_1, \alpha_0\}, \quad \operatorname{supt}(e) = \varnothing.$$

We identify $A_\gamma$ with the direct sum $\bigoplus_{\alpha<\gamma} Z_\alpha$ of infinite cyclic groups and, for each $\alpha < \gamma$, fix some bijection $f_\alpha : X_\alpha \to Z_\alpha \setminus \{e_\alpha\}$, $e_\alpha$ is the identity of $Z_\alpha$. We define a bijection $f : G \to A_\gamma$ putting $f(e) = (e_\alpha)_{\alpha<\gamma}$ and, for $g \in G \setminus \{e\}$,

$$f(g) = f(x_{s(g)} \dots x_1 x_0) = (f_{\alpha_{s(g)}}(x_{s(g)}), \dots, f_{\alpha_1}(x_1), f_{\alpha_0}(x_0)).$$

We show that the bijection $f$ is an asymorphism between the balleans $(G, [G]^{<\kappa})$ and $(A_\gamma, [A_\gamma]^{<\kappa})$.

To show that $f^{-1}$ is a $\prec$-mapping, we take $a \in A_\gamma$, $K \in [A_\gamma]^{<\kappa}$ and choose a set $I \in [\gamma]^{<\kappa}$ such that $K \subseteq \bigoplus_{\alpha \in I} Z_\alpha$. We denote $X = \bigoplus_{\alpha \in I} Z_\alpha$, $b = \operatorname{pr}_I a$, $c = \operatorname{pr}_{\gamma \setminus I} a$. Then we have

$$\begin{aligned} f^{-1}(Ka) \setminus f^{-1}(Xc) &= f^{-1}(X) f^{-1}(c) \\ &= f^{-1}(X)(f^{-1}(b))^{-1} f^{-1}(b) f^{-1}(c) \\ &= f^{-1}(X)(f^{-1}(b))^{-1} f^{-1}(a) \\ &\subseteq f^{-1}(X)(f^{-1}(X))^{-1} f^{-1}(a) \end{aligned}$$

and it is suffices to note that $f^{-1}(X)(f^{-1}(X))^{-1} \in [G]^{<\kappa}$.

The verification that $f$ is a $\prec$-mapping is more delicate. We take an arbitrary $F \in [G]^{<\kappa}$ and denote by $Y$ the smallest subgroup of $G$ containing $F$ and such that if $g \in Y$ and $\alpha \in \operatorname{supt}(g)$, then $X_\alpha \subseteq Y$. We show that $Y \in [G]^{<\kappa}$. Indeed, $Y$ can be obtained in the following way. For a subset $A$ of $G$, we denote by $\langle A \rangle$ the subgroup generated by $A$ and $h(A) = A \cup \bigcup \{X_\alpha : \alpha \in \operatorname{supt}(g), g \in A\}$. We put $S_0 = \langle F \rangle$, $Y_0 = h(S_0)$ and inductively $S_{n+1} = \langle Y_n \rangle$, $Y_{n+1} = h(S_n)$. Then $Y = \bigcup_{n \in \omega} Y_n$.

To conclude the proof, we take an arbitrary $g \in G$, put $I = \bigcup_{t \in Y} \operatorname{supt}(t)$, and write $g = g_0 g_1$ where $\operatorname{supt}(g_0) \subseteq I$, $\operatorname{supt}(g_1) \subseteq \gamma \setminus I$. Then we have

$$f(Fg) \subseteq f(Y g_0 g_1) \subseteq f(Y g_1) \subseteq f(Y) f(g_1) \subseteq f(Y) f(g). \qquad \square$$

*Proof of Theorem* 2. We are going to get a contradiction assuming only that there is a $\prec$-mapping $f : G \to F_\gamma$ such that $F_\gamma = Kf(G)$ for some $K \in [F_\gamma]^{<\kappa}$. This time the Abelian group $G$ is written additively. In view of Theorem 1, we may suppose that $G$ is a group of exponent 2. Since $f$ is a $\prec$-mapping, for every $y \in G$, there exists $K_y \in [F_\kappa]^{<\kappa}$ such that, for each $x \in G$, we have

$$f(y+x) \in K_y f(x).$$

We note that the family $\{K_y : y \in G\}$ can be chosen so that $f(y) \in K_y$ (just join $f(y)$ to $K_y$) and for some cardinal $\delta$, $\kappa \le \delta < \gamma$,

(5) $|K_y| < \delta$ for each $y \in G$.

