Number of Sylow subgroups in finite groups

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Abstract. Denote by $\nu_p(G)$ the number of Sylow p -subgroups of G . It is not difficult to see that $\nu_p(H) \leq \nu_p(G)$ for $H \leq G$, however $\nu_p(H)$ does not divide $\nu_p(G)$ in general. In this paper we reduce the question whether $\nu_p(H)$ divides $\nu_p(G)$ for every $H \leq G$ to almost simple groups. This result substantially generalizes the previous result by G. Navarro and also provides an alternative proof of Navarro's theorem.

1 Introduction

Throughout this paper, all groups are finite. Let G be a group and p a prime. Denote by $\nu_p(G)$ the number of Sylow p -subgroups of G . It is a trivial exercise to check that $\nu_p(H) \leq \nu_p(G)$ for every subgroup H of G . However $\nu_p(H)$ does not necessarily divide $\nu_p(G)$ in general. For example, let $G = A_5$ and $H = A_4$, then $\nu_3(G) = 10$ and $\nu_3(H) = 4$. In 2003, G. Navarro [6] proved that if G is p -solvable, then $\nu_p(H)$ divides $\nu_p(G)$ for every $H \leq G$.

We say that a group G satisfies $\text{DivSyl}(p)$ if $\nu_p(H)$ divides $\nu_p(G)$ for every $H \leq G$. Our goal is to prove that G satisfies $\text{DivSyl}(p)$ provided every nonabelian composition factor of G satisfies $\text{DivSyl}(p)$ (see Theorem 1.1 below). This result substantially generalizes the result by Navarro since in p -solvable groups all non-abelian composition factors evidently satisfy $\text{DivSyl}(p)$. Our technique can also be applied to derive Navarro's theorem, so we provide an alternative proof for Navarro's theorem.

In order to formulate the main theorem, we need to recall some definitions.

Let A, B, H be subgroups of G such that $B \trianglelefteq A$. Define the *normalizer* $N_H(A/B)$ of A/B in H by $N_H(A) \cap N_H(B)$. If $x \in N_H(A/B)$, then x induces an automorphism of A/B acting by $Ba \mapsto Bx^{-1}ax$. Thus there exists a homomorphism $N_H(A/B) \rightarrow \text{Aut}(A/B)$. The image of the homomorphism is denoted by $\text{Aut}_H(A/B)$ and is called the *group of H -induced automorphisms on A/B* ,

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while the kernel of the homomorphism is denoted by $C_H(A/B)$. If $B = 1$, then we simply write $\text{Aut}_H(A)$. The groups of induced automorphisms were introduced by F. Gross in [1], where the author says that this notion is taken from unpublished lectures by H. Wielandt. Clearly $C_H(A/B) = C_G(A/B) \cap H$, so

$$\begin{aligned}\text{Aut}_H(A/B) &= N_H(A/B)/C_H(A/B) \\ &\simeq N_H(A/B)C_G(A/B)/C_G(A/B) \leq \text{Aut}_G(A/B),\end{aligned}$$

that is, $\text{Aut}_H(A/B)$ can be embedded into $\text{Aut}_G(A/B)$ in a natural way. Hence, without loss of generality, we may assume that $\text{Aut}_H(A/B)$ is a subgroup of $\text{Aut}_G(A/B)$.

A composition series is called an *(rc)-series* if it is a refinement of a chief series. (This notion was introduced by V. A. Vedernikov in [11].)

Theorem 1.1 (Main theorem). *Let*

$$1 = G_0 < G_1 < \cdots < G_n = G$$

be an (rc)-series of G . Assume that, for each nonabelian G_i/G_{i-1} and for every p -subgroup P of $\text{Aut}_G(G_i/G_{i-1})$, the group $P(G_i/G_{i-1})$ satisfies $\text{DivSyl}(p)$. Then G satisfies $\text{DivSyl}(p)$.

Notice that if G_i/G_{i-1} (in the notation of the theorem) is nonabelian simple, it has a natural embedding into $\text{Aut}_G(G_i/G_{i-1})$ as a group of its inner automorphisms, and we identify G_i/G_{i-1} with a normal subgroup of $\text{Aut}_G(G_i/G_{i-1})$ of all inner automorphisms.

At the end of the paper, we discuss possible ways of extending this statement and also explain how Navarro's result can be derived from this theorem.

2 Preliminaries

We write $H \leq G$ and $H \trianglelefteq G$ if H is a subgroup of G and H is a normal subgroup of G , respectively, and p denotes a prime number. A natural number n is called a p -number (respectively a p' -number) if it is a power of p (respectively if it is coprime to p). We denote by n_p and $n_{p'}$ the p - and p' -part of the natural number n . A group is said to be a p -group (respectively a p' -group) if its order is a p -number (respectively a p' -number). A group G is called p -solvable if it possesses a subnormal series with all sections either p - or p' -groups. If $\varphi : X \rightarrow Y$ is a map, then $\varphi(x)$ denotes the image of $x \in X$, while for $\sigma \in \text{Sym}_k$ we write $i\sigma$ for the image of $i \in \{1, \dots, k\}$ under σ . By $\text{Syl}_p(G)$ we denote the set of all

Sylow p -subgroups of G . All unexplained notation and definitions can be found in [4].

