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Direct decomposition theory under near-isomorphism for a class of infinite rank torsion-free abelian groups

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In memory of our good friend and colleague Rüdiger Göbel

Abstract. Near-isomorphism is known as the right concept for classification theorems in the theory of almost completely decomposable groups. As a natural generalization the authors extended in [6] the notion of near-isomorphism to Abelian groups of arbitrary rank. In this article we investigate non-isomorphic direct decompositions of a class of infinite rank torsion-free abelian groups which were defined in [4] as special epimorphic images of so-called almost rigid groups. A complete classification of such decompositions up to near-isomorphism is given.

1 Introduction

In the theory of Abelian groups we know many examples where a classification of groups up to some reasonable variant of *isomorphism* has successfully been obtained. In particular, a characterization by *numerical* invariants has been of interest. For instance, *completely decomposable groups* (direct sums of subgroups of the rationals) were classified up to group isomorphism by so-called *types* and *ranks* of their homogeneous components. In the torsion case and later on in the mixed case Ulm's invariants and Warfield's invariants were used to classify *totally projective Abelian p-groups* and *Warfield groups* up to isomorphism. A very good source on these results are the two volumes by Fuchs [7] and Loth [9]. Focusing on the class of torsion-free Abelian groups however, it is in general very difficult (and very often impossible) to obtain such a classification. A very good example where a modified notion of isomorphism led to success in this respect is the class of *almost completely decomposable groups* (acd-groups) which are finite extensions of completely decomposable groups of finite rank (see Mader [10]). It turned out that for acd-groups the concept of *near-isomorphism* was more suit-

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able than classical isomorphism. Recall that two torsion-free abelian groups X and Y of finite rank are *nearly-isomorphic*, in symbols $X \cong_{\mathrm{nr}} Y$, if and only if for every prime p there is a monomorphism $\varphi_p : X \to Y$ such that $Y/\varphi_p(X)$ is finite with zero p-torsion part. Near-isomorphism is indeed an equivalence relation (so $X \cong_{\mathrm{nr}} Y$ implies $Y \cong_{\mathrm{nr}} X$) and from many perspectives almost completely decomposable groups under near-isomorphism are very attractive, in particular, concerning their direct decompositions and the links with their endomorphism rings: nearly-isomorphic acd-groups have isomorphic regulators, nearly isomorphic endomorphism rings (considered as Abelian groups), and sometimes even allow a sort of Baer–Kaplansky theorem. Finally, their direct decompositions in the case of so-called crq-groups (i.e. finite cyclic extensions of completely decomposable groups) can be classified up to near-isomorphism in terms of the so-called *type-invariants* (see the standard book by Mader [10] on acd-groups and the references in there).

There are several ways to extend the class of almost completely decomposable groups to infinite rank, e.g., by looking at continuous increasing chains of acd-groups or by considering torsion-free Abelian groups of infinite rank with all their finite rank pure fully invariant subgroups isomorphic to almost completely decomposable groups – called *local acd-groups*. In each case it is also desirable to have an appropriate notion of near-isomorphism. Such concept was introduced by the authors in [6, Definition 2.3, Proposition 3.4] by defining two torsion-free Abelian groups X and Y of arbitrary rank to be *nearly-isomorphic* ($X \cong_{\mathrm{nr}} Y$) if for every prime p there exist monomorphisms $\eta_p : X \to Y$ and $\mu_p : Y \to X$ such that

(1) $Y/X\eta_p$ and $X/Y\mu_p$ are torsion,

(2) $(Y/X\eta_p)_p = 0 = (X/Y\mu_p)_p$,

(3) for all finite rank pure subgroups $X' \subseteq X$ and $Y' \subseteq Y$ the quotients $(X'\eta_p)_*^Y/X'\eta_p$ and $(Y'\mu_p)_*^X/Y'\mu_p$ are finite.

It was shown in [6] that this notion extends the classical notion of near-isomorphism for finite rank groups (forced by condition (3) above) and for completely decomposable groups it is equivalent to isomorphism as one should expect. Thus keeping the name near-isomorphism is justified.

Our goal in this paper is to classify up to near-isomorphism all possible direct decompositions of countable rank torsion-free Abelian groups from a special class defined in [4]. The first results in this direction were obtained in [3] and later on in [4] for a class $\mathcal{H}_{\mathfrak{T}}$ of Abelian groups where a group X in $\mathcal{H}_{\mathfrak{T}}$ is generated by a completely decomposable subgroup $R(X) = \bigoplus_{\tau \in T_{\mathrm{cr}}(X)} C_\tau^X$ of X – the so-called *regulator* – with critical typeset $T_{\mathrm{cr}}(X)$ an antichain and τ-homogeneous completely decomposable pure subgroups C_τ^X of X such that the quotient $X/R(X)$ is a direct sum of bounded p-groups. Moreover, the *relations* defining X with respect to $R(X)$ are of a *finite character* (see [4] for further details). A complete classification up to near-isomorphism was obtained in the series of papers [1, 3, 6] for different subclasses of $\mathcal{H}_{\mathfrak{T}}$ assuming that the ranks of the homogeneous components ${C_\tau}^X$ of $R(X)$ are finite (this class was denoted by $\mathcal{H}'_{\mathfrak{T}}$ in [4]): the class of *rigid* groups, i.e. groups $X \in \mathcal{H}'_{\mathfrak{T}}$ such that the rank of any homogeneous component is one, and the class of *almost rigid groups*, i.e. groups $X \in \mathcal{H}'_{\mathfrak{T}}$ with homogeneous components of finite rank, the critical typeset $T_{\mathrm{cr}}(X)$ is countable and all primary components of $X/R(X)$ are cyclic groups. In fact, almost rigid groups have a decomposition into a direct sum of a completely decomposable group and a rigid summand from $X \in \mathcal{H}'_{\mathfrak{T}}$ which explains the name *almost rigid*. In particular, it was shown in [3] and [1, Theorem 3.4] that for countable rank groups from the class $\mathcal{H}'_{\mathfrak{T}}$, in particular for almost rigid groups, a classical theorem by Arnold can be extended: If X and Y are nearly-isomorphic and $X = X_1 \oplus X_2$, then there is also a decomposition $Y = Y_1 \oplus Y_2$ such that $X_1 \cong_{\mathrm{nr}} Y_1$ and $X_2 \cong_{\mathrm{nr}} Y_2$ (for the reader's convenience we should say that the class of countable rank groups from $\mathcal{H}'_{\mathfrak{T}}$ was denoted by $\mathcal{C}'$ in the paper [1] published in Russian). Moreover, rigid groups and also almost rigid groups from $\mathcal{H}'_{\mathfrak{T}}$ were classified up to near-isomorphism by their regulators and certain numerical invariants m_τ (see [3] and [4]) and a decomposition theory was constructed in [3] in terms of the numerical invariants as a development of the combinatorial approach invented for some almost completely decomposable groups in [5] and [10, Chapter 13]. Finally, the authors and Göbel passed to certain epimorphic images of rigid groups and also almost rigid groups, called *proper $\mathcal{B}^{(1)}$ asd-groups* and *proper $\mathcal{B}^{(1)}$ alr-groups*, which are obtained by factoring out a subgroup K of an (almost) rigid group X. The subgroup K is of the form $K = \bigoplus_{i\in\omega} K(A_i)^{l_i}$, where $K(A_i) = \langle \sum_{\tau\in T_i} \alpha_\tau a_\tau \rangle$ and the regulator of X satisfies

$$R(X) = \bigoplus_{i\in\omega} A_i = \bigoplus_{i\in\omega} \bigoplus_{\tau\in T_i} \tau a_\tau$$

(for a more detailed definition see Definition 2.2 below). The integers l_i, α_i and the sets of types $T_i \subseteq T_{\mathrm{cr}}(X)$ are called a *triple of partitions, coefficients, and parameters for X*. For proper $\mathcal{B}^{(1)}$ asd-groups and proper $\mathcal{B}^{(1)}$ alr-groups a classification up to near-isomorphism is available (see [4]) in the sense that two such groups $G = X/K_X$ and $H = Y/K_Y$ are nearly-isomorphic if and only if their pre-images X and Y are nearly-isomorphic and the triples of partitions, coefficients, and parameters for X and Y coincide.