If $\kappa < \gamma$, then (5) is evident with $\delta = \kappa$. We consider the case $\gamma = \kappa$ and $\kappa$ is singular. If (5) could not be satisfied, then, for every $\delta < \gamma$, there exists $a_\delta \in G$ such that

$$|\{f(a_\delta + x)(f(x))^{-1} : x \in G\}| \ge \delta.$$

We denote $Y_\delta = \{f(a_\delta + x)(f(x))^{-1} : x \in G\}$ and use singularity of $\gamma$ to choose a subset $\Delta$ of $\gamma$ such that $|\Delta| < \gamma$ and the set $\{|Y_\delta| : \delta \in \Delta\}$ is cofinal in $\gamma$. We put $A = \{a_\delta : \delta \in \Delta\}$. Since $|A| \le |\Delta| < \gamma$ and $f$ is a $\prec$-mapping, there is a subset $Y$ of $F_\gamma$ such that $|Y| < \kappa$ and $\{f(A + x)(f(x))^{-1} : x \in G\} \subseteq Y$. Then we have got a contradiction with $Y_\delta \subseteq Y$ for each $\delta \in \Delta$, $|Y| < \gamma$ and $|Y_\delta| \ge \delta$.

We consider $F_\gamma$ as the group of all reduced group words over the alphabet $\kappa$. For $g \in F_\gamma$, we denote by $\operatorname{alp} g$ the set of all letters $\alpha < \gamma$ such that $\alpha$ or $\alpha^{-1}$ occurs in $g$, and for subset $S$ of $F_\gamma$, we put $\operatorname{alp} S = \bigcup_{g \in S} \operatorname{alp} g$.

Now we show how to find $a, b \in G$ such that

(6) $\operatorname{alp} f(a) \setminus \operatorname{alp} K_b \ne \varnothing$,

(7) $\operatorname{alp} f(b) \setminus \operatorname{alp} K_a \ne \varnothing$.

Since $F_\gamma = Kf(G)$ for some $K \in [F_\gamma]^{<\kappa}$, we can choose a subset $A \subseteq G$ such that $|A| = \delta$ and $|\operatorname{alp} f(A)| = \delta$. It suffices to take $S \subset f(G)$ such that $|S| = |\operatorname{alp} S| = \delta$ and choose $A \subset G$ such that $f(A) = S$. We take $c \in f(G)$ such that

$$\operatorname{alp} c \setminus \bigcup_{t \in A} \operatorname{alp} K_t \ne \varnothing$$

and find $b \in G$ such that $f(b) = c$. Since $|\operatorname{alp} f(A)| = \delta$, there is $a \in A$ such that $\operatorname{alp} f(a) \setminus \operatorname{alp} K_b \ne \varnothing$. Clearly, $\operatorname{alp} f(b) \setminus \operatorname{alp} K_a \ne \varnothing$.

We put $f(a+b) = z$ and note that

(8) $f(b) = f(a + (a+b)) \in K_a z$,

(9) $f(a) = f(b + (a+b)) \in K_b z$.

We take $u \in \operatorname{alp} f(a) \setminus \operatorname{alp} K_b$, $v \in \operatorname{alp} f(b) \setminus \operatorname{alp} K_a$. Then $u \in \operatorname{alp} z$, $v \in \operatorname{alp} z$. If $u$ occurs in $z$ before $v$, then, by (9), $v \in \operatorname{alp} f(a)$ so $v \in \operatorname{alp} K_a$ contradicting the choice of $v$. If $v$ occurs in $z$ before $u$, then, by (8), $u \in \operatorname{alp} f(b)$ so $u \in \operatorname{alp} K_b$ contradicting the choice of $u$. □

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