J. Group Theory **20** (2017), 371–383
DOI 10.1515/jgth-2016-0045

Acylindrical hyperbolicity, non-simplicity and SQ-universality of groups splitting over $\mathbb{Z}$

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Communicated by James Howie

Abstract. We show, using acylindrical hyperbolicity, that a finitely generated group splitting over $\mathbb{Z}$ cannot be simple. We also obtain SQ-universality in most cases, for instance a balanced group (one where if two powers of an infinite order element are conjugate then they are equal or inverse) which is finitely generated and splits over $\mathbb{Z}$ must either be SQ-universal or it is one of exactly seven virtually abelian exceptions.

1 Introduction

An infinite word hyperbolic group can never be simple. Indeed it was shown in [13] and independently in [8] that a non-elementary word hyperbolic group G is SQ-universal, that is every countable group embeds in a quotient of G, and this implies that G has uncountably many normal subgroups so is very far from being simple. A generalisation of word hyperbolicity is that of being hyperbolic relative to a collection of proper subgroups, with non-trivial amalgamated free products $A *_C B$ and HNN extensions $H *_C$ over finite groups C being examples. It was shown in [1] that non-elementary groups which are hyperbolic relative to a collection of proper subgroups are SQ-universal too, thus in particular this result holds for HNN extensions $H *_C$ with C finite and $C < H$, which was not previously known. However, well before this it was established (see [15]) that all non-trivial free products are SQ-universal (excluding of course $C_2 * C_2$), and then in [10] the SQ-universality of amalgamated free products $A *_C B$, where C is finite and has index greater than 2 in one factor and at least 2 in the other, was proven. In fact, the bulk of the work here is in showing that these groups cannot be simple.

In contrast all HNN extensions $H *_C$ are easily seen to be non-simple because such a group will always surject $\mathbb{Z}$. However, it is possible for an amalgamated free product $A *_C B$ to be simple even if finitely generated, as first shown in [5] from 1953 by R. Camm, where A, B are finitely generated free groups but C is not finitely generated. More recently, the striking examples of Burger and Mozes in [4] are simple groups where A, B are as before, yet C is a subgroup of finite index in both A and B. Now given that $A *_C B$ can never be simple if C is trivial or finite, but it can be simple for other choices of amalgamated subgroup, this surely begs the natural question of whether such groups can be simple in the case where C is infinite cyclic (or virtually infinite cyclic). This seems even more natural in light of the fact that groups of the form $A *_{\mathbb{Z}} B$ and $H*_{\mathbb{Z}}$ have been much studied, especially in the context of JSJ splittings, but we can find no instance in the literature of this question of simplicity even being raised, let alone any direct partial results. (Of course some cases can be deduced from other work, for instance if A, B are word hyperbolic groups and the conditions for the Bestvina–Feighn combination theorem are satisfied then $A *_{\mathbb{Z}} B$ will also be word hyperbolic, so not simple. Or if A, B are free groups then $A *_{\mathbb{Z}} B$ will have a presentation with more generators than relators so will surject $\mathbb{Z}$.) We can further ask if any group of the form $A *_{\mathbb{Z}} B$ is SQ-universal, outside of a small collection of examples akin to the virtually cyclic case when C is finite. We can also ask about the SQ-universality of HNN extensions of the form $H*_{\mathbb{Z}}$, again allowing a limited list of exceptions.

A more recent development is that of a group being acylindrically hyperbolic, which is a further generalisation of being hyperbolic relative to proper subgroups. It is shown in [7, Theorem 8.1 (a)] that acylindrical hyperbolicity also implies SQ-universality (here the definition is set up so that there is no such thing as an elementary acylindrically hyperbolic group) and in [11] a fairly general condition for a finite graph of groups to be acylindrically hyperbolic is given. However, unlike when C is finite, we can have groups $A *_{\mathbb{Z}} B$ or $H*_{\mathbb{Z}}$ which are SQ-universal but not acylindrically hyperbolic. Indeed, the only criterion we will need here for showing a group G is not acylindrically hyperbolic is that of an s-normal subgroup H of G, which means that $gHg^{-1} \cap H$ is infinite for all $g \in G$. If G is acylindrically hyperbolic, then an s-normal subgroup H must be too, so possessing an infinite cyclic s-normal subgroup is an obstruction to acylindrical hyperbolicity but need not be for SQ-universality, for instance $F_n \times \mathbb{Z}$ for $n \geq 2$ is SQ-universal and of the form $A *_{\mathbb{Z}} B$ and $A*_{\mathbb{Z}}$ but is not acylindrically hyperbolic.

