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A group of generalized finitary automorphisms of an abelian group

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Abstract. We study the group $\mathrm{IAut}(A)$ generated by the inertial automorphisms of an abelian group A, that is, automorphisms γ with the property that each subgroup H of A has finite index in the subgroup generated by H and $H\gamma$. Clearly, $\mathrm{IAut}(A)$ contains the group $\mathrm{FAut}(A)$ of finitary automorphisms of A, which is known to be locally finite. In a previous paper, we showed that $\mathrm{IAut}(A)$ is (locally finite)-by-abelian. In this paper, we show that $\mathrm{IAut}(A)$ is also metabelian-by-(locally finite). More precisely, $\mathrm{IAut}(A)$ has a normal subgroup Γ such that $\mathrm{IAut}(A)/\Gamma$ is locally finite and the derived subgroup Γ' is an abelian periodic subgroup all of whose subgroups are normal in Γ. In the case when A is periodic, $\mathrm{IAut}(A)$ turns out to be abelian-by-(locally finite) indeed, while in the general case it is not even (locally nilpotent)-by-(locally finite). Moreover, we provide further details about the structure of $\mathrm{IAut}(A)$.

1 Introduction

A subgroup H of a group G is called *inert* if it is commensurable with its conjugates $H^g := g^{-1}Hg$ for each $g \in G$ (see [1]), that is, $H \cap H^g$ has finite index in both H and H^g. Groups whose subgroups are all inert have been considered in [2] under the name TIN-groups. The class of TIN-groups contains Tarski monsters and FC-groups (see [19]), that is groups in which each element has finitely many conjugates, so it is a highly complex class. However there are no infinite locally-finite simple TIN-groups, as shown in [2]. This result has recently been extended to simple locally graded groups in [15]. On the other hand, soluble TIN-groups have been studied in [19] under some finiteness conditions.

According to [13], if H is a subgroup of an abelian group $(A, +)$ and γ is endomorphism of A, then H is called γ-inert if H has finite index in $H + H\gamma$. The interest in the concept of γ-inert subgroups comes from the study of the dynamical properties of a given endomorphism (see [14] and the bibliography therein). A subgroup H is called fully inert if it is γ-inert for every endomorphism γ of A. Fully inert subgroups of divisible abelian groups are investigated in [13]. Then from [11] we recall a definition.

Definition. An endomorphism γ of an abelian group $(A, +)$ is called *inertial endomorphism* if $(H + H\gamma)/H$ is finite for each subgroup H. An inertial endomorphism which is bijective is called *inertial automorphism*.

The set of inertial endomorphisms of an abelian group A is a subring of the ring of all endomorphisms of A. We featured this subring in [9].

Inertial automorphisms of an abelian group may be used as a tool in the study of soluble groups whose many subgroups are inert. Recall that in [10] we have studied finitely generated groups in which subnormal subgroups are inert. Also recall that, on the other hand, in [12] it has been shown that if in a (non-abelian) group G all finitely generated subgroups H are strongly inert, that is the subgroups H and H^g both have finite index in their join $\langle H, H^g \rangle$ for each $g \in G$, then the group G is (locally finite)-by-abelian. Note that strongly inert subgroups are inert.

Recall that, when G is any group, an automorphism of G is called *finitary* if it acts as the identity map on a subgroup of finite index in G. Clearly finitary automorphisms of an abelian group are inertial. In [3] it has been shown that the group $\mathrm{FAut}(G)$ of all *finitary* automorphisms of a group G is both (locally finite)-by-abelian and abelian-by-(locally finite). When G is abelian, then $\mathrm{FAut}(G)$ is even locally finite ([21]).

In this paper we study the group $\mathrm{IAut}(A)$ *generated by all inertial automorphisms* of an abelian group A. Clearly $\mathrm{IAut}(A)$ contains both $\mathrm{FAut}(A)$ and the subgroup $\mathrm{PAut}(A)$ of automorphisms of A leaving each subgroup invariant. In [11] we have shown that $\mathrm{IAut}(A)$ is locally (center-by-finite), hence (locally finite)-by-abelian. Here we consider the question of when $\mathrm{IAut}(A)$ is abelian-by-(locally finite). From the main results of this paper, it follows:

Corollary A. *If A is a periodic abelian group, then* $\mathrm{IAut}(A)$ *is center-by-(locally finite).*

Example B. The group $\mathrm{IAut}(\mathbb{Z}(p^\infty) \oplus \mathbb{Z})$ is not (locally nilpotent)-by-(locally finite), provided p is an odd prime.

Recall that a group G is said to be *metabelian* if its derived subgroup G' is abelian.

Corollary C. *If A is an abelian group, then* $\mathrm{IAut}(A)$ *is metabelian-by-(locally finite).*

Outline of the paper. Since some of our statements and proofs are rather technical, in Section 2 we fix notation in a rather accurate way, recall some definitions and results, and introduce concepts that will be used in the paper.

In Section 3 we highlight the role played by stability groups with respect to finitary and inertial automorphisms of A since in our investigations we shall look for abelian normal subgroups Σ of $\mathrm{IAut}(A)$ such that the automorphisms induced by $\mathrm{IAut}(A)$ via conjugation on Σ are inertial. We also describe the group $\mathrm{FAut}(A)$ in the case when A is of a type which is relevant to our purposes (Proposition 3.6) and in the case when A splits on its torsion subgroup (Proposition 3.8).

In Section 4 by Theorem 4.2 we give a rather complete description of $\mathrm{IAut}(A)$ when A is periodic. It follows Corollary A. In Example 4.4 we see that it may happen that there is no abelian normal subgroup Λ of $\mathrm{IAut}(A)$ such that $\mathrm{IAut}(A) = \Lambda \cdot \mathrm{FAut}(A)$.

In Section 5 by Theorem 5.1 we treat the case in which A is non-periodic. It follows Corollary C. In Example 5.2 we see that $|\,\mathrm{IAut}(\mathbb{Z}(p^\infty) \oplus \mathbb{Q}^{(p)})| = 2$.

In the final Section 6 we describe the group $\mathrm{IAut}(A)$ in detail in cases when A splits on its torsion subgroup, see Theorem 6.1 and Theorem 6.2. Thus we have Example B. Then in Example 6.3 we consider a group A for which the group $\mathrm{IAut}(A)$ is very large.

2 Preliminaries

Notation. First of all, note that, since we are going to apply methods and results from [11] where we used the fact that inertial endomorphisms form a ring, in this paper we will regard abelian groups as *right* modules over their endomorphism ring and reserve the letter A for abelian groups, which are *additively written*.

It shall be remarked that in the previous paper [8] we gave a slightly more restrictive definition of inertial automorphism (and we used different notation). This is not a major problem since in the present paper we just treat the problem in greater generality, see Fact 2.12 below.

For undefined terminology, notation and basic facts we refer to [17] or [18]. The letter $\mathbb{P}$ denotes the set of all prime numbers. If $n \in \mathbb{Z}$, then $\pi(n)$ denotes the set of prime divisors of n. If $\pi \subseteq \mathbb{P}$, then $A_\pi := \langle A_p | p \in \pi \rangle$ is called the π-component of A and π' is the set of primes not in π. Moreover, $T(A)$ and $D(A)$, respectively, denote the torsion subgroup and the maximum divisible subgroup of the abelian group A.

If $nA = 0$ for some $0 \neq n \in \mathbb{N}$, we say that A is bounded. If p is a prime, by exponent $m = \exp(A)$ of a p-group A we mean the smallest m such that $p^m A = 0$, if A is bounded. Otherwise we say that A is unbounded and write $\exp(A) = \infty$. Furthermore, $\mathbb{Q}^{(\pi)}$ is the additive group of rational numbers whose denominator n is such that $\pi(n) \subseteq \pi$ and $\mathbb{Z}(p^\infty) := \mathbb{Q}^{(p)}/\mathbb{Z}$, while $\mathbb{Z}(n)$ is the ring of integers modulo n. Also, $r_0(A)$ (resp. $r_p(A)$) denotes the torsion-free rank (resp. the p-rank) of A, i.e., the cardinality of a maximal independent subset of A of elements with infinite order (resp. with p-power order). The rank of A is $r_0(A) + \sup_{p\in\mathbb{P}} r_p(A)$.

Let $\Gamma \leq \mathrm{Aut}(A)$ and $X \leq A$. As usual, if $a\gamma \in X$ for each $a \in X$ and $\gamma \in \Gamma$, we say that X is Γ-invariant and denote by $\gamma_{|X}$ the restriction of γ to X. If we have $A = A_1 \oplus A_2$, where for $i = 1, 2$ the subgroup A_i is $\langle\gamma\rangle$-invariant and $\gamma_{|A_i} = \gamma_i \in \mathrm{Aut}(A_i)$, we write $\gamma = \gamma_1 \oplus \gamma_2$ *with respect to* $A = A_1 \oplus A_2$.