In this paper we extend the numerical approach for (almost) rigid groups to the proper $\mathcal{B}^{(1)}$ alr-groups using the near-isomorphism invariants m_τ from [4]: Two proper $\mathfrak{B}^{(1)}$alr-groups B and C are nearly-isomorphic if and only if $R(B) \cong R(C)$ and $m_\tau(B) = m_\tau(C)$ for each critical type $\tau \in \mathfrak{T}$ (Theorem 3.3). Moreover, two classification theorems up to near-isomorphism of direct decompositions of proper $\mathcal{B}^{(1)}$ alr-groups are obtained by carrying over the near-isomorphism classifications for almost rigid groups to the class of proper $\mathcal{B}^{(1)}$ alr-groups (see Lemma 4.4 and Theorem 4.6).

The main result of the present paper is the direct decomposition classification in Theorem 4.6 in terms of the numerical invariants for proper $\mathcal{B}^{(1)}$ alr-groups. These groups have many common features with almost rigid groups. However, the big difference is that they are torsion extensions of direct sums of *strongly indecomposable groups* while almost rigid groups are torsion extensions of completely decomposable groups.

2 Preliminaries and notation

All definitions and group properties, which form the basis of our considerations, can be found in the two standard books by Fuchs [7] and Mader [10]. The latter contains special notions of the theory of almost completely decomposable groups which we will use and extend in this paper to some classes of infinite rank groups X of *ring type*, i.e. groups having critical typesets $T_{\mathrm{cr}}(X)$ consisting only of *idempotent types* τ. Such idempotent types can be represented by *characteristics* with only $0'$s and ∞'s as entries, see [10, p. 13], [7, Section 85]). We write $\tau(p) = \infty$ if τ has ∞ as entry at position p and $\tau(p) = 0$ for the other p. Then τa is a τ-homogeneous group, containing the element a, that is divisible only by primes p with $\tau(p) = \infty$.

Moreover, the groups X under consideration are *block-rigid* (i.e. with antichains as their critical typesets) and, therefore, for any $\tau \in T_{\mathrm{cr}}(X)$ the fully invariant pure subgroup $X(\tau)$ is a homogeneous group of type τ denoted by X_τ, and which will be called a *τ-homogeneous component* of X. As for other notation, we use $Y \subseteq_* X$ to indicate that Y is a pure subgroup of the torsion-free group X, and T_p for the *p-component* in the primary decomposition of a torsion group T into primary summands. In most cases we say "group" for a torsion-free abelian group X because it is our main object and denote the subgroup Y of $\mathbb{Q}Y$ such that $qY = X$ by $\frac{X}{q}$ with natural q.

We now recall the basic definitions needed in the sequel. Definition 2.1 in [4] of the class $\mathcal{H}_{\mathfrak{T}}$ with the restrictions before Observation 2.2 in [4] lead to the following definition.

Definition 2.1. A torsion-free abelian group X belongs to the class $\mathcal{H}'_{\mathfrak{T}}$ if there exists a completely decomposable subgroup $R(X) = \bigoplus_{\tau \in T_{\mathrm{cr}}(R(X))} C_\tau^X$ of X such that the following conditions are satisfied:

(1) $\mathfrak{T} = T_{\mathrm{cr}}(R(X))$ is an antichain of idempotent types,

(2) $R(X)_\tau := C_\tau^X \subseteq_* X$ is a finite rank pure and τ-homogeneous completely decomposable subgroup of X for all $\tau \in \mathfrak{T}$,

(3) $X/R(X) = \bigoplus_{p \in P_X} T_p^X$ for some set of primes P_X and p^{γ_p}-bounded p-groups T_p^X,

(4) for every $p \in P_X$ the set $\{q \in P_X : [T_p^X] \cap [T_q^X] \neq \varnothing\}$ is finite; here, $[T_p^X]$ is the minimal subset $\mathfrak{T}_p \subseteq \mathfrak{T}$ such that

$$T_p^X \subseteq \left(\left(\bigoplus_{\tau \in \mathfrak{T}_p} C_\tau^X \right)_* + R(X) \right) / R(X).$$

We will be interested in factor groups of *almost rigid groups*, a subclass of the class $\mathcal{H}'_{\mathfrak{T}}$. Recall from [4] that a group $X \in \mathcal{H}'_{\mathfrak{T}}$ is called an *almost rigid group* (*alr-group*) if the set $\mathfrak{T}$ is countable and all primary components T_p^X of $X/R(X)$ are cyclic groups. Knowing their classification from [3] we will use the almost rigid groups as preimages of the groups, which are to be classified here up to near-isomorphism. The following recalls [4, Definitions 5.1, 5.3]:

Definition 2.2. Let $X \in \mathcal{H}'_{\mathfrak{T}}$ be an almost rigid group and let $(\alpha_\tau : \tau \in \mathfrak{T})$ be integers such that the following conditions hold:

(1) $R(X) = \bigoplus_{i \in \omega} A_i^{l_i}$ (where $l_i \in \mathbb{N}$) with $A_i = \bigoplus_{\tau \in T_i} \tau a_\tau$, $\mathfrak{T} = \bigcup_{i \in \omega} T_i$ the countable disjoint union of finite subsets $T_{\mathrm{cr}}(A_i) = T_i \subset \mathfrak{T}$ and $\mathrm{rk}\, A_i = |T_i| \neq 2$ for any $i \in \omega$,

(2) for any $i \in \omega$ and any $\tau \in T_i$ there exists a prime p such that $\tau(p) = \infty$ and $\sigma(p) \neq \infty$ for all $\sigma \neq \tau$, $\sigma \in T_i$,

(3) if $i \in \omega$ and $|T_i| \neq 1$, then $\bigcap_{\tau \neq \sigma, \tau \in T_i} \tau = \mathbb{Z}$ for any $\sigma \in T_i$,

(4) if $i \in \omega$ and $|T_i| \neq 1$, then α_τ $(\tau \in T_i)$ is not p-divisible if $\sigma(p) = \infty$ for some $\sigma \in T_i$; if $|T_i| = 1$, then $\alpha_\tau = 0$ with $T_i = \{\tau\}$,

(5) if $i \in \omega$ and $|T_i| \neq 1$, then $\gcd(\{\alpha_\tau : \tau \neq \sigma, \tau \in T_i\}) = 1$ for any $\sigma \in T_i$,

(6) $|[T_p^X] \cap T_i| \leqslant 1$ for any $p \in P_X$ and $i \in \omega$,

(7) if $i \in \omega$ and $|[T_p^X] \cap T_i| = 1$ for some $p \in P_X$, then each α_τ is relatively prime to p for any $\tau \in T_i$,

(8) if $i \in \omega$ and $|[T_p^X] \cap T_i| = 1$, then $\tau(p) \neq \infty$ for any $\tau \in T_i$.

Let $K \subseteq X$ be defined as follows:

$$K = \bigoplus_{i\in\omega} K^{l_i}(A_i) \subset X \quad \text{with} \quad K(A_i) = \left\langle \sum_{\tau\in T_i} \alpha_\tau a_\tau \right\rangle \subset A_i.$$

The partition $(T_i : i \in \lambda)$ and the two sets of integers $(\alpha_\tau : \tau \in \mathfrak{T})$, $(l_i : i \in \omega)$ will be called a *triple of partition, coefficients and parameters* for the group X. Moreover, the group $B = X/K$ is called a *proper* $\mathfrak{B}^{(1)}$*alr-group* and the canonical epimorphism $\phi : X \mapsto B = X/K$ will be called a *regular representation* of the proper $\mathfrak{B}^{(1)}$alr-group B. The *regulator* of B is defined as $R(B) = R(X)\phi$.

Note that for simplicity we use $K^{l_i}(A_i)$ to denote the direct sum $(K(A_i))^{l_i}$ of l_i copies of $K(A_i)$ in the above Definition 2.2.

Since we intend to classify proper $\mathfrak{B}^{(1)}$alr-groups in terms of numerical invariants we need to use the invariants $m_\tau(X)$, $\tau \in T_{\mathrm{cr}}(X)$, of almost rigid groups defined in [3, Definition 3.2] and [4, Definition 5.4].