In this paper we apply the sufficient result in [11] on acylindrical hyperbolicity of the fundamental group of a finite graph of groups with arbitrary vertex and edge groups to show in Corollary 2.5 that if G is finitely generated and is the fundamental group of a reduced non-trivial graph of groups with one edge group being virtually infinite cyclic, then it is not simple; indeed, it always has a non-trivial normal subgroup of infinite index so cannot be just infinite either.

The method of proof here involves distinguishing between balanced and non-balanced elements x of infinite order in a group G, where x being balanced means that if $gx^m g^{-1} = x^n$, then $|m| = |n|$. We then have a version Δ_x^G of the modular homomorphism of x in G, with the domain being all elements $g \in G$ such that $g\langle x\rangle g^{-1} \cap \langle x\rangle$ is non-trivial. This is a subgroup of G containing the centraliser of x but here it will only be applied when this subgroup is all of G. In this case, if x is balanced in the finitely generated group G then we obtain an infinite cyclic normal subgroup of G and if not we have a surjection from G to $\mathbb{Z}$.

As for the question of SQ-universality, it would be good if we could say that a finitely generated group G of the form $A *_{\mathbb{Z}} B$ or $H*_{\mathbb{Z}}$ is always SQ-universal or one in a list of small examples. In Section 3 we first find these small examples, in that we give all non-trivial amalgamated free products $A *_{\mathbb{Z}} B$ and HNN extensions $H*_{\mathbb{Z}}$ which do not contain a non-abelian free group. This list turns out to consist of the soluble Baumslag–Solitar groups and exactly six other examples. Unfortunately, we cannot quite show that all other finitely generated groups of this form are SQ-universal as it is unclear how to proceed when mixing balanced and unbalanced elements. However for a finitely generated group G of the form $A *_{\langle c\rangle} B$ where c has infinite order and G is not one of these six small examples, we show in Theorem 3.3 that G is indeed SQ-universal unless $\langle c\rangle$ is s-normal in G and c is balanced in one of the factors but not balanced in the other. Proposition 3.4 even has some partial results on SQ-universality in this case.

It is a similar story for finitely generated HNN extensions $G = H*_{\mathbb{Z}}$ with stable letter t such that $tat^{-1} = b$ for a, b infinite order elements of H, where Theorem 3.5 says that G is SQ-universal (or $\mathbb{Z}^2$ or the Klein bottle group) unless $\langle a\rangle$ is s-normal in G and a is balanced in H but where $a^r = b^s$ holds in H with $|r| \neq |s|$ (so that a is not balanced in G). Again we get partial results on SQ-universality in this leftover case from Proposition 3.6. We finish with Corollary 3.7 which states that if every infinite order element of a finitely generated group G is balanced (which is true of most groups occurring in practice) and G splits over $\mathbb{Z}$, then either G is $\mathbb{Z}^2$ or one of these six small exceptions, or G is acylindrically hyperbolic, or G has an infinite cyclic normal subgroup Z such that G/Z is acylindrically hyperbolic. In the last two cases G is SQ-universal, so for balanced finitely generated groups G splitting over $\mathbb{Z}$ we have a complete result: G must be SQ-universal or is isomorphic to one of exactly seven exceptions.

2 Acylindrical hyperbolicity of graphs of groups with infinite cyclic edge groups

In [11] a subgroup H of a group G is called *weakly malnormal* in G if there is $g \in G$ such that $gHg^{-1} \cap H$ is finite, and *s-normal* in G otherwise. In that paper this concept was introduced in the context of acylindrical hyperbolicity, with a group being acylindrically hyperbolic implying that it is SQ-universal and in particular is not a simple group. The paper gives sufficient conditions under which the fundamental group of a finite graph of groups is acylindrically hyperbolic, which we now review. Note that if a group G is acylindrically hyperbolic, then any s-normal subgroup of G is itself acylindrically hyperbolic (in particular, it cannot be a cyclic subgroup) by [14, Corollary 1.5].