In the sequel, commutators are calculated in the holomorph group $A \rtimes \mathrm{Aut}(A)$. Moreover, if φ is an endomorphism of the additive abelian group A and $a \in A$, we use the notation $[a, \varphi] := a\varphi - a = a(\varphi - 1)$.

Now we recall some known facts and introduce some notions which will be used in the rest of the paper.

Power automorphisms of an abelian group. Recall that an automorphism leaving every subgroup invariant is called a *power automorphism*. Clearly, the set $\mathrm{PAut}(A)$ of power automorphisms of an abelian group A is a subgroup of $\mathrm{Aut}(A)$.

If A is an abelian p-group and $\alpha = \sum_{i=0}^{\infty} \alpha_i p^i$ (with $0 \leq \alpha_i < p$) is an invertible p-adic integer, then, by a slight abuse of notation, one defines the power automorphism α of A by setting $a\alpha := (\sum_{i=0}^{k-1} \alpha_i p^i)a$, for any $a \in A$ of order p^k. This gives an action on A of the group $\mathcal{U}_p$ of units of the ring of p-adic integers, whose image is known to be the whole $\mathrm{PAut}(A)$. If $\exp(A) = \infty$, this action is faithful and, if $\exp(A) =: e < \infty$, its kernel is $\{\alpha \in \mathcal{U}_p \mid \alpha \equiv 1 \bmod p^e\}$. Thus we recall the following fundamental facts (see [18, Section 13.4.3 and point 9 of Exercises 13.4]).

Fact 2.1. Let A be an abelian group.

(1) If A is a p-group and $\exp(A) = \infty$, then $\mathrm{PAut}(A)$ is isomorphic to $\mathcal{U}_p$.

(2) If A is a p-group and $e := \exp(A) < \infty$, then $\mathrm{PAut}(A)$ is isomorphic to the group of units of the ring $\mathbb{Z}(p^e)$.

(3) If A is any periodic abelian group, then $\mathrm{PAut}(A)$ is isomorphic to the cartesian product of all the groups $\mathrm{PAut}(A_p)$ where A_p is the p-component of A and p ranges in $\mathbb{P}$.

(4) If A is non-periodic, then $\mathrm{PAut}(A) = \{\pm 1\}$, i.e. $\mathrm{PAut}(A)$ consists only of the identity map and the map $x \mapsto -x$.

(5) In any case $\mathrm{PAut}(A)$ is a central subgroup of $\mathrm{Aut}(A)$.

Invertible multiplications of an abelian group. According to [11], let us consider a generalization of power automorphisms.

Definition 2.2. An automorphism γ is called an *invertible multiplication* of A if one of the following holds:

(a) A is periodic and γ is a power automorphism of A,

(b) A is non-periodic and

(∗) there exist coprime integers m, n such that $(na)\gamma = ma$, for each $a \in A$.

Note that if (∗) holds, even in the case that A is periodic, we necessarily have $mnA = A$ and $A_{\pi(mn)} = 0$. Moreover, if A is periodic, then clearly (∗) implies (a). In any case, by abuse of notation, we will write $\gamma = m/n$ and say that γ acts on A as multiplication by m/n. Furthermore, if A is a p-group, $\alpha \in \mathcal{U}_p$ and $a\gamma = a\alpha$ as defined above, we will write $\gamma = \alpha$ and say that γ acts on A as multiplication by α.

We warn the reader that we are using the word "multiplication" in a way different from [17] and that also in [8] the terminology is somewhat different. Moreover, sometimes we will omit writing the word "invertible".

Proposition 2.3. *The invertible multiplications of an abelian group A form a central subgroup of* $\mathrm{Aut}(A)$.

Proof. If A is periodic, the statement follows from Fact 2.1. Let then A be non-periodic. It is clear that the set of invertible multiplications is a subgroup of $\mathrm{Aut}(A)$. Let then $\gamma, \delta \in \mathrm{Aut}(A)$ where $(na)\gamma = ma$, for each $a \in A$, with m, n coprime. Then for each $x \in A$ we have

$$n(x\delta\gamma - x\gamma\delta) = n(x\delta)\gamma - (nx\gamma)\delta = m(x\delta) - (mx)\delta = 0.$$

It follows

$$x\delta\gamma = x\gamma\delta,$$

since $A_{\pi(nm)} = 0$. Thus $\delta\gamma = \gamma\delta$. □

Inertial automorphisms of an abelian group. In a lemma, we recall now some non-elementary facts that will be used in the sequel. They follow from [11, Propositions 2.2, 2.3, 3.3 and Theorem A].

Lemma 2.4. *Let γ be an automorphism of an abelian group A.*

(1) *If A is torsion-free and γ is inertial, then γ is an invertible multiplication.*

(2) *If $r_0(A) = \infty$, then γ is inertial if and only if there are a subgroup A_0 of finite index in A and an integer m such that γ acts as invertible multiplication by m on A_0.*

(3) *If* $0 < r_0(A) < \infty$, *then* γ *is inertial if and only if*

   (a) *there is a torsion-free* $\langle\gamma\rangle$*-invariant subgroup* V, *which is finitely generated as a* $\mathbb{Z}\langle\gamma\rangle$*-submodule, and a rational number* m/n *(with* m *and* n *coprime integers) such that* γ *acts on* V *as the invertible multiplication by* m/n,

   (b) γ *is inertial on the periodic group* A/V,

   (c) A_π *is bounded and* A/A_π *is* π*-divisible, where* $\pi := \pi(mn)$.

(4) *If* A *is periodic, then* γ *inertial if and only if* γ *is inertial on each* p*-component* A_p *of* A *and acts as invertible multiplication on all but finitely many of them.*

(5) *If* A *is a* p*-group, then* γ *is inertial if and only if either* γ *acts as an invertible multiplication on a subgroup* A_0 *of finite index in* A *or*

   (a) $D := D(A) \neq 0$ *has finite rank and* A/D *is infinite and bounded,*

   (b) *there is a* $\langle\gamma\rangle$*-invariant subgroup* A_1 *of finite index in* A *such that* γ *acts as an invertible multiplication (by possibly different* p*-adics) on both* A_1/D *and* D.

Notice that if V is as in (2) above, then there are finitely many elements $a_i \in A$ such that, in the holomorph group of A, we have that $V = \langle a_1, \ldots, a_n\rangle^\Gamma$ is isomorphic to the direct sum of $r_0(A)$ copies of $\mathbb{Q}^{(\pi)}$.

Corollary 2.5. *Invertible multiplications of an abelian group* A *with* $r_0(A) < \infty$ *are inertial.*

Proof. If A is periodic, the statement follows from Fact 2.1. Let then $\gamma = m/n$ on A and $V = F^{\langle\gamma\rangle}$ where $F \neq 0$ is a free subgroup of A such that A/F is periodic. Then V/F is contained in the π-component of A/F. Hence V is torsion-free, as $A_{\pi(mn)} = 0$. Moreover $\gamma = m/n$ on A/V, thus γ acts as a power automorphism on A/V. Thus, we may apply Lemma 2.4 (3) and deduce that γ is inertial. □

Now we state some facts that follow from [11, Theorem B and Corollary B].

Fact 2.6. Let A be an abelian group. Then:

(1) the subgroup IAut(A) is the set of the products $\gamma_1\gamma_2^{-1}$ where γ_1 and γ_2 are both inertial automorphisms,

(2) if $r_0(A) < \infty$, then IAut(A) is the set of inertial automorphisms,

(3) the group IAut$(A)/$ FAut(A) is abelian.

The group $Q(A)$ of automorphisms of a non-periodic abelian group A. Now we define a group $Q(A)$ if inertial automorphisms of A of a particular type.

Lemma 2.7. *Let A be a non-periodic abelian group and let p be a prime such that A/A_p is p-divisible and one of the following holds:*

(i) *A_p is finite,*

(ii) *$r_0(A)$ is finite and A_p is bounded.*

Then there is a unique subgroup C such that $A = A_p \oplus C$ and the automorphism $\gamma_{(p)} := 1 \oplus p$ (with respect to this decomposition) is an inertial automorphism of A.

Proof. The existence of C follows from the fact that A_p is bounded (see [17]). For the uniqueness, if C_1 is in the same conditions as C, we have that the group $C/C \cap C_1$ is trivial, as it is both bounded and p-divisible. Since C is a fully invariant subgroup of A, each $\gamma_{(p)}$ is inertial by Lemma 2.4. □

Definition 2.8. If $\pi_*(A)$ is the set of primes p as in Lemma 2.7 above, then we define $Q(A) := \langle \gamma_{(p)} \mid p \in \pi_*(A) \rangle \times \{\pm 1\}$, where 1 and -1 denote the maps id_A and $-\mathrm{id}_A$ resp.

Proposition 2.9. *If A is a non-periodic abelian group, then $Q(A)$ is isomorphic to the multiplicative group of units of $\mathbb{Q}^{(\pi_*(A))}$ and is a central subgroup of* $\mathrm{IAut}(A)$.