Recall that for an almost rigid group $X \in \mathcal{H}'_{\mathfrak{T}}$ there are elements $u_p \in X$ such that

$$\bigoplus_{p\in P} \langle u_p + R(X)\rangle = X/R(X) \quad \text{with} \quad |u_p + R(X)| = p^{\gamma_p}$$

for some integers γ_p since the primary components of $X/R(X)$ are assumed to be cyclic. Writing

$$p^{\gamma_p} u_p = \sum_{\tau\in\mathfrak{T}} u_{\tau p} \text{ with } u_{\tau p} \in R(X)_\tau$$

we have

$$\overline{p^{\gamma_p} u_p} = \sum_{\tau\in\mathfrak{T}} \overline{u_{\tau p}}, \quad \overline{u_{\tau p}} \in R(X)_\tau / p^{\gamma_p} R(X)_\tau,$$

where $\bar{\ } : R(X) \to R(X)/p^{\gamma_p} R(X)$ is the canonical epimorphism. The numerical invariants $m_\tau(X)$ of X were then defined as

$$m_\tau(X) = \prod_{p\in P} |\overline{u_{\tau p}}| \quad \text{for } \tau \in \mathfrak{T}.$$

The appearance of these numbers for proper $\mathfrak{B}^{(1)}$alr-groups will be discussed in detail below, but it was already shown in [4, Definition-Lemma 5.6] that conditions (6) to (8) of Definition 2.2 can be reformulated using the numerical invariants m_τ:

Lemma 2.3. *Conditions* (6)–(8) *of Definition* 2.2 *are equivalent to the following ones:*

(6') *if there exists* $i \in \omega$ *with* $\tau \in T_i$ *and* $\sigma \in T_i$, *then* $\gcd(m_\tau(X), m_\sigma(X)) = 1$,

(7') *if there exists* $i \in \omega$ *with* $\tau \in T_i$ *and* $\sigma \in T_i$, *then* α_τ *is relatively prime to* $m_\sigma(X)$, (8') *if there exists* i *with* $\tau \in T_i$ *and* $\sigma \in T_i$, *then* $\tau(p) \neq \infty$ *for any prime divisor* p *of* $m_\sigma(X)$.

The basis of our desired decomposability criterion for proper $\mathfrak{B}^{(1)}$alr-groups will be the corresponding decomposability criterion [3, Theorem 5.2] for almost rigid groups.

Theorem 2.4 ([3, Theorem 5.2]). *Let* X *be an almost rigid group with* $T_{\mathrm{cr}}(X) = T$ *and* $m_\tau(X) = m_\tau$. *Then there exists a decomposition* $X = \bigoplus_{i\in I} X_i$ *into almost rigid summands with products* $m_\tau(X) = \prod_{i\in I} m_{\tau i}$, *where* $m_{\tau i} = m_\tau(X_i)$ *if and only if the following compatibility conditions hold:*

(D1) *The numbers* $m_{\tau i}$ *and* $m_{\sigma j}$ *are relatively prime if* $i \neq j$ *and* $\tau, \sigma \in T_{\mathrm{cr}}(X)$.

(D2) $|\{i \in I : m_{\tau i} > 1\}| \leqslant \mathrm{rk}(X(\tau))$ *for any* $\tau \in T_{\mathrm{cr}}(X)$.

In order to carry over this decomposition result for almost rigid groups to their images, the proper $\mathfrak{B}^{(1)}$alr-groups, we will need the so-called *admissible decompositions* for almost rigid groups (see [4, Definitions 5.8, 5.9]):

Definition 2.5. Let $X \in \mathcal{H}'_{\mathfrak{T}}$ be an almost rigid group that has a decomposition $X = \bigoplus_{j\in I} X_j$. If X_j $(j \in I)$ are rigid groups, having regulators isomorphic to direct sums of groups from the set $\{A_i, i \in \omega\}$, then this decomposition will be called an *admissible decomposition* of X.

Among all the admissible decompositions of almost rigid groups there are special ones, determined uniquely up to near-isomorphism and called *main decompositions*:

Definition 2.6. Let $X \in \mathcal{H}'_{\mathfrak{T}}$ be an almost rigid group. An admissible decomposition $X = X' \oplus F$, where X' is a rigid group satisfying:

(1) $T_{\mathrm{cr}}(X') \supseteq T_i$ if and only if $m_\tau(X) \neq 1$ for some $\tau \in T_i$,

(2) $m_\tau(X') = m_\tau(X)$ for all $\tau \in T_{\mathrm{cr}}(X')$

will be called a *main admissible decomposition* of X.

It will be shown next that we are interested only in admissible decompositions of an almost rigid group X. In fact, given X and a proper $\mathfrak{B}^{(1)}$alr-group $B = X/K$ that has a decomposition, then our strategy is to see that there is an admissible decomposition of X such that the direct summands can be considered as preimages of the summands of B. More precisely, we need some restrictions on the admissible decompositions of X which will lead us to the notion of *K-admissible decompositions* in the later sections.

3 Near-isomorphism of $\mathfrak{B}^{(1)}$alr-groups in terms of the invariants

Our aim in this section is to introduce numerical invariants for proper $\mathfrak{B}^{(1)}$alr-groups and to obtain a near-isomorphism criterion for these groups formulated in terms of these invariants. This will be done in complete analogy to [4, Theorem 5.15], which is the main result of that paper.

Let $X \in \mathcal{H}'_{\mathfrak{T}}$ be an almost rigid group and $B = X/K$ a proper $\mathfrak{B}^{(1)}$alr-group with $\phi : X \to B = X/K$ the regular presentation of B. For simplicity we denote $P = P_X$ in (3) of Definition 2.1. It is almost evident and easy to recall from [4, equations (2), (3), and (12)] that there exists a uniquely determined *factor primary representation* of X in the following form:

$$X = \sum_{p \in P} X_p \quad \text{with} \quad X_p = X \cap \frac{R(X)}{p^{\gamma_p}} \in \mathcal{H}'_{\mathfrak{T}} \text{ and } X_p/R(X) \cong T_p^X. \quad (3.1)$$

This implies the *factor primary representation of B*:

$$B = X/K = \sum_{p \in P} B_p \quad \text{with} \quad B_p = X_p\phi = X_p/K.$$

Besides X_p with $p \in P$, we introduce the set of finite rank fully invariant subgroups of X

$$X'_p = [T_p^X]_*^{X_p}.$$

Their images $B'_p = X'_p\phi$ are isomorphic to X'_p and they are block-rigid crq-groups with primary regulator quotient. It follows that there exist elements $u'_p \in X$ such that $X'_p = \langle R(X'_p), u'_p\rangle$ with $|X'_p/R(X'_p))| = p^{\gamma_p}$ and, clearly,

$$m_\tau(X_p) = m_\tau(X'_p) = m_\tau(B_p)$$

with $m_\tau(X'_p) = 1$ if $\tau \notin T_{\text{cr}}(X'_p)$.

Recall from [4, Theorem 5.13] the following

Theorem 3.1. *Let $\phi : X \mapsto B$ be a regular representation of a $\mathfrak{B}^{(1)}$alr-group B with the triple $(T_i : i \in \lambda)$, $(\alpha_\tau : \tau \in \mathfrak{T})$ and $(l_i : i \in \omega)$ of partition, coefficients and parameters for the group X. Then there exists a main admissible decomposition $X = X' \oplus F$ with $m_\tau(X') = m_\tau(X)$ and the corresponding decomposition $B = B' \oplus H$ such that $X'\phi = B'$, $F\phi = H$.*

In [4, Definition 5.14] this fact motivated a new definition as follows,

Definition 3.2. Given a proper $\mathfrak{B}^{(1)}$alr-group B, a decomposition $B = B' \oplus H$ with a rigid $\mathfrak{B}^{(1)}$alr-group B' such that $B/R(B) \cong B'/R(B')$ will be called a *main decomposition* of B.

Now, if $B = B' \oplus H$ is the main decomposition of the proper $\mathfrak{B}^{(1)}$alr-group B, induced by a main admissible decomposition $X = X' \oplus F$, which means that $B' = X'\phi$ and $H = F\phi$, then let

$$X' = \sum_{p \in P} X_p \quad \text{with} \quad X_p = X' \cap \frac{R(X')}{p^{\gamma_p}} \text{ and } X_p/R(X') \cong T_p^X$$

be the factor primary representation of the rigid group X', see (3.1).

Clearly,

$$m_\tau(X') = \prod_{p \in P} m_\tau(X_p) = \prod_{p \in P} m_\tau(X_p'')$$

with finite rank rigid crq-subgroup

$$X_p'' = \Big(\bigoplus_{\tau \in [T_p]} X'(\tau) \Big)_*^{X_p}$$

of X' for each $p \in P$.