Given a graph of groups $G(\Gamma)$ with connected graph Γ and fundamental group G, an edge e is called good if both edge inclusions into the vertex groups at either end of e give rise to proper subgroups, otherwise it is bad. A reducible edge is a bad edge which is not a self loop. Given a finite graph of groups, we can contract the reducible edges one by one until none are left, whereupon we say $G(\Gamma)$ is reduced. This process does not affect the fundamental group G and the new vertex groups will form a subset of the original vertex groups. It could be that we are left with a single vertex and no edges, in which case we say that the graph of groups $G(\Gamma)$ was trivial with G equal to the remaining vertex group. We then have:

Theorem 2.1 ([11, Theorem 4.17]). *Suppose that $G(\Gamma)$ is a finite reduced graph of groups which is non-trivial and which is not just a single vertex with a single bad edge. If there are edges e, f of Γ (not necessarily distinct) with edge groups G_e, G_f and an element $g \in G$ such that $G_f \cap gG_eg^{-1}$ is finite then G is either virtually cyclic or acylindrically hyperbolic.*

This immediately gives rise to two corollaries, one for amalgamated free products and one for HNN extensions:

Corollary 2.2. (i) ([11, Corollary 2.2]) *If $G = A *_C B$ is a non-trivial amalgamation of any two groups A, B (meaning $C \neq A, B$), then G is acylindrically hyperbolic (or virtually cyclic) if C is not s-normal in G.*

In particular, if C is virtually infinite cyclic, then G is acylindrically hyperbolic exactly when C is not s-normal in G.

(ii) ([11, Corollary 2.3]) *If $G = H*_{tAt^{-1}=B}$ is a non-ascending HNN extension of any group H (meaning $H \neq A, B$), then G is acylindrically hyperbolic if A is not s-normal in G.*

In particular, if A (and thus B) is virtually infinite cyclic, then G is acylindrically hyperbolic exactly when A is not s-normal in G.

The following is a definition from [16]:

Definition 2.3. A group G is called *balanced* if for any element x in G of infinite order we have that x^m conjugate to x^n implies that $|m| = |n|$.

Here we will also define: A *balanced element* in a group G is an element x in G of infinite order such that if we have $m, n \in \mathbb{Z}$ with x^m conjugate to x^n in G then $|m| = |n|$.

Thus a group is balanced if and only if all its elements of infinite order are balanced. Examples of balanced groups are all word hyperbolic groups (see for instance [3, Chapter III.Γ, Corollary 3.10]) all 3-manifold groups by [9], groups acting properly and cocompactly on a CAT(0) space ([3, Chapter III.Γ, Theorem 1.1]) and groups that are subgroup separable by [16, Lemma 4.12]. As for unbalanced groups, by far the most common examples are groups containing a Baumslag–Solitar subgroup

$$\mathrm{BS}(m,n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle,$$

where $|m| \neq |n|$. Thus an unbalanced group such as $\mathrm{BS}(1,2)$ can certainly be linear but any subgroup S of $\mathrm{GL}(n,\mathbb{Z})$ is balanced: say we have $gx^m g^{-1} = x^n$ for $g, x \in S$ and $|m| \neq |n|$; then we know $\langle g, x \rangle$ will be residually finite. This implies that the kernel of the natural homomorphism from $\mathrm{BS}(m,n)$ onto $\langle g, x \rangle$ must contain the finite residual of $\mathrm{BS}(m,n)$ so $\langle g, x \rangle$ is soluble by [12]. By results of Malce'ev, a soluble subgroup of $\mathrm{GL}(n,\mathbb{Z})$ is polycyclic and a polycyclic group is subgroup separable.

Suppose now that we have a cyclic subgroup $\langle x \rangle$ which is s-normal in a group G (and hence x has infinite order). This means that for any $g \in G$, there exist non-zero integers m, n such that $gx^m g^{-1} = x^n$. Although m and n of course depend on g and are not even uniquely defined for a particular g, it is easily checked that the map Δ_x^G from G to the non-zero multiplicative rationals $\mathbb{Q}^*$ given by $\Delta_x^G(g) = m/n$ is well defined and is even a homomorphism, which we call the *modular homomorphism* of x in G (in line with other cases such as in generalised Baumslag–Solitar groups). In particular, the element x is balanced in G if and only if the image $\Delta_x^G(G)$ of the modular homomorphism is contained in $\{\pm 1\}$. This generalises to virtually infinite cyclic subgroups V, because then $\langle x \rangle$ is also s-normal in G for any infinite order $x \in V$ and the corresponding modular homomorphism Δ_V^G does not depend on which x is taken.