Proof. Apply Lemma 2.7, Proposition 2.3 and note that the rule $-1 \mapsto -\mathrm{id}_A$ and $p \mapsto \gamma_{(p)}$ (for each p) defines the wished isomorphism. □

The group $\mathrm{IAut}_1(A)$ and finitary automorphisms of an abelian group A. To answer the question in the Introduction, we will reduce a to a somewhat smaller group of automorphisms.

Definition 2.10. Let $\mathrm{IAut}_1(A)$ be the set of *inertial automorphisms of the abelian group A that act as the identity map on $A/T(A)$.*

Clearly $\mathrm{IAut}_1(A)$ is a normal subgroup of $\mathrm{IAut}_1(A)$. When A is non-periodic, we have

$$\mathrm{IAut}_1(A) \cap \mathrm{PAut}(A) = 1.$$

When A is periodic, it holds

$$\mathrm{IAut}_1(A) = \mathrm{IAut}(A).$$

Note that in the sequel we will apply the basic fact (1) below without reference.

Proposition 2.11. *Let A be an abelian group. Then:*

(1) *an automorphism γ of A is finitary if and only if the subgroup $A(\gamma - 1)$ is finite,*

(2) $\mathrm{FAut}(A) \leq \mathrm{IAut}_1(A)$,

(3) $\mathrm{FAut}(A) = \mathrm{IAut}_1(A)$, *provided* $r_0(A) = \infty$.

Proof. To prove statement (1), consider the endomorphism $\gamma - 1$ of A and that $A(\gamma - 1) \simeq \ker(\gamma - 1)$. To prove (2) note that if $\gamma \in \mathrm{FAut}(A)$, then by (1) we have $A(\gamma - 1) \leq T(A)$. So that γ acts trivially on $A/T(A)$. Finally, if $r_0(A) = \infty$, then each $\gamma \in \mathrm{IAut}(G)$ acts as multiplication by $m \in \mathbb{Z}$ on a subgroup with finite index in A by Lemma 2.4 (2). By considering the endomorphism $\gamma - m$ we have that $A(\gamma - m) \simeq A/\ker(\varphi - m)$ is finite. Therefore γ acts as the multiplication by m on $A/T(A)$. If $\gamma \in \mathrm{IAut}_1(A)$, then $m = 1$ and $\gamma \in \mathrm{FAut}(A)$. □

Almost-power automorphisms of an abelian group. Recall that the group of the so-called *almost-power* automorphisms of A, that is, automorphisms γ such that every subgroup of A contains a $\langle\gamma\rangle$-invariant subgroup of finite index was introduced in [16], where generalized soluble groups in which subnormal subgroups are normal-by-finite (or core-finite, according to the terminology of [4] and [6]) are studied.

Clearly an almost-power automorphism γ has the following property:

(†) *H and $H\gamma$ are commensurable for each $H \leq A$,*

which is stronger than the property of being inertial as defined in the Introduction.

Actually, in [8] we called *inertial* an automorphism γ with (†), while the definition of *inertial* that we are using in the present paper is different (and is the same as in [11]). However, there is no risk of misunderstandings, since applying results from [8] we have the following.

Fact 2.12. Let A be an abelian group. Then:

(1) if A is periodic, then all inertial automorphisms are almost-power,

(2) if A is non-periodic, then the group of almost-power automorphism of A is $\mathrm{IAut}_1(A) \times \{\pm 1\}$,

(3) if $r_0(A) < \infty$, then all inertial automorphisms γ have the property (†),

(4) if $r_0(A) = \infty$, then an automorphism γ of A with the property (†) is almost-power.

3 Finitary automorphisms and stability groups

We begin this section by stating some basic facts that perhaps are already known (see [5]).

If $\Gamma \le \mathrm{Aut}(A)$ and X is a subgroup of A, then we denote by $\mathrm{St}_\Gamma(A, X)$ the stability group in Γ of the series $A \ge X \ge 0$, that is, the set of $\gamma \in \Gamma$ such that $X \ge [A, \gamma] := A(\gamma - 1)$ and $[X, \gamma] = 0$. When $\Gamma = \mathrm{Aut}(A)$, then we will write $\mathrm{St}(A, X) := \mathrm{St}_{\mathrm{Aut}(G)}(A, X)$.

If X is Γ-invariant, then each $\gamma \in \Gamma$ acts via conjugation on the abelian Γ-invariant subgroup $\Sigma := \mathrm{St}(A, X)$ of Γ, according to the rule $\sigma \mapsto \gamma^{-1}\sigma\gamma =: \sigma^\gamma$ for each $\sigma \in \Sigma$. Similarly, γ acts on the additive group $\mathrm{Hom}(A/X, X)$ of homomorphisms $A/X \to X$ by a corresponding formula, i.e. $\varphi \mapsto \gamma_{|A/X}^{-1}\varphi\gamma_{|X}$, where $\varphi \in \mathrm{Hom}(A/X, X)$ and $\gamma_{|A/X}$ denotes the group isomorphism induced by γ on A/X.

Thus, by extending the above actions (and the natural action of $\mathbb{Z}$ on any abelian group) both $\mathrm{St}(A, X)$ and $\mathrm{Hom}(A/X, X)$ are equipped with a structure of right $\mathbb{Z}\Gamma$-module, where $\mathbb{Z}\Gamma$ denotes the group ring constructed in the usual way (see [18]).

Fact 3.1. Let A be an abelian group, $\Gamma \le \mathrm{Aut}(A)$ and X a Γ-invariant subgroup of A. For each $\sigma \in St_\Gamma(A, X)$, let $\sigma\mathcal{H}$ be the well-defined homomorphism

$$\bar{a} \in A/X \mapsto a\sigma - a \in X.$$

Then the map $\mathcal{H} : \sigma \in \mathrm{St}(A, X) \mapsto \sigma\mathcal{H} \in \mathrm{Hom}(A/X, X)$ is an isomorphism of $\mathbb{Z}\Gamma$-modules, so that for each $\gamma \in \mathrm{Aut}(A)$ we have

$$(\sigma^\gamma)\mathcal{H} = \gamma_{|A/X}^{-1}(\sigma\mathcal{H})\gamma_{|X}.$$

Proof. Let $\sigma, \tau \in \mathrm{St}(A, X)$, and $\bar{a} \in A/X$. On the one hand

$$\bar{a}(\sigma\mathcal{H} + \tau\mathcal{H}) = (a\sigma - a) + (a\tau - a).$$

On the other hand we have $(a\sigma - a)\tau - (a\sigma - a) = 0$, thus

$$\bar{a}(\sigma\tau\mathcal{H}) = a\sigma\tau - a = a\tau + a\sigma - 2a.$$

So $\mathcal{H}$ is a group isomorphism, the inverse being the map

$$\varphi \in \mathrm{Hom}(A/X, X) \mapsto \nu\varphi\iota + \mathrm{id} \in \mathrm{St}(A, X),$$

where ν is the canonical homomorphism $A \to A/X$, ι is the embedding of X in A and id is the identity of A. Finally, for each $\gamma \in \mathrm{Aut}(A)$ and $\bar{a} \in A/X$, we have $\bar{a}(\sigma^\gamma)\mathcal{H} = a\sigma^\gamma - a = a\gamma^{-1}(\sigma - 1)\gamma = \bar{a}(\gamma_{|A/X}^{-1}(\sigma\mathcal{H})\gamma_{|X})$ and $\mathcal{H}$ is an isomorphism of $\mathbb{Z}\Gamma$-modules. □

By this argument we have two technical lemmas. For the first one see [7].

Lemma 3.2. *Let A be an abelian group, $\sigma, \gamma \in \mathrm{Aut}(A)$ and $m_1, m_2 \in \mathbb{Z}$. If σ stabilizes a series $0 \leq A_1 \leq A$, where $\gamma = m_1$ on A_1 and $\gamma^{-1} = m_2$ on A/A_1, then $\sigma^\gamma = \sigma^{m_1 m_2}$.*

Proof. Applying Fact 3.1 with $X := A_1$ and $\Gamma = \langle \gamma \rangle$, for any $\bar{a} \in A/X$, we have

$$\begin{aligned} \bar{a}(\sigma^\gamma \mathcal{H}) &= \bar{a}\gamma_{|A/X}^{-1}(\sigma\mathcal{H})\gamma_{|X} \\ &= (m_2\bar{a})(\sigma\mathcal{H})\gamma_{|X} \\ &= m_1((m_2\bar{a})\sigma\mathcal{H}) \\ &= \bar{a}(m_1 m_2(\sigma\mathcal{H})). \end{aligned}$$

Since $\mathcal{H}$ is an isomorphism from a multiplicative to an additive group, it follows that $\sigma^\gamma = \sigma^{m_1 m_2}$. □

Our next lemma deals with the case when A splits over X and will be used several times. In such a situation, once we have fixed a direct decomposition $A = X \oplus K$, we have an embedding $\mathrm{Aut}(K) \to \mathrm{Aut}(A)$ given by $\gamma \mapsto 1 \oplus \gamma$.