Let $B' = \sum_{p \in P} B_p'$ with $B_p' = B' \cap \frac{R(B')}{p^{\gamma_p}}$. Define in B' uniquely determined crq-subgroups $B_p'' = X_p''\phi \cong X_p''$, then $m_\tau(X_p) = m_\tau(X_p'') = m_\tau(B_p'')$, $\tau \in \mathfrak{T}$.

We may define the set of numbers

$$m_\tau(B') = \prod_{p \in P} m_\tau(B_p'')$$

with the property $m_\tau(B') = m_\tau(X')$, $\tau \in \mathfrak{T}$. Furthermore, $m_\tau(X') = m_\tau(X)$ and we are allowed to put

$$m_\tau(B) = m_\tau(B') = m_\tau(X). \tag{3.2}$$

Recall again from [3, Theorem 4.12] that two almost rigid groups X and Y are nearly-isomorphic if and only if their regulators are isomorphic, in symbols $R(X) \cong R(Y)$, and $m_\tau(X) = m_\tau(Y)$ for each critical type $\tau \in \mathfrak{T}$ and using the fact that by [4, Theorem 5.15] two proper $\mathfrak{B}^{(1)}$alr-groups are nearly-isomorphic if and only if their preimages in regular representations are nearly-isomorphic and the triples of partition, coefficients and parameters coincide for them we immediately obtain the following *near-isomorphism criterion*:

Theorem 3.3. *Let B and C be proper $\mathfrak{B}^{(1)}$alr-groups. Then $B \cong_{\mathrm{nr}} C$ if and only if $R(B) \cong R(C)$ and $m_\tau(B) = m_\tau(C)$ for each critical type $\tau \in \mathfrak{T}$.*

Having this result we are permitted to call the numbers $m_\tau(B)$ not only type-invariants of a proper $\mathfrak{B}^{(1)}$alr-group B, but also *near-isomorphism invariants* of B, $\tau \in \mathfrak{T}$. The same terminology is appropriate for almost rigid groups, which are preimages of $\mathfrak{B}^{(1)}$alr-groups.

4 Direct decompositions of $\mathfrak{B}^{(1)}$alr-groups

We are now approaching our main result which gives a combinatorial decomposability criterion for proper $\mathfrak{B}^{(1)}$alr-groups based on that for almost rigid groups, see Theorem 2.4. In connection with Definition 2.2 we introduce some more definitions. Recall that a proper $\mathfrak{B}^{(1)}$alr-group was defined as the quotient of an almost rigid group X and some subgroup K of the following form:

$$K = \bigoplus_{i \in \omega} K^{l_i}(A_i) \subset X \quad \text{with} \quad K(A_i) = \left\langle \sum_{\tau \in T_i} \alpha_\tau a_\tau \right\rangle \subset A_i,$$

where $R(X) = \bigoplus_{i \in \omega} A_i^{l_i}$ with $A_i = \bigoplus_{\tau \in T_i} \tau a_\tau$.

Definition 4.1. Any set of elements $(A_i^{l_i})_K = \{(a_\tau^n)_{n=1,\ldots,l_i} \in A_i^{l_i} : \tau \in T_i\}$ will be called a *K-basis* of $A_i^{l_i}$ if the following hold:

(1) $A_i^{l_i}(\tau) = \bigoplus_{n \leqslant l_i} \tau a_\tau^n$ with $\tau \in T_i$,

(2) $K^{l_i}(A_i) = \langle \{\sum_{\tau \in T_i} \alpha_\tau a_\tau^n : n = 1, \ldots, l_i\} \rangle$.

Definition 4.2. Automorphisms $\Psi \in \mathrm{Aut}(X)$ will be called *K-basic* automorphisms of X if for any $i \in \omega$ they act on each τ-homogeneous fully invariant subgroup $A_i^{l_i}(\tau)$ as multiplication by the same matrix D_i ($|\det D_i| = 1$) with respect to a K-basis $\{(a_\tau^n)_{n=1,\ldots,l_i} \in A_i^{l_i} : \tau \in T_i\}$.

As was promised above we strengthen the definition of admissible decompositions as follows, see Definition 2.5.

Definition 4.3. Let B be a proper $\mathfrak{B}^{(1)}$alr-group and let $\phi : X \mapsto B$ be its regular representation. We say that $X = \bigoplus_{j \in J} X_j$ is a *K-admissible decomposition* of X if it is admissible and obtained by a K-basic automorphism of X.

At this point we would like to remark that the K-admissibility of a decomposition $X = \bigoplus_{j \in J} X_j$ of the almost rigid group X is equivalent to the existence of the corresponding induced decomposition $K = \bigoplus_{j \in J} K_j$ with $K_j \subset X_j$ (see Definitions 4.1 and 4.2). It then follows that the proper $\mathfrak{B}^{(1)}$alr-group $B = X/K$ is also decomposable as $B = \bigoplus_{j \in J} X_j/K_j$. This is the reason why we needed the K-admissible automorphisms and this leads to the following very natural

Lemma 4.4 (Decomposability Criterion 1). *Let B be a proper $\mathfrak{B}^{(1)}$alr-group and $\phi : X \mapsto B$ its regular representation with the triple $(T_i : i \in \lambda)$, $(\alpha_\tau : \tau \in \mathfrak{T})$ and $(l_i : i \in \omega)$ of partition, coefficients and parameters for the group X. Then one has $B = B_1 \oplus B_2$ if and only if there exists a K-admissible decomposition $X = X_1 \oplus X_2$ such that $B_j = X_j \phi$ for $j = 1, 2$.*

Proof. The sufficiency of the condition is trivial by the remark above.

Let $B = B_1 \oplus B_2$. This implies the existence of the corresponding decomposition of its regulator, fully invariant subgroup $\mathcal{A} = G_1 \oplus G_2$ with $G_1 = \mathcal{A} \cap B_1$ and $G_2 = \mathcal{A} \cap B_2$. Then we have $B/\mathcal{A} = B_1/G_1 \oplus B_2/G_2$ and B_1 and B_2 are also $\mathfrak{B}^{(1)}$alr-groups. We have $B/\mathcal{A} \cong (X/K)/(R(X)/K) \cong X/R(X)$ which is isomorphic to $\bigoplus_{p \in P} T_p^X$. Then there exists the partition $P = P_1 \bigcup P_2$ such that $B_1/G_1 \cong \bigoplus_{p \in P_1} T_p^X$ and $B_2/G_2 \cong \bigoplus_{p \in P_2} T_p^X$.

For the same reason we have the corresponding decompositions of the groups $\mathcal{A}(T_i)$ with critical typesets T_i, which are direct sums of l_i copies of strongly indecomposable groups $A_i/K(A_i)$, see [7, Section 92] and [8, Section 3.3]. Namely, $\mathcal{A}(T_i) = G_i^1 \oplus G_i^2$ with $G_i^1 = \mathcal{A}(T_i) \cap G_1$ and $G_i^2 = \mathcal{A}(T_i) \cap G_2$ for each $i \in \omega$ since $\mathcal{A}(T_i) = \sum_{\tau \in T_i} \mathcal{A}(\tau)$ are fully invariant in $\mathcal{A}$, see [3, Introduction]. Denote $\mathcal{A}_i = \mathcal{A}(T_i)$ and let $\mathcal{A}_i(\tau)$ be its fully invariant τ-homogeneous subgroup with $\tau \in T_i$. Then $\mathcal{A}_i(\tau) = (\mathcal{A}_i(\tau) \cap B_1) \oplus (\mathcal{A}_i(\tau) \cap B_2)$ with one of the summands allowed to be zero. Recall that $\mathcal{A}_i(\tau) \cong A_i(\tau)$, which implies the corresponding decomposition $R(X) = R_1 \oplus R_2$ of the regulator of X with $G_1 = R_1\phi$ and $G_2 = R_2\phi$.

Clearly, $B_1 = {(G_1)_*}^B$, $B_2 = {(G_2)_*}^B$ and $X = X_1 \oplus X_2$ is a K-admissible direct decomposition such that $X_1 = {(R_1)_*}^X$, $X_2 = {(R_2)_*}^X$ and $X_1\phi = B_1$, $X_2\phi = B_2$. □

Since the number of direct summands of B is at most countable (because it is not greater than the rank of the group), a trivial induction leads to the following.