We can now use the above to show that a finitely generated simple group cannot split over $\mathbb{Z}$ or even a virtually infinite cyclic group. In fact, we actually show that a finitely generated group splitting over a virtually infinite cyclic group has a non-trivial normal subgroup of infinite index.

Theorem 2.4. *Suppose that $G(\Gamma)$ is a reduced non-trivial graph of groups with finitely generated fundamental group G and where one edge group is virtually infinite cyclic. Then one of the following holds:*

(i) *G is acylindrically hyperbolic,*

(ii) *G has a homomorphism onto $\mathbb{Z}$,*

(iii) *G has an infinite cyclic normal subgroup.*

Proof. As G is finitely generated, we can assume that Γ is a finite graph, and then we can remove the edge corresponding to the virtually infinite cyclic group V which will express G either as an amalgamated free product $A *_V B$ for $V \neq A, B$ (if the edge separates Γ) or as an HNN extension $H *_{tAt^{-1}=B}$ for $A, B \cong V$ (if the edge is non-separating). But in the latter case of an HNN extension, G surjects to $\mathbb{Z}$ anyway. We now apply Corollary 2.2 in the case of an amalgamated free product to obtain acylindrical hyperbolicity of G, unless this edge group V is s-normal in G which we now assume.

First suppose that x is also balanced in G, where x is any infinite order element of V. Then for all $g \in G$ we can find integers $k > 0$ and $l \neq 0$, both depending on g, such that $gx^k g^{-1} = x^l$. But x being balanced in G means that $l = \pm k$. So on taking a finite generating set $g_1, \dots, g_s$ for G, we have for each $1 \leq i \leq s$ an integer k_i with $g_i x^{k_i} g_i^{-1} = x^{\pm k_i}$. Consequently, we can find a common power p that works for all of this set and hence for all of G, thus $\langle x^p \rangle$ is normal in G.

If however x is not balanced in G, then the image of the modular homomorphism Δ_x^G of x in G will not be contained in $\{\pm 1\}$. But then $|\Delta_x^G|$ provides a homomorphism from G to the positive rationals $\mathbb{Q}^+$ which is non-trivial, thus the image is an infinite torsion free abelian group which is finitely generated, thus must be $\mathbb{Z}^n$. □

Corollary 2.5. *If G is a finitely generated group splitting non-trivially over a virtually infinite cyclic subgroup V, namely as an amalgamated free product $A *_V B$ with $V \neq A, B$ or as an HNN extension of the form $H *_{tAt^{-1}=B}$ for $A, B \cong V$, then G has infinitely many normal subgroups, at least one of which is infinite and has infinite index in G. Consequently, G is not simple or just infinite and nor is any finite index subgroup of G.*

Proof. On application of Theorem 2.4, either G is SQ-universal in which case it has uncountably many normal subgroups (only countably many of which can be finite or have finite index as G is finitely generated), or it surjects to $\mathbb{Z}$ (therefore to every finite cyclic group) and this kernel is infinite as a virtually cyclic group does not split non-trivially over $\mathbb{Z}$, or there is $p > 0$ such that $\langle x^p \rangle$ is an infinite cyclic normal subgroup of G, hence again this is of infinite index, as are the distinct proper normal subgroups $\langle x^{np} \rangle$ for $n \in \mathbb{N}$. The same holds for any finite index subgroup of G as can been in a variety of ways, not least because it also splits over a virtually infinite cyclic subgroup. □

Notes. (1) It is quite possible in the amalgamated free product case that G has no proper finite index subgroups, for instance if A and B have no proper finite index subgroups then nor will $A * B$ or $A *_C B$.

(2) We do not quite have a definitive result in the case of an infinitely generated group G: here the proof of Theorem 2.4 tells us that either G is acylindrically hyperbolic, or G has an infinite torsion free abelian quotient, or G has an infinite cyclic normal subgroup, or G fails to have the maximum condition on centralisers. Unfortunately, we do not know in the last case if G can also be simple.