Lemma 3.3. *Let $A = X \oplus K$, $\Gamma \leq \mathrm{Aut}(A)$, let X be a Γ-invariant subgroup of A and let $\zeta : A/X \leftrightarrow K$ the natural isomorphism. Furthermore, let $\Sigma := \mathrm{St}_\Gamma(A, X)$, $\Gamma_1 := \{\gamma_{|X} \oplus 1 \mid \gamma \in \Gamma\}$ and $\Gamma_2 := \{1 \oplus \zeta^{-1}\gamma_{|A/X}\zeta \mid \gamma \in \Gamma\}$.*

(1) *If $\Gamma_1 \leq \Gamma$, then $\Gamma = C_\Gamma(X) \rtimes \Gamma_1$ and $C_\Gamma(A/X) = \Sigma \rtimes \Gamma_1$.*

(2) *If $\Gamma_2 \leq \Gamma$, then $\Gamma = C_\Gamma(A/X) \rtimes \Gamma_2$ and $C_\Gamma(X) = \Sigma \rtimes \Gamma_2$.*

(3) *If $\Gamma_1\Gamma_2 \leq \Gamma$, then $\Gamma = \Sigma \rtimes (\Gamma_1 \times \Gamma_2)$.*

Proof. To prove the first statement, for each $\gamma \in \Gamma$, let $\bar{\gamma} := \gamma_{|X} \oplus 1 \in \Gamma_1$. Then $\delta := \gamma\bar{\gamma}^{-1} \in C_\Gamma(X)$, hence $\Gamma = C_\Gamma(X)\Gamma_1$. Moreover $C_\Gamma(X)$ is normal in Γ, as X is Γ-invariant, and $C_\Gamma(X) \cap \Gamma_1 = 1$, as it centralizes both X and K. If $\gamma \in C_\Gamma(A/X)$, then $\delta \in \Sigma$ and hence (1) is proved.

Statement (2) can be proved similarly and then (3) follows directly. □

Proposition 3.4. *Let A be an abelian group and $T := T(A)$.*

(1) *If $r_0(A) < \infty$, then the automorphisms induced by $\mathrm{FAut}(A)$ via conjugation on $\mathrm{St}(A, T)$ are finitary.*

(2) *If $r_0(A) = \infty$ and the quotient A/T is free abelian, then there is $\gamma \in \mathrm{FAut}(A)$ which induces via conjugation on $\mathrm{St}(A, T)$ a non-finitary automorphism, provided $\mathrm{FAut}(T) \neq 1$.*

Proof. (1) Denote $\bar{A} = A/T$ and fix $\gamma \in \mathrm{FAut}(A)$. Then $\gamma_{|A/T} = 1$ by Proposition 2.11. So by Fact 3.1 with $X := T$, for each $\sigma \in \mathrm{St}(A, T)$ we have

$$\begin{aligned}[\sigma, \gamma]\mathcal{H} &= (\sigma^{-1}\sigma^{\gamma})\mathcal{H} \\ &= -\sigma\mathcal{H} + \sigma^{\gamma}\mathcal{H} \\ &= -\sigma\mathcal{H} + \sigma\mathcal{H}\gamma_{|T} \\ &= \sigma\mathcal{H}(\gamma_{|T} - 1) =: \varphi_{\sigma}.\end{aligned}$$

Thus, we have to check that the set $\{\varphi_\sigma \mid \sigma \in \mathrm{St}(A, T)\}$ is finite. For each σ, we have that $im(\varphi_\sigma) \leq im(\gamma - 1)$ has finite order, say n. On the other hand, $\ker(\varphi_\sigma) \geq n\bar{A}$ and $\bar{A}/n\bar{A}$ is finite since $r_0(\bar{A}) < \infty$.

(2) If $A = T \oplus K$, where K is free abelian on the infinite $\mathbb{Z}$-basis $\{a_i\}$, take $\gamma_0 \in \mathrm{FAut}(T) \setminus \{1\}$. Let $t \in T$ such that $t\gamma_0 \neq t$ and $\gamma := \gamma_0 \oplus 1$. For each i define $\sigma_i \in \mathrm{St}(A, T)$ by the rule $a_i(\sigma_i - 1) := t$ and $a_j(\sigma_i - 1) := 0$ if $j \neq i$. Then there are infinitely many $[\sigma_i, \gamma]$, as $a_i \notin \ker([\sigma_i, \gamma]\mathcal{H}) \ni a_j$ for each $i \neq j$. □

Example 3.5. If $A = \mathbb{Z}(p^\infty) \oplus \mathbb{Q}^{(p)}$, then $\mathrm{St}(A, T(A)) \not\leq \mathrm{FAut}(A) = 1$.

Proof. By Fact 3.1 we have that $\mathrm{St}(A, T(A)) \simeq \mathrm{Hom}(\mathbb{Q}^{(p)}, \mathbb{Z}(p^\infty))$ is infinite, while by using Proposition 2.11 (1) it is easy to check that $\mathrm{FAut}(A) = 1$. (see also Example 5.3). □

Despite the above example, we will see that for some relevant characteristic subgroups X of A, we have $\mathrm{St}(A, X) \leq \mathrm{FAut}(A)$, provided that one of the following holds:

- A/X is bounded and X has finite rank, as in Propositions 3.6 and 4.1 (2),
- A/X has finite rank and X is bounded, as in Theorem 6.1,
- A/X is finitely generated and X is periodic, as in Theorem 6.2.

Proposition 3.6. *Let A be an abelian p-group such that $D := D(A)$ has finite rank and A/D is bounded. Then $\Sigma := \mathrm{St}(A, D)$ is a bounded abelian p-group and there is a subgroup $\Phi \simeq \mathrm{FAut}(A/D)$ such that*

$$\mathrm{FAut}(A) = \Sigma \rtimes \Phi,$$

where the automorphisms induced by Φ via conjugation on Σ are finitary and this action is faithful.

Proof. First note that if $\sigma \in \Sigma$, then $[A, \sigma] = A(\sigma - 1)$ is finite, since it is both of finite rank and bounded. Hence, we have $\sigma \in \mathrm{FAut}(A)$. Consider a decomposition $A = D \oplus B$ and apply Lemma 3.3, with $X = D$ and $\Gamma = \mathrm{FAut}(A) = C_\Gamma(X)$. Put $\Phi := \Gamma_2$. Then $\mathrm{FAut}(A) = \Sigma \rtimes \Phi$, as claimed.

Let $\gamma \in \Phi$. We have to show that set $\{[\sigma, \gamma] \mid \sigma \in \Sigma\}$ is finite. Since $\gamma_{|D} = 1$, as in Proposition 3.4, we have

$$\begin{aligned}[\sigma,\gamma]\mathcal{H} &= (\sigma^{-1}\sigma^{\gamma})\mathcal{H}\\ &= -\sigma\mathcal{H} + \sigma^{\gamma}\mathcal{H}\\ &= -\sigma\mathcal{H} + \gamma^{-1}_{|A/D}\sigma\mathcal{H}\\ &= (\gamma^{-1}_{|A/D} - 1)\sigma\mathcal{H} =: \varphi_\sigma.\end{aligned}$$

Thus, we have to count how many homomorphisms φ_σ there are. On the one hand, $\ker(\varphi_\sigma)$ contains $\ker(\gamma^{-1}_{|A/D} - 1)$ which has finite index in A/D. On the other hand, the image of each φ_σ is contained in the finite subgroup $D[p^m]$, where p^m is a bound for A/D. Therefore, there are only finitely many φ_σ, once γ is fixed.

Let us check that the action is faithful. Let $1 \neq \gamma \in \Phi$ and let $b \in B$ with maximal order and $b \neq b\gamma$. Then we have $B = \langle b \rangle \oplus B_0$ for some $B_0 \leq B$ and we can write $b\gamma = nb + b_0$ with $n \in \mathbb{Z}, b_0 \in B_0$. If $b \neq nb$, then there is $\sigma \in \Sigma$ such that $B_0(\sigma - 1) = 0$ and $b(\sigma - 1) = d$ where $d \in D$ has the same order as b. Thus, by Fact 3.1, $b\gamma(\sigma^\gamma - 1) = b\gamma(\gamma^{-1}(\sigma - 1)) = d$, while $b\gamma(\sigma - 1) = nd$. Therefore, we have $\sigma^\gamma \neq \sigma$. Similarly, if $b = nb$, then there is $\sigma \in \Sigma$ such that $b(\sigma - 1) = 0$ and $b_0(\sigma - 1) = d_1$ of order p. Then we have $b\gamma(\sigma^\gamma - 1) = 0$, while $b\gamma(\sigma - 1) = d_1$ and again $\sigma^\gamma \neq \sigma$. □

We now see that, in Proposition 3.6 the picture may be rather complicated. Recall that the FC-center of a group is the set of elements with finitely many conjugates or – equivalently – whose centralizer has finite index.