Corollary 4.5. *Let B be a proper $\mathfrak{B}^{(1)}$alr-group and let $\phi : X \mapsto B$ be its regular representation with the triple $(T_i : i \in \lambda)$, $(\alpha_\tau : \tau \in \mathfrak{T})$ and $(l_i : i \in \omega)$ of partition, coefficients and parameters for the group X. Then $B = \bigoplus_{j \in J} B_j$ if and only if there exists a K-admissible decomposition $X = \bigoplus_{j \in J} X_j$ such that $B_j = X_j\phi$, $j \in J$.*

Now we need some preparation for transforming the obtained decomposability criterion into the one based on the numerical invariants of $\mathfrak{B}^{(1)}$alr-groups which will be our main result.

Let M denote the set enumerating the groups A_i to write $i \in M$ for our convenience. Then $R = R(X) = \bigoplus_{i \in M} A_i^{l_i}$, see Definition 2.2. For any filtration $M = \bigcup_{k \in I} M_k$ with $M_k \subset M_{k+1}$ and $M_0 = \varnothing$ there exists a corresponding filtration $X = \bigcup_{k \in I} X^k$ with the ascending chain of fully invariant pure subgroups

$$X^k = {(R(X^k))_*}^X$$

such that $R(X^k) = \bigoplus_{i \in M_k} {A_i}^{l_i}$ and $R(X) = \bigcup_{k \in I} R(X^k)$. Note that all X^k are crq-groups. Without loss of generality assume that it is a special filtration characterized by the condition that $m_\sigma(X)$ and $m_\tau(X)$ are relatively prime for any $\sigma \in T_{\text{cr}}(X^{k-1})$ and $\tau \in \mathfrak{T} \setminus T_{\text{cr}}(X^k)$.

For any $M' \subset M$ introduce a K-basis of $\bigoplus_{i\in M'} A^{l_i}$ as $\bigcup_{i\in M'}(A^{l_i})_K$, see Definition 4.1.

Denote $T^k = T_{\text{cr}}(X^k)$, $k \in I$. Recall from [3, Lemma 4.6] that

$$m_\tau(X^k) = m_\tau(X)$$

if $\tau \in T^{k-1}$. Moreover, near-isomorphism invariants of the groups X^k can be calculated on the basis of [3, Proposition 2.11], namely,

$$m_\tau(X^k) = \frac{m_\tau(X)}{\gcd(m_\tau(X), Q^{k+1}/Q^k)} \tag{4.1}$$

with $Q^k = \exp X^k/R(X^k)$, $k \in I$.

If $X = X_1 \oplus X_2$, then

$$X^k = Y_1^k \oplus Y_2^k \quad \text{with} \quad Y_1^k = X_1 \cap X^k \text{ and } Y_2^k = X_2 \cap X^k,\ k \in I, \tag{4.2}$$

as X^k is fully invariant in X. Moreover, near-isomorphism invariants of Y_1^k and Y_2^k can also be calculated in the following way:

$$m_\tau(Y_1^k) = \frac{m_\tau(X_1)}{\gcd(m_\tau(X_1), Q_1^{k+1}/Q_1^k)} \quad \text{with} \quad Q_1^k = \exp X_1^k/R(X_1^k),\ k \in I,$$

and

$$m_\tau(Y_2^k) = \frac{m_\tau(X_2)}{\gcd(m_\tau(X_2), Q_2^{k+1}/Q_1^k)} \quad \text{with} \quad Q_2^k = \exp X_2^k/R(X_2^k),\ k \in I,$$

if we naturally denote

$$X_1^k = \left(\left(\bigoplus_{i\in M_k} A_i^{l_i}\right) \cap X^k\right)_*^{X_1} \text{ and } X_2^k = \left(\left(\bigoplus_{i\in M_k} A_i^{l_i}\right) \cap X^k\right)_*^{X_2}.$$

In the proof of the next theorem we need to construct decompositions of the proper $\mathfrak{B}^{(1)}$alr-group $B = X/K$ with type invariants induced by the decomposition of the almost rigid group $X = X_1 \oplus X_2$. So, we will say that the invariants $m_\tau(Y_1^k)$ and $m_\tau(Y_2^k)$ are *predicted* (or even a decomposition of X^k is predicted in the sense that its regulator and near-isomorphism invariants are known).

The next ingredient of the algorithm will be the *extension* of a decomposition (4.2) from X^k to X^{k+1}, which means that $Y_1^{k+1} \cap X^{k-1} = Y_1^k \cap X^{k-1}$ and $Y_2^{k+1} \cap X^{k-1} = Y_2^k \cap X^{k-1}$.

Following [2] we say that a direct sum of strongly indecomposable groups is a strongly decomposable group. It is associated with completely decomposable groups which are direct sums of rank-one groups.

Finally, we have approached the main result summarizing the above discussion. Let

$$X = \bigcup_{k \in I} X^k$$

be a special filtration of X with an ascending chain of fully invariant pure subgroups X^k satisfying the following condition: $m_\sigma(X)$ and $m_\tau(X)$ are relatively prime for any $\sigma \in T_{\text{cr}}(X^{k-1})$ and $\tau \in \mathfrak{T} \setminus T_{\text{cr}}(X^k)$.

The following main Theorem 4.6 is a generalization of the corresponding Theorem 5.2 from [3] on almost rigid groups to the class of proper $\mathfrak{B}^{(1)}$alr-groups. Its proof is quite technical because it needs a double induction. First, we restrict ourselves to the case of two summands and induct on $k \in I$. Then we induct on the number of the summands.

Theorem 4.6 (Decomposability Criterion 2). *Let B be a proper $\mathfrak{B}^{(1)}$alr-group with $m_\tau = m_\tau(B)$. Then there exists a decomposition*

$$B = \bigoplus_{j \in J} B_j \oplus C \tag{4.3}$$

into rigid summands B_j with $m_{\tau j} = m_\tau(B_j)$ and strongly decomposable C if and only if the following hold:

(1) $m_\tau = \prod_{j \in J} m_{\tau j}$ *for any* $\tau \in \mathfrak{T}$,

(2) $\gcd(m_{\tau j}, m_{\sigma k}) = 1$ *if* $j \neq k$,

(3) $|\{j : \prod_{\tau \in T_i} m_{\tau j} > 1\}| \leqslant l_i$, $i \in \omega$, $j \in J$.

Proof. Let $\phi : X \mapsto B$ be the regular representation of B. On the basis of Decomposability Criterion 1 (Lemma 4.4) we have that the decomposition

$$B = \bigoplus_{j \in J} B_j \oplus C$$

with $m_{\tau j} = m_\tau(B_j)$ satisfying (1)–(3) exists if and only if there is a K-admissible decomposition

$$X = \bigoplus_{j \in J} X_j \oplus H'$$

with $m_{\tau j} = m_\tau(X_j)$, $B_j = X_j\phi$ and $C = H'\phi$, see (3.2).

Note that the required admissible decomposition (4.3) of X with conditions (1)–(3) is one of the decompositions described in the decomposability criteria [3, Theorem 5.2]. Then we need only to show that it can be obtained by a K-basic automorphism of X. Moreover, the necessity of these conditions also follows from the mentioned criteria.

To prove that conditions (1)–(3) are sufficient for the existence of the required decomposition of B we take its main decomposition $B = B' \oplus H$, which exists by Theorem 3.1, and obtain a decomposition (4.3) on this basis. Without loss of generality assume that $\operatorname{lcm}_{\tau\in T_i} m_\tau \neq 0$ for any i with $T_i \subset T_{\mathrm{cr}}(B)$. Since there exists a corresponding admissible main decomposition

$$X = X' \oplus H \tag{4.4}$$

with $X'\phi = B'$ and $H\phi = F$, we need to obtain from (4.4) by a K-basic automorphism an admissible decomposition

$$X = \bigoplus_{j\in J} X_j \oplus H' \tag{4.5}$$

into rigid alr-groups X_j of rank more than one and $m_\tau(X_j) = m_{\tau j}$. We may restrict ourselves to the situation when $\operatorname{lcm}_{\tau\in T_i} m_\tau(X_j) \neq 1$ for each i satisfying $T_i \subset T_{\mathrm{cr}}(X_j)$ with $j \in J$.