3 SQ-universality

We can now ask whether the stronger property of being SQ-universal holds for finitely generated groups splitting over a virtually infinite cyclic group, given that this is true for all acylindrically hyperbolic groups. However, this will fail for the groups mentioned below in Proposition 3.2, because being SQ-universal implies containing a non-abelian free group. Hence we first consider this case by using the following well known proposition.

Proposition 3.1. *The following statements hold.*

(i) *If $G = A *_C B$ is a non-trivial amalgamated free product so that $C \neq A, B$, then G contains a non-abelian free group, unless neither of A and B do and C has index 2 in both A and B, in which case G does not.*

(ii) *If $G = H *_{tAt^{-1}=B}$ is an HNN extension, then G contains a non-abelian free group, unless H does not and at least one of A and B is equal to H, in which case G does not.*

Proof. Part (i) is [2, Lemma 1], whereas for (ii) we can take the group $\langle t, xtx^{-1}\rangle$, where t is the stable letter and $x \in H \setminus (A \cup B)$, which is seen to be a rank 2 free group by using reduced forms of the HNN extension. □

Thus now we would like to see which groups can have such splittings despite not containing a non-abelian free group. At this point we restrict the edge group to $\mathbb{Z}$ so as to obtain a complete list of exceptions.

Proposition 3.2. *If the group G splits as an HNN extension over $\mathbb{Z}$ but does not contain a non-abelian free group, then G is isomorphic to one of the soluble Baumslag–Solitar groups* $\mathrm{BS}(1, n)$ *for $n \in \mathbb{Z} \setminus \{0\}$.*

If the group G splits as a non-trivial amalgamated free product over $\mathbb{Z}$ but does not contain a non-abelian free group, then G is isomorphic to one of the following (mutually non-isomorphic) six groups:

(i) *The group $\langle a, b \mid a^2 = b^2\rangle$ which is the Klein bottle group K (and is also* $\mathrm{BS}(1, -1)$*, so is the only group here splitting over $\mathbb{Z}$ both as an HNN extension and an amalgamated free product).*

(ii) *The group* $\langle s, t, c \mid [s, t], c^2, csc^{-1} = t, ctc^{-1} = s\rangle$ *which is* $\mathbb{Z}^2 \rtimes C_2$*, where* C_2 *swaps this standard generating set of* $\mathbb{Z}^2$*.*

(iii) *The group* $\langle a, b, c \mid a^2 = b^2, c^2, cac^{-1} = b, cbc^{-1} = a\rangle$ *which is* $K \rtimes C_2$*, where the generators for* K *in* (i) *are swapped.*

(iv) *The group* $\langle c, d, z \mid c^2, d^2, [c, z], [d, z]\rangle$ *which is* $(C_2 * C_2) \times \mathbb{Z}$*.*

(v) *The group* $\langle t, c, d \mid c^2, d^2, ctc^{-1} = t^{-1}, dtd^{-1} = t\rangle$ *which is* $\mathbb{Z} \rtimes (C_2 * C_2)$*, where one* C_2 *factor inverts* $\mathbb{Z}$ *and one fixes* $\mathbb{Z}$*.*

(vi) *The group* $\langle s, t, c \mid [s, t], c^2, csc^{-1} = s^{-1}, ctc^{-1} = t^{-1}\rangle$ *which is* $\mathbb{Z}^2 \rtimes C_2$ *where* C_2 *inverts the elements of* $\mathbb{Z}^2$*.*

Proof. The HNN extension case follows directly from Proposition 3.1 (ii) on putting A or B equal to $\mathbb{Z}$. In the amalgamated free product case we have that A and B contain $\mathbb{Z}$ with index 2, meaning that they are isomorphic to (a) $\mathbb{Z}$ itself, (b) $\mathbb{Z} \times C_2$ or (c) $C_2 * C_2$. Now $\mathbb{Z}$ and $C_2 * C_2$ each have a unique index 2 subgroup that is isomorphic to $\mathbb{Z}$ whereas $\mathbb{Z} \times C_2$ has two, but these two are equivalent under an automorphism. Therefore we can write out standard presentations for all three groups and then for the six possible amalgamations, which we have done above in the order (a)-(a), (a)-(b), (a)-(c), (b)-(b), (b)-(c), (c)-(c). We have also performed some tidying up of the resulting presentations, and calculated the abelianisation of these groups and their index 2 subgroups which distinguishes these six groups, as well as distinguishing the last five from any Baumslag–Solitar group. □

We now consider all other finitely generated free products amalgamated over $\mathbb{Z}$, where we can prove SQ-universality in most cases.