Example 3.7. There is an abelian p-group A such that $D := D(A)$ has finite rank, A/D is bounded and $\Sigma := \mathrm{St}(A, D)$ is not contained in the FC-center of $\mathrm{FAut}(A)$.

Proof. Write $A = D \oplus B_0$, where $D \simeq \mathbb{Z}(p^\infty)$ and $B_0 = \bigoplus_i \langle b_i \rangle \leq B$ is infinite and homogeneous. Fix $\sigma \in \Sigma$ such that $b_1(\sigma - 1) = d$, where d is an element of D of order p, and $\sigma - 1 = 0$ on $D \oplus \bigoplus_{j \neq 1} \langle b_j \rangle$. For each i consider $\gamma_i \in \mathrm{FAut}(A)$ switching $b_i \leftrightarrow b_1$ and acting trivially on $D \oplus (\bigoplus_{j \notin \{1,i\}} \langle b_j \rangle)$. Then

$$\sigma^{\gamma_i} = \gamma_i^{-1}(\sigma - 1) + 1.$$

Hence, $b_i\sigma^{\gamma_i} = d + b_i$ and $b_j\sigma^{\gamma_i} = b_j$ for each $j \neq i$. □

We now use a similar argument in the case when $X = T(A)$.

Proposition 3.8. *Let A be an abelian group with $r_0(A) < \infty$ such that A/T is finitely generated (resp. $T := T(A)$ is bounded). Then $\Sigma := \mathrm{St}(A, T)$ is a periodic (resp. bounded) abelian group and there is a subgroup $\Phi_1 \simeq \mathrm{FAut}(T)$ such that*

$$\mathrm{FAut}(A) = \Sigma \rtimes \Phi_1,$$

where Φ_1 induces via conjugation on Σ finitary automorphisms.

In the case when $A/T \neq 0$ is finitely generated, this action by conjugation is faithful.

Proof. In any case, we can write $A = T \oplus K$, where $r := r_0(K) < \infty$. Recall that $\Sigma \simeq \mathrm{Hom}(A/T, T)$. Note that $\Sigma \leq \mathrm{FAut}(A)$. In fact, if $\sigma \in \Sigma$, then we have that $\sigma - 1 \in \mathrm{Hom}(A/T, T)$ and $A(\sigma - 1)$ is an abelian group which is both finitely generated and periodic (resp. finite rank and bounded). Hence, $A(\sigma - 1)$ is finite that is $\sigma \in \mathrm{FAut}(A)$.

Clearly $\Phi_1 := \{\varphi \oplus 1 \mid \varphi \in \mathrm{FAut}(T)\} \simeq \mathrm{FAut}(T)$ and $\Phi_1 \leq \mathrm{FAut}(A)$. Then, by Lemma 3.3 (1) we have that $\mathrm{FAut}(A) = \Sigma \rtimes \Phi_1$. By Proposition 3.4, Φ_1 induces via conjugation on Σ finitary automorphisms.

If A/T is finitely generated, then $\Sigma \simeq \mathrm{Hom}(A/T, T)$ is a periodic abelian group which is naturally isomorphic to the direct sum of r copies of T as a right $\mathrm{Aut}(A)$-module. Therefore, the action of Φ_1 on Σ is faithful. □

Example 3.9. In the notation of Proposition 3.8, if $A = \mathbb{Z}_{12} \oplus \mathbb{Q}^{(2)}$, then Σ is not self-centralizing in $\mathrm{FAut}(A)$, that is, the action that Φ_1 induces via conjugation on Σ is not faithful.

Proof. We have $\mathrm{FAut}(A)/\Sigma \simeq \mathcal{U}\mathbb{Z}_{12}$ and $\Sigma \simeq \mathbb{Z}_3$. □

4 The group IAut(A) when A is periodic

To give a detailed description of $\mathrm{IAut}(A)$ when A is a p-group, let us introduce some terminology.

Definition. If A is a p-group, by the *essential exponent* $e = \mathrm{eexp}(A)$ of A we mean the smallest e such that $p^e A$ is finite, or $e = \infty$ if A is unbounded. In the former case, this is equivalent to saying that $A = A_0 \oplus A_1 \oplus A_2$, where A_0 is finite, $\exp(A_1) < e \leq \exp(A_0)$ and A_2 is the sum of infinitely many cyclic groups of order p^e. In [8] we called *critical* a p-group of type $A = B \oplus D$ with B infinite but bounded and $D \neq 0$ divisible with finite rank (see Lemma 2.4 (5)).

Critical groups will be a tool to describe $\mathrm{IAut}(A)$ when A is periodic.

Proposition 4.1. *Let A be an abelian p-group and $D := D(A)$.*

(1) *If A is non-critical, then*

$$\mathrm{IAut}(A) = \mathrm{PAut}(A) \cdot \mathrm{FAut}(A),$$

where $\mathrm{PAut}(A) \cap \mathrm{FAut}(A)$ *is either trivial or cyclic of order p^{m-e}, according as A is unbounded or $m :=$* $\exp(A) < \infty$ *and $e :=$* $\mathrm{eexp}(A)$.

(2) *If $A = D \oplus B$ is critical, let*

$$\Delta := \{1 \oplus n \mid n \in \mathbb{Z} \setminus p\mathbb{Z}\},$$
$$\Phi := \{1 \oplus \varphi_0 \mid \varphi_0 \in \mathrm{FAut}(B)\},$$
$$\Psi := \{1 \oplus \gamma_0 \mid \gamma_0 \in \mathrm{IAut}(B)\}.$$

Then

$$\mathrm{IAut}(A) = \mathrm{PAut}(A) \times (\mathrm{FAut}\, A \cdot \Delta).$$

Moreover, $\mathrm{FAut}\, A \cdot \Delta = C_{\mathrm{IAut}(A)}(D) = \Sigma \rtimes \Psi$, *where* $\mathrm{FAut}(A) = \Sigma \rtimes \Phi$ *and the following hold:*

(i) $\Sigma := \mathrm{St}(A, D)$ *is an infinite abelian p-group,* $\exp(\Sigma) = \exp(B) =: m' < \infty$ *and* $\mathrm{eexp}(\Sigma) = \mathrm{eexp}(B) =: e'$,

(ii) $\Psi = \Phi\Delta \simeq \mathrm{IAut}(B)$, *where* $[\Phi, \Delta] = 1$ *and Ψ induces via conjugation on Σ inertial automorphisms and this action is faithful,*

(iii) $\Delta \simeq \mathrm{PAut}(B) \simeq \mathcal{U}(\mathbb{Z}(p^{m'}))$, *each $\delta_n := 1 \oplus n \in \Delta$ acts via conjugation on Σ as multiplication by n and* $\mathrm{FAut}(A) \cap \Delta$ *has order $p^{m'-e'}$,*

(iv) $\Phi \simeq \mathrm{FAut}(B)$ *and Φ induces via conjugation on Σ finitary automorphisms.*

Proof. Let $\gamma \in \Gamma := \mathrm{IAut}(A)$.

(1) If A is non-critical, then, according to Lemma 2.4 (5), there exist a p-adic α and a subgroup A_0 of finite index in A such that $\gamma_{|A_0} = \alpha$. Thus, $\gamma^{-1}\alpha$ acts on A_0 as the identity map, that is, $\gamma^{-1}\alpha \in \mathrm{FAut}(A)$. Hence,

$$\mathrm{IAut}(A) = \mathrm{PAut}(A) \cdot \mathrm{FAut}(A).$$

Further, if the multiplication by the p-adic number $\beta = \sum_i \beta_i p^i$ $(0 \le \beta_i < p)$ is in $\mathrm{PAut}(A) \cap \mathrm{FAut}(A)$, then it is trivial on a subgroup B of finite index in A. Therefore, if $\exp(A) = \infty$, then $\exp(B) = \infty$ and $\beta = 1$. Otherwise, $\exp(B) \ge e$ and then $\beta \equiv 1 + \sum_{i=e+1}^{m-1} \beta_i p^i \mod p^m$ (see Fact 2.1 and the definition of action given before). Thus, there are at most p^{m-e} choices for such a β. On the other hand, the invertible multiplication on A by a p-adic number $\beta \equiv 1 \mod p^e$ is a finitary automorphisms since it acts trivially on $A[p^e]$.

(2) Let $A = D \oplus B$ be critical. By Lemma 2.4 (5) there exists an invertible p-adic α such that $\gamma_{|D} = \alpha$. Thus, we get $\gamma_1 := \gamma\alpha^{-1} \in C_\Gamma(D)$. Clearly, we have $\mathrm{PAut}(A) \cap C_\Gamma(D) = 1$, so that $\mathrm{IAut}(A) = \mathrm{PAut}(A) \times C_\Gamma(D)$.