More precisely, our purpose is to get the corresponding decomposition

$$R(X) = \left(\bigoplus_{j\in J} R_j\right) \oplus H' \quad \text{with} \quad X_j = (R_j)_*{}^X.$$

We may also assume that $H' = 0$. If $H' \neq 0$, then H contains a completely decomposable summand $H'' \cong H'$, therefore H'' and H' can be removed from (4.4) and (4.5) accordingly and in this case for any $i \in M$ there exists $\tau \in T_i$ such that $m_\tau(X) \neq 1$. Moreover, without loss of generality assume that there is no non-trivial partition $M = M_1' \cup M_2'$ such that $\gcd(m_\tau(B), m_\sigma(B)) = 1$ whenever $\tau \in \bigcup_{i\in M_1'} T_i$ and $\sigma \in \bigcup_{j\in M_2'} T_j$. Otherwise group B is a direct sum of proper $\mathfrak{B}^{(1)}$alr-groups with independent direct decomposition constructions. It follows from (3.2) that the same *connection condition* should be assumed for group X, that is for any partition $M = M_1' \cup M_2'$ there exist $\tau \in \bigcup_{i\in M_1'} T_i$ and $\sigma \in \bigcup_{j\in M_2'} T_j$ such that $m_\tau(X)$ and $m_\sigma(X)$ are not relatively prime.

So, we need to have the necessary decompositions of alr-groups.

Case I. Let us first consider the case $|J| = 2$, that is $B = B_1 \oplus B_2$ and $l_i \leqslant 2$ for all $i \in M$. By Lemma 4.4 we need a decomposition $X = X_1 \oplus X_2$ with not necessarily indecomposable rigid summands X_1, X_2 and invariants $m_{\tau 1} = m_\tau(X_1)$, $m_{\tau 2} = m_\tau(X_2)$, $\tau \in \mathfrak{T}$. For the corresponding decomposition of the regulator $R(X) = R_1 \oplus R_2$ such that $X_1 = (R_1)_*{}^X$ and $X_2 = (R_2)_*{}^X$ we have that for each $\tau \in \mathfrak{T}$ the following holds:

$$R(\tau) = R_1(\tau) \oplus R_2(\tau)$$

with $X_1 = (\bigoplus_{\tau\in\mathfrak{T}} R_1(\tau))_*{}^X$ and $X_2 = (\bigoplus_{\tau\in\mathfrak{T}} R_2(\tau))_*{}^X$ (remark that $R_1(\tau)$ or $R_2(\tau)$ is allowed to be zero). We are concentrated on the admissible decompositions of X, therefore, if $R_j(\tau) \neq 0$ for some $\tau \in \mathfrak{T} \cap T_i$ then $R_j(\sigma) \neq 0$ for any $\sigma \in \mathfrak{T} \cap T_i$ with $i \in M$, $j = 1, 2$.

We now intend to obtain by a K-basic automorphism such a decomposition

$$X = X_1 \oplus X_2 \tag{4.6}$$

from the main decomposition (4.4) with $m_{\tau 1} = m_\tau(X_1)$, $m_{\tau 2} = m_\tau(X_2)$.

Clearly there exists a K-basis $R(X)_K = \{a_\tau^n : n = 1, 2\}$ of $R(X)$ such that

$$X' = \Big\langle \bigoplus_{\tau \in T_{\mathrm{cr}}(X)} a_\tau^1 \Big\rangle_*^X \quad \text{and} \quad H = \Big\langle \bigoplus_{\tau \in T_{\mathrm{cr}}(X)} a_\tau^2 \Big\rangle \tag{4.7}$$

with $a_\tau^2 = 0$ if $\tau \notin T_{\mathrm{cr}}(H)$, see (4.4).

Let X^1 be an arbitrary fully invariant pure subgroup of X with $|M_1| > 1$. We are now concentrated on X^2 with the regulator $R(X^2) = \bigoplus_{i \in M_2} {A_i}^{l_i}$. Recall that $M_1 \subset M_2$.

Since X^2 is fully invariant in X, the above decomposition $X = X_1 \oplus X_2$, which is to be constructed, reflects on the group X^2 in the way that it must have the corresponding decomposition $X^2 = {Y_1}^2 \oplus {Y_2}^2$ with ${Y_1}^2 = X_1 \cap X^2$, ${Y_2}^2 = X_2 \cap X^2$. Similarly, $X^3 = {Y_1}^3 \oplus {Y_2}^3$ with ${Y_1}^3 = X_1 \cap X^3$, ${Y_2}^3 = X_2 \cap X^3$ etc.

Recall from [3, Lemma 4.6] that we have $m_\tau(X^2) = m_\tau(X)$ if $\tau \in T^1$, and $m_\tau(X^3) = m_\tau(X)$ if $\tau \in T^2$.

Our first object now is to get the necessary decomposition of X^2. To this end, Let $Q = \exp X^2/R(X^2)$, $Q_0 = \exp X^1/R(X^1)$ and $\widetilde{Q} = \exp X^3/R(X^3)$. These numbers can be viewed as $Q = \mathrm{lcm}_{\tau \in T^2}\, m_\tau(X^2)$, $Q_0 = \mathrm{lcm}_{\tau \in T^1}\, m_\tau(X^1)$ and $\widetilde{Q} = \mathrm{lcm}_{\tau \in T^3}\, m_\tau(X^3)$. We also have that

$$m_\tau(X^2) = \frac{m_\tau(X^3)}{\gcd(m_\tau(X^3), \widetilde{Q}/Q)} = \frac{m_\tau(X)}{\gcd(m_\tau(X), \widetilde{Q}/Q)},$$

see [3, Proposition 2.11] and (4.1). It is clear that

$$Q = Q_1 Q_2$$

with $Q_1 = \exp {Y_1}^2/R({Y_1}^2)$, $Q_2 = \exp {Y_2}^2/R({Y_2}^2)$, and also

$$\widetilde{Q} = \widetilde{Q}_1 \widetilde{Q}_2$$

with $\widetilde{Q}_1 = \exp {Y_1}^3/R({Y_1}^3)$, $\widetilde{Q}_2 = \exp {Y_2}^3/R({Y_2}^3)$. Evidently, $\gcd(\widetilde{Q}_1, \widetilde{Q}_2)$ is equal to 1, see conditions (1)–(2) for $m_\tau(X_j) = m_\tau(B_j)$.

By number theory there exist relatively prime integers μ_1 and μ_2 such that $\mu_1 Q_2 + \mu_2 Q_1 = 1$, that is

$$\mu_1 Q_2 \equiv 1 \ (\mathrm{mod}\ Q_1), \quad \mu_2 Q_1 \equiv 1 \ (\mathrm{mod}\ Q_2). \tag{4.8}$$

Moreover, we can find integers u and v such that $u(\mu_1 Q_2)Q_2 - v(\mu_2 Q_1)Q_1 = 1$.

Determine a new K-basis of $R(X^2)$ as follows:

$$b^1_\tau = \mu_1 Q_2 a^1_\tau + vQ_1 a^2_\tau \quad \text{and} \quad b^2_\tau = \mu_2 Q_1 a^1_\tau + uQ_2 a^1_\tau, \quad \tau \in T_{\text{cr}}(X^2). \tag{4.9}$$

For each $\tau \in T_{\text{cr}}(X^2)$ the matrix of linear transformation of $X^2(\tau) = X(\tau)$ is

$$D = \begin{pmatrix} \mu_1 Q_2 & \mu_2 Q_1 \\ vQ_1 & uQ_2 \end{pmatrix} \quad \text{with} \quad \det D = 1.$$

If $l_i = 1$ for some i, that is $a^2_\tau = 0$ for all $\tau \in T_i$, then $m_\tau(X^2)$ coincides with $m_\tau({Y_1}^2)$ or $m_\tau({Y_2}^2)$ and we take accordingly

$$b^1_\tau = a^1_\tau, \quad b^2_\tau = 0 \quad \text{or} \quad b^2_\tau = a^1_\tau, \quad b^1_\tau = 0, \quad \text{with} \quad \tau \in T^2 = T_{\text{cr}}(X^2),$$

see (4.8). Therefore

$$R(X^2) = \bigoplus_{\tau \in T^2} (\tau b^1_\tau \oplus \tau b^2_\tau).$$

Denote ${Y_1}^2 = (\bigoplus_{\tau\in T^2} \tau b^1_\tau)_*{}^X$ and ${Y_2}^2 = (\bigoplus_{\tau\in T^2} \tau b^2_\tau)_*{}^X$. By construction,

$$[{Y_1}^2 : R({Y_1}^2)] = Q_1 \quad \text{and} \quad [{Y_2}^2 : R({Y_2}^2)] = Q_2,$$

then $[{Y_1}^2 \oplus {Y_2}^2 : R(X^2)] = Q$ and $X^2 = {Y_1}^2 \oplus {Y_2}^2$. Hence the K-basic automorphism Ψ of X, which acts on each τ-homogeneous component of rank 2 as multiplication by the matrix D with respect to $\{a^1_\tau, a^2_\tau\}$ and is identity, if $a^2_\tau = 0$, led us to the predicted K-admissible decomposition of X^2 determined by (4.6).