Theorem 3.3. *Suppose that* G *is finitely generated and is equal to a non-trivial amalgamated free product* $A *_C B$ *for* $C = \langle c\rangle \cong \mathbb{Z}$*. If* $\langle c\rangle$ *is not* s*-normal in* G*, or* $\langle c\rangle$ *is* s*-normal in* G *and* c *is balanced in* G *but* G *is not one of the six groups listed in Proposition* 3.2*, or if* $\langle c\rangle$ *is* s*-normal in* G *but* c *is not balanced in* A *and not balanced in* B*, then* G *is SQ-universal.*

Proof. By Corollary 2.2 (i) we know that G is acylindrically hyperbolic and hence SQ-universal unless $\langle c\rangle$ is s-normal in G, which will be assumed for the rest of the proof. First suppose that c is balanced in G, so as in the proof of Theorem 2.4 we have $p > 0$ such that $\langle c^p\rangle$ is normal in G, and hence in A and B. This means that on quotienting both A and B by this infinite cyclic normal subgroup to obtain $\overline{A}$ and $\overline{B}$, in each of which the image $\overline{c}$ has order p, our amalgamated free product $A *_C B$ factors through $\overline{A} *_{\overline{C}} \overline{B}$, where $\overline{C} = \langle\overline{c}\rangle$ is cyclic of order p. Moreover,

$\langle c^p \rangle$ is contained in C, so that the indices $[\overline{A} : \overline{C}] = [A : C]$ and $[\overline{B} : \overline{C}] = [B : C]$ are unchanged. But $[A : C], [B : C] > 1$ by assumption and at least one of these indices is greater than two (or else we are back in the case of Proposition 3.2), so by [10] we have that $\overline{A} *_{\overline{C}} \overline{B}$ is SQ-universal and hence G is also.

Now we assume that c is not balanced in A, nor in B. This means that the modular homomorphisms $|\Delta_c^A|$ from A to $\mathbb{Q}^+$ and $|\Delta_c^B|$ from B to $\mathbb{Q}^+$ are both maps to infinite torsion free abelian groups with $\langle c \rangle$ in the kernel, thus we can put them together to obtain a homomorphism from G to the free product of these infinite abelian groups, thus here G is SQ-universal. □

Thus we are left with c being balanced in A but not in B (or vice versa), whereupon we cannot always say here that $A *_C B$ is SQ-universal. However, there are subcases where we can obtain this conclusion so we look at this in a little more detail. By D. E. Cohen's comment ([6, Section 3]) we have that in an amalgamated free product $G = A *_C B$, if G and C are finitely generated, then so are A and B.

Proposition 3.4. *Suppose that $G = A *_C B$ is a non-trivial amalgamated free product with $C = \langle c \rangle$ infinite cyclic and G finitely generated. Suppose also that $\langle c \rangle$ is s-normal in G (thus in A and in B) and c is balanced in A but not in B, so that (as Cohen's comment tells us A is finitely generated) there is $p > 0$ with $\langle c^p \rangle$ normal in A. If there exists an integer $k > 0$ such that*

(i) *the normal closure $\langle\langle c^k \rangle\rangle^A$ of the element c^k in A does not contain any of the elements $c, c^2, \dots, c^{k-1}$, with the same holding for the normal closure $\langle\langle c^k \rangle\rangle^B$ of c^k in B,*

(ii) *the quotient of A by the normal closure $\langle\langle c^k \rangle\rangle^A$ has order greater than k,*

then G is SQ-universal.

In particular, if $\langle\langle c \rangle\rangle^A$ is not equal to A, so that (ii) *holds for $k = 1$ with* (i)*, then holding automatically, or if there is k which is a multiple of p such that* (i) *holds just for $\langle\langle c^k \rangle\rangle^B$, then G is SQ-universal.*

Proof. Condition (i) implies that in both quotients $A/\langle\langle c^k \rangle\rangle^A$ and $B/\langle\langle c^k \rangle\rangle^B$, the element c has order exactly k. If so, then $B/\langle\langle c^k \rangle\rangle^B$ is infinite because it surjects $B/\langle\langle c \rangle\rangle^B$ which itself surjects an infinite abelian group using $|\Delta_c^B|$, as c is not balanced in B. In particular, the image of $\langle c \rangle$ in this quotient $B/\langle\langle c^k \rangle\rangle^B$ of B has infinite index. Hence the amalgamated free product $(A/\langle\langle c^k \rangle\rangle^A) *_{\langle c \rangle} (B/\langle\langle c^k \rangle\rangle^B)$ is a quotient of $A *_C B$ and the former is SQ-universal by [10] provided only that $\langle c \rangle$ has index greater than 1 in $A/\langle\langle c^k \rangle\rangle^A$, which is condition (ii).