Again by Lemma 2.4 (5), γ_1 acts by multiplication by an integer n on a subgroup of finite index in $A[p^{m'}]$ where $A[p^{m'}] \geq B$. Therefore, if $\delta_n := 1 \oplus n \in \Delta$ with respect to $A = D \oplus B$, we have $\gamma_1\delta_n^{-1} \in \mathrm{FAut}(A)$. Hence, $C_\Gamma(D) = \mathrm{FAut}(A) \cdot \Delta$.

It is routine to verify that (i) holds, since $\Sigma := \mathrm{St}(A, D) \simeq \mathrm{Hom}(B, D)$. By Proposition 3.6, (iv) holds as well. By Lemma 3.3 (with $X := D$, $K := B$ and so $\Gamma_2 = \Psi$), we have $C_\Gamma(D) = \Sigma \rtimes \Psi$ as stated in (2). Then, applying part (1) of the statement to B, we have $[\Phi, \Delta] = 1$ and $\Psi = \Phi\Delta = \Delta\Phi$ as stated in (ii). Moreover, $\mathrm{FAut}(A) \cap \Delta$ has order $p^{m'-e'}$.

By Lemma 3.2, we have that Δ acts on Σ as in (iii). Thus, the whole of $\Psi = \Delta\Phi$ acts via conjugation on Σ inducing inertial automorphisms and (ii) holds.

It remains to show that Ψ acts faithfully on Σ. Let $\varphi\delta_n \in C_\Psi(\Sigma)$ with $\varphi \in \Phi$ and $\delta_n := 1 \oplus n \in \Delta$. On one hand, δ_n acts via conjugation on Σ as the multiplication by n by (iii). On the other hand, δ_n is finitary on Σ by (iv). Since $\mathrm{eexp}(\Sigma) = \mathrm{eexp}(B)$ by (i), then multiplication by n is finitary on B. Thus, we have $\delta_n \in C_\Phi(\Sigma) = 1$ by Proposition 3.6. □

In next statement we regard $\mathrm{FAut}(A_\pi)$ as naturally embedded in $\mathrm{FAut}(A)$.

Theorem 4.2. *Let A be a periodic abelian group. Then there is a subgroup Δ of* $\mathrm{IAut}(A)$ *which is a direct product of finite abelian groups and such that*

$$\mathrm{IAut}(A) = \mathrm{PAut}(A) \cdot \mathrm{FAut}(A) \cdot \Delta,$$

where Δ is trivial if A contains no non-trivial divisible subgroups.

Moreover, there are a set π of primes and subgroups Σ, Ψ of $\mathrm{IAut}(A)$ *such that Σ is an abelian π'-group with bounded primary components and*

$$\mathrm{FAut}(A) \cdot \Delta = \mathrm{FAut}(A_\pi) \times (\Sigma \rtimes \Psi),$$

where the automorphisms induced by Ψ via conjugation on Σ are inertial and this action is faithful.

Proof. From Lemma 2.4 (4) we know that the group $\mathrm{IAut}(A)$ may be identified with $\mathrm{PAut}(A) \cdot \mathrm{Dr}_{p \in \mathbb{P}}\ \mathrm{IAut}(A_p)$. Apply Proposition 4.1 to each A_p. Let π be the set of primes p for which A_p is not critical. If $p \in \pi$, we have

$$\mathrm{IAut}(A_p) = \mathrm{PAut}(A_p) \cdot \mathrm{FAut}(A_p).$$

Otherwise, for each $p \notin \pi$, there are subgroups $\Delta_p, \Sigma_p, \Psi_p$ corresponding to Δ, Σ, Ψ in Proposition 4.1 such that $\mathrm{IAut}(A_p) = \mathrm{PAut}(A_p) \cdot \mathrm{FAut}(A_p) \cdot \Delta_p$ and

$\mathrm{FAut}(A_p) \cdot \Delta_p = \Sigma_p \rtimes \Psi_p$. Now it is routine to verify that the statement follows by setting $\Delta := \mathrm{Dr}_{p \notin \pi} \Delta_p$, $\Sigma := \mathrm{Dr}_{p \notin \pi} \Sigma_p$, $\Psi := \mathrm{Dr}_{p \notin \pi} \Psi_p$, and recalling that $\mathrm{FAut}(A) = \mathrm{Dr}_{p \in \mathbb{P}} \mathrm{FAut}(A_p)$ as $A = \bigoplus_{p \in \mathbb{P}} A_p$. □

Proof of Corollary A. In the notation of Theorem 4.2, we have that Δ is periodic abelian, $\mathrm{PAut}(A)$ is central and $\mathrm{FAut}(A)$ is locally finite as recalled in Section 2 and the Introduction, respectively. □

Lemma 4.3. *If B_0 is a subgroup of finite index in a bounded abelian group B, then there are subgroups B_1 and B_2 such that B_2 is finite, $B_1 \geq B_0$ and $B = B_1 \oplus B_2$.*

Proof. Clearly there is a finite subgroup F such that $B = B_0 + F$. Since B_0 is separable and $B_0 \cap F$ is finite, then there is a finite subgroup $B_3 \geq B_0 \cap F$ such that $B_0 = B_1 \oplus B_3$ for some $B_1 \leq B_0$. Fix B_1 and $B_2 := B_3 + F$. On the one hand

$$B_1 + B_2 = B_1 + B_3 + F = B_0 + F = B.$$

On the other hand, by Dedekind's law,

$$\begin{aligned} B_1 \cap B_2 &= B_1 \cap (B_3 + F) \\ &= B_1 \cap (B_0 \cap (B_3 + F)) \\ &= B_1 \cap (B_3 + (B_0 \cap F)) \\ &= B_1 \cap B_3 = 0. \end{aligned}$$

□

Example 4.4. If A is a critical p-group (with $p \neq 2$) and $\Lambda \lhd \mathrm{IAut}(A)$ is such that $C_\Gamma(D) = \mathrm{FAut}(A) \cdot \Lambda$, then Λ is neither finite nor locally nilpotent.

Proof. We use the same notation as in Proposition 4.1. Let $n \in \mathbb{N}$ be a primitive root of $1 \mod p^{m'}$ and consider $\delta := 1 \oplus n \in \Delta$ with respect to $A = D \oplus B$. Since $\Delta \leq C_\Gamma(D) = \mathrm{FAut}(A) \cdot \Lambda$, we can assume that $\varphi \in \mathrm{FAut}(A)$ and $\lambda \in \Lambda$. Hence, $\delta = \lambda = n$ on some subgroup B_0 of finite index in B. By Lemma 4.3, $B = B_1 \oplus B_2$ with $B_1 \leq B_0$ and B_2 finite. Put $A_1 := D + B_1$ and note that $\lambda_{|A_1} = 1 \oplus n$ with respect to $A_1 = D \oplus B_1$.

It is sufficient to show that $\langle \lambda \rangle^{\Gamma_1}$ is infinite and not locally nilpotent, where Γ_1 is the group of (inertial) automorphisms of A of type $\gamma_1 \oplus 1$ with respect to $A = A_1 \oplus B_2$, with $\gamma_1 \in \mathrm{IAut}(A_1)$. Thus, we may assume $A_1 = A$ and $\Gamma := \Gamma_1$. Then multiplication by n is in Λ and $\Lambda = \Delta^\Gamma$.

We claim that $\Delta^\Gamma = \Sigma \rtimes \Delta$. In fact, by Proposition 4.1 we have that the group $\Delta \simeq \mathcal{U}(\mathbb{Z}_{p^{m'}})$ acts faithfully by multiplication on the infinite abelian p-group Σ of exponent m' and then $\Sigma = [\Sigma, \Delta]$ and $\Delta^\Gamma = \Sigma\Delta$, as claimed. Thus, Δ^Γ is not locally nilpotent, since the action of Δ on Σ is fixed-point-free. □

Remark that, in Theorem 4.2, when we consider the action of the above Ψ on the p-component Σ_p of the group Σ, we are concerned with subgroups of $\mathrm{IAut}(\Sigma_p) = \mathrm{PAut}(\Sigma_p) \cdot \mathrm{FAut}(\Sigma_p)$, where Σ_p is a bounded abelian p-group and $\mathrm{PAut}(\Sigma_p)$ is finite abelian.

5 The group IAut(A) when A is non-periodic

Let us state now our main results in the non-periodic case. Recall that metabelian groups G in which each subgroup of G' is normal in G are called *KI-groups* and have been studied in a series of papers (see [20] and the references therein).