The next purpose is to get the admissible decomposition of $X^3 = {Y_1}^3 \oplus {Y_2}^3$ determined by (4.6) which preserves the decomposition of X^1 in the following sense:

$$X^1 \cap {Y_1}^3 = X^1 \cap {Y_1}^2 \quad \text{and} \quad X^1 \cap {Y_2}^3 = X^1 \cap {Y_2}^2. \tag{4.10}$$

Let α and β be some integers such that

$$\alpha(\mu_1 \widetilde{Q}_2) - \beta(\mu_2 \widetilde{Q}_1) = 1.$$

Put $-z = \alpha(\mu_1 \widetilde{Q}_2)$ and $y = \beta(\mu_2 \widetilde{Q}_1)$ we have $-y - z = 1$ and also

$$1 + y = -\alpha\mu_1 \widetilde{Q}_2 \quad \text{and} \quad 1 + z = -\beta\mu_2 \widetilde{Q}_1.$$

It is important that the two numbers $1 + y$ and $1 + z$ are relatively prime and $\gcd(1+y, \mu_2 Q_1) = 1$ as well as $\gcd(1+z, \mu_1 Q_2) = 1$ by construction as $Q_1 | \widetilde{Q}_1$ and $Q_2 | \widetilde{Q}_2$. Then the numbers $\mu_1 Q_2(1 + y)$ and $\mu_2 Q_1(1 + z)$ are relatively prime and there exist integers $\widetilde{u}, \widetilde{v}$ such that the matrix

$$\widetilde{D} = \begin{pmatrix} \mu_1 Q_2(1 + y) & \mu_2 Q_1(1 + z) \\ \widetilde{v}\widetilde{Q}_1 & \widetilde{u}\widetilde{Q}_2 \end{pmatrix} \quad \text{has} \quad \det \widetilde{D} = 1$$

and

$$\widetilde{Q}_1 | y, \quad \widetilde{Q}_2 | z. \tag{4.11}$$

Now we are able to construct the required decomposition of X^3. Determine a K-basis of $R(X^3)$ as follows:

$$b_\tau^1 = \mu_1 Q_2(1+y)a_\tau^1 + \widetilde{v}\widetilde{Q}_1 a_\tau^2 \quad \text{and} \quad b_\tau^2 = \mu_2 Q_1(1+z)a_\tau^1 + \widetilde{u}\widetilde{Q}_2 a_\tau^1$$

for each $\tau \in T_{\mathrm{cr}}(X^3) \setminus T_{\mathrm{cr}}(X^1)$.

If $l_i = 1$ for some i, that is $a_\tau^2 = 0$ for $\tau \in T_i$, then $m_\tau(X^3)$ coincides with $m_\tau({Y_1}^3)$ or $m_\tau({Y_2}^3)$ and we take accordingly

$$b_\tau^1 = a_\tau^1, \quad b_\tau^2 = 0 \quad \text{or} \quad b_\tau^2 = a_\tau^1, \quad b_\tau^1 = 0, \quad \tau \in T_{\mathrm{cr}}(X^3) \setminus T_{\mathrm{cr}}(X^1).$$

Simultaneously, b_τ^1 and b_τ^2 are left the same as in (4.9) if $\tau \in T_{\mathrm{cr}}(X^1)$ by the condition (4.11) associated with matrices D and $\widetilde{D}$.

We have

$$R(X^3) = \bigoplus_{\tau \in T^3} (\tau b_\tau^1 \oplus \tau b_\tau^2).$$

Denote ${Y_1}^3 = (\bigoplus_{\tau \in T^3} \tau b_\tau^1)_*{}^X$ and ${Y_2}^3 = (\bigoplus_{\tau \in T^3} \tau b_\tau^2)_*{}^X$.

Since the elements of the first lines of the matrices D and $\widetilde{D}$ are congruent modulo Q_1 and Q_2 accordingly by (4.11) as $Q_1 | \widetilde{Q_1}$ and $Q_2 | \widetilde{Q_2}$, we obtain that $[{Y_1}^3 : R({Y_1}^3)] = \widetilde{Q}_1$ and $[{Y_2}^3 : R({Y_2}^3)] = \widetilde{Q}_2$, which means

$$[{Y_1}^3 \oplus {Y_2}^3 : R(X^3)] = \widetilde{Q} \quad \text{and} \quad X^3 = {Y_1}^3 \oplus {Y_2}^3.$$

Hence the K-basic automorphism Ψ of X^3, which acts on each τ-homogeneous component of rank 2 as multiplication by the matrix D or $\widetilde{D}$ depending on $\tau \in T_{\mathrm{cr}}(X^1)$ or $\tau \in T_{\mathrm{cr}}(X^3) \setminus T_{\mathrm{cr}}(X^1)$ with respect to $\{a_\tau^1, a_\tau^2\}$ and it is identity, if $a_\tau^2 = 0$, led us to the required admissible decomposition of X^3 determined by (4.6).

Continuing the process we take the groups X^2, X^3 and X^4 and consider them as X^1, X^2 and X^3 respectively to apply for them the above construction. Instead of D we take the constructed matrix $\widetilde{D}$, the numbers $\widetilde{Q}_1$ and $\widetilde{Q}_2$ will serve as Q_1 and Q_2, the numbers $\widetilde{v}$ and $\widetilde{u}$ will serve as v and u and the numbers $\mu_1 Q_2(1+y)/\widetilde{Q}_2$ and $\mu_2 Q_1(1+z)/\widetilde{Q}_1$ will work as the new μ_1 and μ_2 accordingly. Due to the above connection restriction we get the required decomposition of $A_i^{l_i}$ for any $i \in M$ in finitely many steps.

Case II. Consider the general case $|J| > 2$, which implies for each $i \in M$ that l_i can be any finite positive integer. Under the condition $H' = 0$ we have to construct a decomposition

$$X = \bigoplus_{j \in J} X_j \tag{4.12}$$

into rigid summands of rank more than one with a set of near-isomorphism invariants satisfying conditions (1)–(3) on the basis of the main K-admissible decomposition (4.4). Without loss of generality assume that the rigid summands X_j are listed in (4.12) in the following order: $(X_1, \dots, X_{j_2})$ are all the summands having non-zero intersection with X^2, $((X_1, \dots, X_{j_2}), \dots, X_{j_3})$ are all the summands having non-zero intersection with X^3 and so on. Assume that $j_2 > 2$ (we are able to choose X^2 satisfying this condition, otherwise the decomposition has been already constructed in Case I). The case $j_{k+1} = j_k$ is allowed for some (or even all) $k \in I$.

As above, $\{a_\tau^n : \tau \in T_i,\, n = 1, \dots, l_i,\, i \in M\}$ is a K-basis of $R(X)$. It follows from the decomposition in (4.12) that $l_i \leqslant j_k$ if $i \in M_k$ because, by construction, $X^k \subset (X_1 \oplus \cdots \oplus X_{j_k})$. Let k_i be the minimal natural number such that $i \in M_{k_i}$ (equivalently, k_i is the natural number satisfying $i \in M_{k_i} \setminus M_{k_i - 1}$ or, the same, $T_i \subset T_{\mathrm{cr}}(X^{k_i}) \setminus T_{\mathrm{cr}}(X^{k_i - 1})$). Then we may rewrite the K-basis of each ${A_i}^{l_i}$, $i \in M$, in the following way:

$$\begin{aligned}({A_i}^{l_i})_K = \{a_\tau^n : \tau \in T_i,\, i \in M_k \setminus M_{k-1},\, n = 1, \dots, j_k,\, k \geqslant 1 \\ \text{with } a_\tau^n = 0 \text{ if and only if } T_i \cap T_{\mathrm{cr}}(X_n) = \varnothing\}.\end{aligned}$$

First we should concentrate on X^1, X^2 and X^3 again as in Case I. The main decomposition of X^2 determined by the main admissible decomposition (4.4) is

$$X^2 = X'^2 \oplus \Bigg(\bigoplus_{\tau \in T_i,\, T_i \subset T^1,\, n = 2, \dots, j_1} \tau a_\tau^n \Bigg) \oplus \Bigg(\bigoplus_{\tau \in T_i,\, T_i \subset T^2 \setminus T^1,\, n = 2, \dots, j_2} \tau a_\tau^n \Bigg)$$

with $X'^2 = X' \cap X^2$, see (4.7). Denote the groups $Y_j^k = X_j \cap X^k$ ($j \in J, k \leqslant 3$) with the predicted invariants, which are under construction for the required decomposition (4.12).