As for the particular cases mentioned, taking $k = 1$ gives us the free product $(A/\langle\langle c \rangle\rangle^A) * (B/\langle\langle c \rangle\rangle^B)$. If the left-hand factor is non-trivial, then this will be a non-trivial free product not equal to $C_2 * C_2$, and therefore is SQ-universal.

Meanwhile, if c has order exactly k in $B/\langle\langle c^k\rangle\rangle^B$ for p dividing k, then we know c also has order k in $A/\langle\langle c^k\rangle\rangle^A$ because $\langle c^p\rangle$ and then $\langle c^k\rangle$ is normal in A, with $[A/\langle c^k\rangle : C/\langle c^k\rangle] = [A : C] > 1$ so (ii) holds as well. □

We now move to HNN extensions.

Theorem 3.5. *Suppose that G is a finitely generated group which is an HNN extension $H*_{tAt^{-1}=B}$ with base H and stable letter t conjugating the infinite cyclic subgroup $A = \langle a\rangle$ of H to $B = \langle b\rangle$ via $tat^{-1} = b$. Then G is SQ-universal or $\mathbb{Z}^2$ or the Klein bottle group, with the possible exception of when $\langle a\rangle$ is s-normal in H and there exist integers r, s with $a^r = b^s$ for $|r| \neq |s|$ but a (equivalently b) is balanced in H.*

Proof. By Corollary 2.2 we have that G is acylindrically hyperbolic and hence SQ-universal unless $\langle a\rangle$ is s-normal in G, which we assume for the rest of the proof. Thus $\langle a\rangle$ is s-normal in H too and also there are non-zero integers r, s with $a^r = b^s$ because otherwise $A \cap B$ is trivial with A conjugate to B in G, contradicting A being s-normal in G. Now our HNN extension G factors through the HNN extension with base $H/\langle\langle a, b\rangle\rangle^H$ and trivial edge subgroups, which is the free product $(H/\langle\langle a, b\rangle\rangle^H) * \mathbb{Z}$ and thus is SQ-universal provided the first factor is non-trivial. But if a is not balanced in H, then a, and hence also b because of the relation $a^r = b^s$, will be in the kernel of the modular homomorphism $|\Delta_a^H|$ which here has infinite image in $\mathbb{Q}^+$, so $H/\langle\langle a, b\rangle\rangle^H$ is non-trivial.

Now say a is balanced in H but $|r| = |s|$. Then as a is s-normal in H too, we have $p > 0$ such that $\langle a^p\rangle$, and hence $\langle a^k\rangle$ for any multiple k of p, is normal in H as before by using D. E. Cohen's comment applied to HNN extensions (namely G and A being finitely generated imply the base H is finitely generated). Hence $\langle a^{p|r|}\rangle = \langle b^{p|r|}\rangle$ is normal in H with a and b both having order exactly $p|r|$ in this quotient, so G factors through the HNN extension $(H/\langle a^{p|r|}\rangle)*_{tAt^{-1}=B}$, where now A and B both map to finite cyclic groups of order $p|r|$. This HNN extension is SQ-universal by [1, Corollary 1.3] unless we have $A/\langle a^{p|r|}\rangle = B/\langle b^{p|r|}\rangle = H/\langle a^{p|r|}\rangle$, whereupon we must have had $A = B = H \cong \mathbb{Z}$ initially. This means that G was $\langle t, h \mid tht^{-1} = h^{\pm 1}\rangle$, so G is $\mathbb{Z}^2$ or the Klein bottle group. □

As any HNN extension over $\mathbb{Z}$ excluded by Theorem 3.5 will have the edge generator a (equivalently b) balanced in the base group but not balanced in the HNN extension because we would have $ta^st^{-1} = a^r$ for $|r| \neq |s|$, we see that (just as in Theorem 3.3) it is the "mixed" case which is troublesome in proving SQ-universality. Nevertheless, here we can still obtain partial results similar to those in Proposition 3.4.