Theorem 5.1. *Let A be a non-periodic abelian group. Then there is a central subgroup $Q(A)$ of* IAut(A)*, which is isomorphic to a multiplicative group of rational numbers, such that*

$$\mathrm{IAut}(A) = \mathrm{IAut}_1(A) \times Q(A).$$

Moreover, there is a normal subgroup Γ of $\mathrm{IAut}_1(A)$ *such that:*

(i) $\mathrm{IAut}_1(A)/\Gamma$ *is locally finite,*

(ii) Γ *is a KI-group with periodic derived subgroup.*

In particular, we have that if A is torsion-free, then $\mathrm{IAut}(A) = Q(A)$ is abelian, as $\mathrm{IAut}_1(A) = 1$. Further, we will prove that in the statement of Theorem 5.1 one may take Γ to be the subgroup of $\mathrm{IAut}_1(A)$ consisting of inertial automorphisms acting by multiplication on $T(A)$. Unfortunately this subgroup need not be locally nilpotent, as in Example B.

Proof. Let $\gamma_{(p)}$ and $Q := Q(A)$ as in Lemmata 2.7, 2.9 and Definition 2.8.

We first consider the case when $r_0(A) = \infty$. Let $\gamma \in \mathrm{IAut}(A)$. By Fact 2.6 (1), we have $\gamma = \gamma_1\gamma_2^{-1}$ with γ_1, γ_2 inertial. Further, by Lemma 2.4 (2), there is a subgroup A_0 with finite index in A such that we have $\gamma_{|A_0} = m/n = p_1^{s_1} \cdots p_t^{s_t} \in \mathbb{Q}$ (m, n coprime, p_i prime, $s_i \in \mathbb{Z}$). Also, $\mathrm{IAut}_1(A) = \mathrm{FAut}(A)$ and $\gamma = m/n$ on A/T as well. If $m/n = 1$, then $\gamma \in \mathrm{FAut}(A)$. If $m/n = -1$, put $\gamma_0 := -1 \in Q$. Otherwise, since γ is invertible, we have that $mA_0 = A_0 = nA_0$. Then, for each $p_i \in \pi := \pi(mn)$, the p_i-component of A is finite and A/T is p_i-divisible. Consider then

$$\gamma_0 := \gamma_{(p_1)}^{s_1} \cdots \gamma_{(p_t)}^{s_t} \in Q.$$

In both cases, $\gamma\gamma_0^{-1} = 1$ on $A_0/(A_0)_\pi$ hence $\gamma\gamma_0^{-1} \in \mathrm{FAut}(A)$. Thus,

$$\mathrm{IAut}(A) = \mathrm{IAut}_1(A) \times Q(A).$$

Moreover, (i) and (ii) are true with $\Gamma = 1$, since $\mathrm{IAut}_1(A) = \mathrm{FAut}(A)$ is locally finite.

Let then $r_0(A) < \infty$ and $\gamma \in \mathrm{IAut}(A)$. By Fact 2.6 (2), γ is inertial. By Lemma 2.4 (3), we have that $\gamma = m/n = p_1^{s_1} \cdots p_t^{s_t} \in \mathbb{Q}$ (m, n coprime, p_i prime, $s_i \in \mathbb{Z}$) on A/T. We also have that, for each $p_i \in \pi := \pi(mn)$, the group A/T is p_i-divisible and A_{p_i} is bounded. Consider

$$\gamma_0 := \gamma_{(p_1)}^{s_1} \cdots \gamma_{(p_t)}^{s_t} \in Q.$$

Clearly $\gamma_0 = m/n$ on A/T. Thus, $\gamma\gamma_0^{-1}$ acts trivially on A/T and

$$\mathrm{IAut}(A) = \mathrm{IAut}_1(A) \times Q(A),$$

as stated.

Let Γ be the preimage of $\mathrm{PAut}(T)$ under the canonical homomorphism $\mathrm{IAut}_1(A) \mapsto \mathrm{IAut}(T)$. Now statement (i) holds, since $\mathrm{IAut}_1(A)/\Gamma$ is locally finite by Theorem 4.2. To check (ii), consider that the derived subgroup Γ' of Γ stabilizes the series $0 \leq T \leq A$ and therefore is abelian. Moreover, by Fact 2.6 (3), the subgroup Γ' consists of finitary automorphisms. Thus, Γ' is torsion and (ii) holds by Lemma 3.2. □

Proof of Corollary C. Apply Theorem 5.1 and note that $Q(A)\Gamma$ is metabelian where $\mathrm{IAut}(A)/Q(A)\Gamma$ is isomorphic to a quotient of $\mathrm{IAut}_1(A)/\Gamma$. □

When A/T is not finitely generated, it may happen that A has very few inertial automorphisms.

Proposition 5.2. *Let A be a π-divisible non-periodic abelian group, where π is a set of primes. If $T := T(A)$ is a π-group, then* $\mathrm{IAut}_1(A) = 1$.

Proof. If $r_0(A) = \infty$, then $\mathrm{IAut}_1(A) = \mathrm{FAut}(A)$. Moreover, if $\gamma \in \mathrm{FAut}(A)$, then $A(\gamma - 1)$ is a finite π-group. Then $A/\ker(\gamma - 1)$ is such. Hence, $A = \ker(\gamma - 1)$ and $\mathrm{FAut}(A) = 1$.

If $r_0(A) < \infty$, by Lemma 2.4 (3) we have $\gamma = 1$ on some free abelian subgroup $V \leq A$ such that A/V is periodic. Moreover, the π-component B/V of A/V is divisible. Then, by Lemma 2.4, parts (4) and (5), we have that γ is a multiplication on B/V. Furthermore, the group $B/(V + T)$ is π-divisible and has non-trivial p-component for each $p \in \pi$, since $(V + T)/T \simeq V$ is free abelian. Thus, from $\gamma = 1$ on B/T it follows that $\gamma = 1$ on $\gamma = 1$. Hence, γ stabilizes the series $0 \leq V \leq B$. However $\mathrm{Hom}(B/V, V) = 0$. Then $\gamma = 1$ on B. Therefore, $\gamma - 1$ induces a homomorphism $A/B \to T$ which is necessarily 0 since A/B is a π'-group. Thus, $\gamma = 1$ on the whole group A. □

From Proposition 5.2 and Lemma 2.4 (3) we have

Example 5.3. If $A = \mathbb{Z}(p^\infty) \oplus \mathbb{Q}^{(p)}$, then $\mathrm{IAut}(A) = \{\pm 1\}$.

6 The group IAut(A) when A splits on $T(A)$

The next two theorems consider cases in which A splits on its torsion subgroup T.

Theorem 6.1. *Let A be an abelian group and $T := T(A)$. If $r_0(A) < \infty$ and T is bounded, then $\Sigma := \mathrm{St}(A, T)$ is a bounded abelian group and there is a subgroup Γ_1 of $\mathrm{IAut}_1(A)$ such that $\Gamma_1 \simeq \mathrm{IAut}(T)$ and*

$$\mathrm{IAut}_1(A) = \Sigma \rtimes \Gamma_1,$$

where Γ_1 induces via conjugation on Σ inertial automorphisms.

Proof. We can write $A = T \oplus K$, where $r := r_0(K) < \infty$. Note that the group $\Sigma \simeq \mathrm{Hom}(A/T, T)$ is a periodic abelian group which is bounded as T.

Clearly $\Gamma_1 := \{\gamma \oplus 1 \mid \gamma \in \mathrm{IAut}(T)\} \simeq \mathrm{IAut}(T)$. If $\gamma \in \mathrm{IAut}(T)$, then $\gamma \oplus 1$ (with respect to $T \oplus K$) is inertial by Lemma 2.4 (5), and so $\Gamma_1 \leq \mathrm{IAut}_1(A)$. Thus, we may apply Lemma 3.3 with $\Gamma := \mathrm{IAut}_1(A)$. We obtain $\mathrm{IAut}_1(A) = \Sigma \rtimes \Gamma_1$, as claimed.

By Proposition 4.1, we have

$$\mathrm{IAut}(T) = \mathrm{FAut}(T) \cdot \mathrm{PAut}(T).$$

Hence, we have $\Gamma_1 = \Phi_1 \Delta_1$, where $\Phi_1 := \{\varphi \oplus 1 \mid \varphi \in \mathrm{FAut}(T)\} \simeq \mathrm{FAut}(T)$ acts by conjugation on Σ by means of finitary automorphisms, by Proposition 3.8, and $\Delta_1 := \{\delta \oplus 1 \mid \delta \in \mathrm{PAut}(T)\} \simeq \mathrm{PAut}(T)$ acts via conjugation on Σ by means of multiplications, by Lemma 3.2. Therefore, the whole Γ_1 induces by conjugation on Σ inertial automorphisms. □

We notice that the action of Γ_1 on Σ in Theorem 6.1 need not be faithful, as already seen in Proposition 3.8.

Theorem 6.2. *Let A be a non-periodic abelian group and $T := T(A)$. If A/T is finitely generated, then $\Sigma := \mathrm{St}(A, T)$ is a periodic abelian group and there is a subgroup Γ_1 of $\mathrm{IAut}_1(A)$ such that $\Gamma_1 \simeq \mathrm{IAut}(T)$ and*

$$\mathrm{IAut}_1(A) = \Sigma \rtimes \Gamma_1$$

where Γ_1 induces via conjugation on Σ inertial automorphisms and this action is faithful.