Let us take a subgroup Z^{12} of X^2 with homogeneous components of rank not greater than 2, which is

$$Z^{12} = X'^2 \oplus \Bigg(\bigoplus_{\tau \in T_i,\, T_i \subset T^1} \tau a_\tau^2 \Bigg) \oplus \Bigg(\bigoplus_{\tau \in T_i,\, T_i \subset T^2 \setminus T^1} \tau a_\tau^2 \Bigg), \tag{4.13}$$

and construct its K-admissible decomposition $Z^{12} = {Y_1}^2 \oplus {Z_1}^2$ with the rigid summand ${Z_1}^2$, uniquely determined up to near-isomorphism by its near-isomorphism invariants

$$m_\tau({Z_1}^2) = \frac{m_\tau(X^2)}{m_\tau({Y_1}^2)}$$

under the restriction in (4.13) that $a_\tau^2 = 0$ if $m_\tau({Z_1}^2) = 0$, see Case I.

The connection condition provides that a special numbering the homogeneous components A_i, $i \in M$, yields that $m_\tau(X^3)$ and $m_\sigma(X^3)$ are not relatively prime for some $\tau \in T^2, \sigma \in T^3 \setminus T^2$. Again using Case I we extend the obtained decomposition of Z^{12} to Z^{13}, which is

$$Z^{13} = X'^3 \oplus \Big(\bigoplus_{\tau \in T^1} \tau a_\tau^2\Big) \oplus \Big(\bigoplus_{\tau \in T^2 \setminus T^1} \tau a_\tau^2\Big) \oplus \Big(\bigoplus_{\tau \in T^3 \setminus T^2} \tau a_\tau^2\Big)$$

with $X'^3 = X' \cap X^3$ and pure fully invariant subgroup Z^{12}. Then it follows that $Z^{13} = Y_1{}^3 \oplus Z_1{}^3$ with the rigid summand $Y_1{}^3$ having the predicted invariants, and rigid summand $Z_1{}^3$, uniquely determined up to near-isomorphism by its regulator and the numbers

$$m_\tau(Z_1{}^3) = \frac{m_\tau(X^3)}{m_\tau(Y_1{}^3)}, \quad \tau \in T^3.$$

It is important that $X^1 \cap Y_1{}^3 = X^1 \cap Y_1{}^2$ and $X^1 \cap Z_1{}^3 = X^1 \cap Z_1{}^2$ (in particular, the possible equalities $Y_1{}^3 = Y_1{}^2$ or $Z_1{}^3 = Z_1{}^2$ do not create any problems), see (4.10).

Since $j_2 > 2$, we have a subgroup of X^2 which is

$$Z^{22} = Z_1{}^2 \oplus \Big(\bigoplus_{\tau \in T^1} \tau a_\tau^3\Big) \oplus \Big(\bigoplus_{\tau \in T^2 \setminus T^1} \tau a_\tau^3\Big)$$

and construct its K-admissible decomposition $Z^{22} = Y_2{}^2 \oplus Z_2{}^2$ with the predicted $Y_2{}^2$ and rigid summand $Z_2{}^2$ satisfying

$$m_\tau(Z_2{}^2) = \frac{m_\tau(Z^{22})}{m_\tau(Y_2{}^2)}, \quad \tau \in T^2.$$

Extending this decomposition to the group

$$Z^{23} = Z_1{}^3 \oplus \Big(\bigoplus_{\tau \in T^1} \tau a_\tau^3\Big) \oplus \Big(\bigoplus_{\tau \in T^2 \setminus T^1} \tau a_\tau^3\Big) \oplus \Big(\bigoplus_{\tau \in T^3 \setminus T^2} \tau a_\tau^3\Big),$$

we obtain $Z^{23} = Y_2{}^3 \oplus Z_2{}^3$ with predicted $Y_2{}^3$ and

$$m_\tau(Z_2{}^3) = \frac{m_\tau(X^3)}{m_\tau(Y_1{}^3) m_\tau(Y_2{}^3)}, \quad \tau \in T^3,$$

satisfying $X^1 \cap Y_2{}^3 = X^1 \cap Y_2{}^2$ and $X^1 \cap Z_2{}^3 = X^1 \cap Z_2{}^2$ (as above, the case $Y_2{}^3 = Y_2{}^2$ is obviously allowed).

Continuing this process in $(j_2 - 1)$ steps we obtain a decomposition

$$X^3 = \bigoplus_{j < j_2} Y_j{}^3 \oplus Z_{j_2}^3 \oplus G$$

with the predicted invariants of rigid groups ${Y_j}^3$ with $j < j_2$, rigid $Z^3_{j_2}$ satisfying

$$m_\tau(Z^3_{j_2}) = \frac{m_\tau(X^3)}{\prod_{j<j_2} m_\tau({Y_j}^3)}$$

and completely decomposable G with $T^2 \cap T_{\mathrm{cr}}(G) = \varnothing$. Note that by construction

$$X^2 = \bigoplus_{j<j_2} {Y_j}^2 \oplus Y^2_{j_2} \quad \text{with} \quad Y^2_{j_2} = Z^3_{j_2} \cap X^2. \tag{4.14}$$

Moreover, ${Y_j}^3 \cap X^1 = {Y_j}^2 \cap X^1$ for each $j < j_2$ and $Z^3_{j_2} \cap X^1 = {Y_{j_2}}^2 \cap X^1$. If $j_2 = j_3$, then $Z^3_{j_2} = Y^3_{j_2} = X_{j_2} \cap X^3$. If $j_2 < j_3$, then the summands ${Y_j}^3$ of X^3 with $j_2 < j \leqslant j_3$ have zero intersection with X^2 and can be obtained on the basis of the main decomposition $Z^3_{j_2} \oplus G$ as above for the group X^2, see (4.14). Finally, we have

$$X^3 = \bigoplus_{j \leqslant j_3} {Y_j}^3 \tag{4.15}$$

The main idea of the described decomposition process is that we are able to get a predicted decomposition of X^2 and then to extend this to X^3 so that the corresponding decomposition of the regulator $\bigoplus_{i \in M_1} A_i^{l_i}$ of X^1 would not be changed.

Case III. Our object now is extending the decomposition (4.15) from X^3 to X^4. Again, we start with the main decomposition

$$X^4 = X'^4 \oplus \bigoplus_{T_i \subset T^4,\, \tau \in T_i,\, n=2,\dots,l_i} \tau a^n_\tau$$

having the rigid summand $X'^4 = X' \cap X^4$.

We take the groups X^2, X^3 and X^4 instead of X^1, X^2 and X^3 and apply for them the above construction which preserves the decomposition of the regulator $\bigoplus_{i \in M_2} A_i^{l_i}$ of X^2 obtained previously.

Inductively, due to the above connection restriction we get the required decomposition of each $A_i^{l_i}$, $i \in M$, in finitely many steps. The proof is completed. □

Remark 4.7. (a) It follows from Theorem 3.3 that conditions (1)–(3) determine the direct summands B_j, $j \in J$, of group X up to near-isomorphism, see (4.5).

(b) In the proof under the natural assumption $H' = 0$ we have the equalities in the condition (3) by construction, as the rank of an arbitrary homogeneous component is equal to the number of direct summands having non-trivial intersection with this component. However, the presence of rank-one summands of H' leads to the required inequalities in the general case.

It is routine to prove the following theorem.

Theorem 4.8 (Indecomposability Criterion). *A proper* $\mathfrak{B}^{(1)}$*alr-group* B *with*

$$T_{\mathrm{cr}}(B) = \bigcup_{i \in M} T_i$$

is indecomposable if and only if it is rigid and its near-isomorphism invariants $m_\tau(B)$ *satisfy the connection condition: for any partition* $M = M_1' \cup M_2'$ *there exist* $\tau \in \bigcup_{i \in M_1'} T_i$ *and* $\sigma \in \bigcup_{j \in M_2'} T_j$ *such that* $m_\tau(B)$ *and* $m_\sigma(B)$ *are not relatively prime*

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