Proposition 3.6. *Suppose that G is a finitely generated group which is an HNN extension $H*_{tAt^{-1}=B}$ with base H and stable letter t conjugating the infinite cyclic subgroup $A = \langle a \rangle$ of H to $B = \langle b \rangle$ via $tat^{-1} = b$ such that there are integers r, s with $a^r = b^s$ for $|r| \neq |s|$ but a (equivalently b) is balanced in H. If there exists an integer $k > 0$ such that*

(i) *the normal closure $\langle\langle a^k, b^k \rangle\rangle^H = \langle\langle a^k \rangle\rangle^H \langle\langle b^k \rangle\rangle^H$ of the elements a^k, b^k in H does not contain any of the elements $a, a^2, \ldots, a^{k-1}$ or $b, b^2, \ldots, b^{k-1}$,*

(ii) *the quotient of H by $\langle\langle a^k, b^k \rangle\rangle^H$ has order greater than k,*

then G is SQ-universal.

In particular, if $\langle\langle a, b \rangle\rangle^H$ is not equal to H, so that (ii) *holds for $k = 1$ with* (i)*, then holding automatically, then G is SQ-universal.*

Proof. On setting $N = \langle\langle a^k, b^k \rangle\rangle^H$ we see that a and b both have exact order k in H/N so the HNN extension $(H/N)*_{tat^{-1}=b}$ is well defined, as a and b both have order k in H/N, and it is a quotient of G. Moreover, neither a nor b generate H/N from (ii) so G is SQ-universal by [1, Corollary 1.3]. □

To finish, if G is finitely generated, splits over $\mathbb{Z}$ and is a balanced group, then we have a complete result. Indeed, we have a trichotomy (reading like [11, Theorem 2.8] but there the groups being considered are all subgroups of fundamental groups of compact 3-manifolds) which says that G is acylindrically hyperbolic, or is so on quotienting out by an infinite cyclic normal subgroup (and therefore is SQ-universal because this is preserved by prequotients), or is one of a very few small exceptions. In fact, we only need the edge group generator to be balanced for this to hold.

Corollary 3.7. *Suppose that G is a finitely generated group which equals the fundamental group of a reduced non-trivial graph of groups where one edge group is infinite cyclic with a generator that is balanced in G. Then one of the following mutually exclusive cases occurs:*

(i) *G is acylindrically hyperbolic,*

(ii) *G has an infinite cyclic normal subgroup Z such that G/Z is relatively hyperbolic but not virtually cyclic and so is acylindrically hyperbolic,*

(iii) *G is isomorphic to $\mathbb{Z}^2$ or one of the six groups listed in Proposition* 3.2.

In particular, G is SQ-universal or virtually abelian.

Proof. On applying Theorem 2.4 we see that if G is not acylindrically hyperbolic, then either it is an HNN extension over $\mathbb{Z}$ or it is an amalgamated free product with an infinite cyclic normal subgroup Z whose generator c^p is a power of the edge group generator c, because we are in the case where c is balanced in G. In the latter case we can apply Theorem 3.3 with $\langle c\rangle$ being s-normal and c balanced in G to conclude that either G/Z is an amalgamation over a finite subgroup which is not virtually cyclic but which is hyperbolic relative to the vertex groups, or G is one of the six groups listed in Proposition 3.2.

If now G is an HNN extension, we are then covered by the proof of Theorem 3.5, unless $\langle a\rangle$ is s-normal in G, whereupon we have $a^r = b^s$, but a balanced in G and $tat^{-1} = b$ tells us that $|r| = |s|$. In this case we conclude that the infinite cyclic subgroup $C = \langle a^{p|r|}\rangle$ is normal not just in the base H but in G too as $a^{p|r|} = b^{\pm p|r|}$ with $ta^{pr}t^{-1} = b^{\pm pr}$. Moreover, the quotient G/Z is again relatively hyperbolic but not virtually cyclic, or G is $\mathbb{Z}^2$ or the Klein bottle group with the latter being the first group listed in Proposition 3.2. □

Acknowledgments. We would like to thank the referee for a thorough reading of the paper.

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Received March 21, 2016; revised August 25, 2016.

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