If in addition T is unbounded, then $\mathrm{IAut}_1(A)$ is not nilpotent-by-(locally finite). Further, if $A_{2'}$ is unbounded, then $\mathrm{IAut}_1(A)$ is not even (locally nilpotent)-by-(locally finite).

Proof. As in the proof of Theorem 6.1, we can write $A = T \oplus K$, where K is finitely generated. The group $\Sigma \simeq \mathrm{Hom}(A/T, T)$ is a periodic abelian group which is isomorphic to the direct sum $\oplus_r T$ of $r := r_0(A) > 0$ copies of T as a right $\mathrm{Aut}(A)$-module.

Clearly $\Gamma_1 := \{\gamma \oplus 1 \mid \gamma \in \mathrm{IAut}(T)\} \simeq \mathrm{IAut}(T)$. If $\gamma \in \mathrm{IAut}(T)$, then $\gamma \oplus 1$ (with respect to $T \oplus K$) is inertial by Lemma 2.4 (5). Hence, $\Gamma_1 \leq \mathrm{IAut}_1(A)$. Thus, we may apply Lemma 3.3 with $\Gamma := \mathrm{IAut}_1(A)$, and we obtain

$$\mathrm{IAut}_1(A) = \Sigma \rtimes \Gamma_1.$$

Let us investigate now the action of Γ_1 via conjugation on Σ. Assume first that T is a p-group. Let $\gamma \in \mathrm{IAut}(T)$. By Proposition 4.1, we have $\gamma = \gamma_0\varphi$, where $\varphi \in \mathrm{FAut}(T)$ and either $\gamma_0 \in \mathrm{PAut}(T)$ or T is a critical p-group and γ_0 induces multiplications on both $D(T)$ and $T/D(T)$. Recall that Σ is $\mathrm{Aut}(A)$-isomorphic to $\oplus_r T$. In the former case, that is if $\gamma_0 \in \mathrm{PAut}(T)$, then $\gamma_0 \oplus 1$ acts via conjugation on Σ as a power automorphism (that is a multiplication). In the latter case, Σ is critical as well and $\gamma_0 \oplus 1$ induces invertible multiplications on both $D(\Sigma)$ and $\Sigma/D(\Sigma)$. Thus, $\gamma_0 \oplus 1$ acts via conjugation on Σ as an inertial automorphism of Σ, by Lemma 2.4 (5). In both cases, by Proposition 3.8, φ acts via conjugation on Σ as a finitary automorphism. Hence, $\gamma \oplus 1$ acts via conjugation on Σ as an inertial automorphism.

In the general case, when T is any periodic group and $\gamma \in \mathrm{IAut}(T)$, then $\gamma \oplus 1$ (with respect to $T \oplus K$) acts via conjugation as an inertial automorphism on all primary components Σ_p of Σ, by what we have seen above and the fact that $\Sigma_p \simeq \mathrm{Hom}(A/T, A_p)$. Similarly, since $\gamma \oplus 1$ acts as a multiplication on all but finitely many primary components A_p of A, it acts the same way on all but finitely many Σ_p. Thus, $\gamma \oplus 1$ is inertial on Σ by Lemma 2.4 (4).

It is clear that the action via conjugation of Γ_1 on Σ is faithful as the standard action of Γ_1 on T is such.

To prove the last part of the statement, note that in the case when T is unbounded, there exists a non-periodic multiplication α of T. Note that the automorphism $\mu := \alpha \oplus 1$ (with respect to $T \oplus K$) belongs to Γ_1. If, by way of contradiction, $\langle \Sigma, \mu \rangle$ is nilpotent-by-(locally finite), then there is $s \in \mathbb{Z} \setminus \{0\}$ such that $\langle \Sigma, \mu^s \rangle$ is nilpotent, so there is $n \in \mathbb{N}$ such that $[\Sigma,_n \mu^s] = 0$, and hence $0 = \Sigma^{(\mu^s - 1)^n} = \Sigma^{(\alpha^s - 1)^n}$. This is a contradiction, since Σ is unbounded as T is.

Finally, if the group $A_{2'}$ is unbounded, then $\Sigma_{2'}$ is unbounded as well. Let α be a non-periodic multiplication of $A_{2'}$. Then $\mu := \alpha \oplus 1 \oplus 1$ with respect to the group $A = A_{2'} \oplus A_2 \oplus K$ acts as non-periodic multiplication (by α) of $\Sigma_{2'}$ acting fixed-point-free on a primary component. Thus, μ (and any non-trivial power of μ as well) does not belong to the locally nilpotent radical R of $\mathrm{IAut}_1(A)$. Therefore, $\mathrm{IAut}_1(A)/R$ is not locally finite. □

Proof of Example B. It follows from the last part of Theorem 6.2. □

Finally, we note that, despite the above propositions, in the general case the group $\mathrm{IAut}_1(A)$ may be large.

Example 6.3. There exists an abelian group A with $r_0(A) = 1$ and $A_p \simeq \mathbb{Z}(p)$ for each prime p such that $\mathrm{IAut}(A) = \mathrm{IAut}_1(A) \times \{\pm 1\}$, $\mathrm{IAut}_1(A) = \Sigma \cdot \mathrm{FAut}(A)$, where $\Sigma := \mathrm{St}_{\mathrm{IAut}(A)}(A, T(A)) \not\leq \mathrm{FAut}(A)$ with the property that $\Sigma \simeq \prod_p \mathbb{Z}(p)$ and $\mathrm{IAut}_1(A)/\,\mathrm{FAut}(A) \simeq \Sigma/T(\Sigma)$ is a divisible torsion-free abelian group with cardinality $2^{\aleph_0}$.

Moreover, any element of $\mathrm{IAut}_1(A)$ induces a finitary automorphism on $T(A)$.

Proof. As in [11, Proposition A], we consider the group $G := B \oplus C$, where $B := \prod_p \langle b_p\rangle$, $C := \prod_p \langle c_p\rangle$, and b_p, c_p have order p, p^2, respectively, and p ranges over all primes. Consider the (aperiodic) element $v := (b_p + pc_p)_p \in G$ and $V := \langle v\rangle$. We have that for each prime p there is an element $d_{(p)} \in G$ such that $pd_{(p)} = v - b_p$. Let $A := V + \langle d_{(p)} \mid p\rangle$ and $T := T(A)$. Then

$$A/T \simeq \langle 1/p \mid p\rangle \leq \mathbb{Q},$$

since A/T has torsion free rank 1 and $v + T$ has p-height 1 for each p. Thus, $T = T(B) \simeq \bigoplus_p \mathbb{Z}(p)$ and the p-component of A/V is generated by $d_{(p)} + V$ and has order p^2, since $pd_{(p)} = v - b_p$.

Then $\Sigma \simeq \prod_p \mathbb{Z}(p)$ and $\Sigma \cap F\,\mathrm{Aut}(A) = T(\Sigma)$, hence $\Sigma \not\leq F\,\mathrm{Aut}(A)$. Moreover, $A = \langle d_{(p)}\rangle + V$, where $V = \langle v\rangle$ is infinite cyclic and $A_p = \langle b_p\rangle$ has order p. Also, $\mathrm{Aut}(A/T) = \{\pm 1\}$ and $\mathrm{IAut}(A) = \mathrm{IAut}_1(A) \times \{\pm 1\}$.

We claim that *if* $\gamma \in \mathrm{IAut}_1(A)$ *induces on* T *a finitary automorphism, then* $\gamma \in \Sigma \cdot \mathrm{FAut}(A)$. In fact, $T\gamma$ is finite, so it is a π-component of A for some finite π. Thus, $\gamma\gamma_0^{-1} \in \Sigma$, where $\gamma_0 := \gamma_{|A_\pi} \oplus 1$ with respect to $A = A_\pi \oplus K$ and clearly $\gamma_0 \in \mathrm{FAut}(A)$.

Finally, we prove the last part of the statement, from which it follows that $\mathrm{IAut}_1(A) = \Sigma \cdot \mathrm{FAut}(A)$. To this end, let $\gamma \in \mathrm{IAut}_1(A)$ and $\varphi := \gamma - 1$. Since $A\varphi \leq T$, there exists a non-zero integer n such that $(nv)\varphi = 0$. We prove that $T\varphi \subseteq A_{\pi(n)}$, which is finite. For any prime p, on the one hand, $nd_{(p)}$ is a p-element modulo $\langle nv\rangle \leq \ker\varphi$, hence $(nd_{(p)})\varphi \in A_p$, that implies

$$(pnd_{(p)})\varphi = p(nd_{(p)})\varphi = 0.$$

On the other hand,

$$(pnd_{(p)})\varphi = n(v - b_p)\varphi = -n(b_p)\varphi.$$

Hence, if $p \notin (n)\varphi$, then $A_p\varphi = 0$. □

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