The following elementary result follows from [2, Theorem 2.1].

Lemma 2.1. *Assume that $A \trianglelefteq G$ and $P \in \text{Syl}_p(G)$. Then*

$$v_p(G) = v_p(G/A) \cdot v_p(PA)$$

for some (hence for every) $P \in \text{Syl}_p(G)$, and $v_p(PA) = |A : N_A(P)|$.

Lemma 2.2. *Assume that $A \trianglelefteq G$ and both AP and G/A satisfy $\text{DivSyl}(p)$. Then G satisfies $\text{DivSyl}(p)$.*

Proof. Let $H \leq G$. Choose $Q \in \text{Syl}_p(G)$ so that $Q \cap H \in \text{Syl}_p(H)$. Since PA and QA are conjugate, QA satisfies $\text{DivSyl}(p)$. Now by Lemma 2.1,

$$v_p(G) = v_p(G/A) \cdot v_p(QA),$$

and

$$\begin{aligned} v_p(H) &= v_p(H/(H \cap A)) \cdot v_p((Q \cap H)(H \cap A)) \\ &= v_p(HA/A) \cdot v_p((Q \cap H)(H \cap A)). \end{aligned}$$

Since $HA/A \leq G/A$ and $(Q \cap H)(H \cap A) \leq QA$, the conditions of the lemma imply $v_p(HA/A)$ divides $v_p(G/A)$ and $v_p((Q \cap H)(H \cap A))$ divides $v_p(QA)$. This implies that $v_p(H)$ divides $v_p(G)$. $\square$

The following lemma is proven in [6], but now it can be easily derived from the above two lemmas and we provide here an alternative proof of it.

Lemma 2.3 ([6, Theorem A]). *Let H be a subgroup of a p -solvable group G . Then $v_p(H)$ divides $v_p(G)$.*

Proof. We argue by induction on $|G|$. Let A be a minimal normal subgroup of G and $H \leq G$. Let $P \in \text{Syl}_p(G)$ with $P \cap H \leq \text{Syl}_p(H)$. By Lemma 2.1, we have

$$v_p(G) = v_p(G/A) \cdot v_p(PA)$$

and

$$v_p(H) = v_p(HA/A) \cdot v_p((H \cap P)(H \cap A)).$$

By induction, $v_p(HA/A)$ divides $v_p(G/A)$. Since G is p -solvable, A is either a p -group or a p' -group.

If A is a p -group, then $v_p((H \cap P)(H \cap A)) = 1$, so $v_p(H)$ divides $v_p(G)$.

If A is a p' -group, then $|A|$ and $|P|$ are coprime, in particular $P \cap A = 1$. Since P normalizes A , we have that $[N_A(P), P] \leq A \cap P = 1$, so $N_A(P) = C_A(P)$. The known properties of coprime action (see, for example, [4, Exercise 3E.4]) and Lemma 2.1 imply that $v_p((H \cap P)(H \cap A)) = |(H \cap A) : C_{H \cap A}(H \cap P)|$ divides $|A : C_A(H \cap P)|$, while $|A : C_A(H \cap P)|$ divides $|A : C_A(P)| = v_p(PA)$. Thus $v_p(H)$ divides $v_p(G)$. $\square$

Recall that a group A is said to be an almost simple group if there exists a simple nonabelian group S such that $S \leq A \leq \text{Aut}(S)$. By the Schreier conjecture $\text{Aut}(S)/S$ is solvable for each simple group S .

Corollary 2.4. *Let A be an almost simple group with simple socle S and $P \in \text{Syl}_p(A)$. If PS satisfies $\text{DivSyl}(p)$, then A satisfies $\text{DivSyl}(p)$.*

Proof. This follows from Lemmas 2.2 and 2.3 since A/S is solvable. $\square$

Lemma 2.5 ([9, Theorem 1]). *Let*

$$1 = G_0 < G_1 < \cdots < G_n = G$$

be an (rc)-series of G and denote the section G_i/G_{i-1} by S_i . Let

$$1 = H_0 < H_1 < \cdots < H_n = G$$

be a composition series of G . Then there exist a permutation $\sigma \in \text{Sym}_n$ such that $\text{Aut}_G(H_i/H_{i-1}) \leq \text{Aut}_G(S_{i\sigma})$ for $i = 1, \dots, n$. Moreover, if the second series is an (rc)-series, then σ can be chosen so that $\text{Aut}_G(H_i/H_{i-1}) \simeq \text{Aut}_G(S_{i\sigma})$ for $i = 1, \dots, n$.

The following simple lemma plays a technical role.

Lemma 2.6. *Let G be a finite group, possessing a normal series*

$$1 \triangleleft H \triangleleft K \triangleleft G$$

such that H is a p -group, K/H is a p' -group and G/K is a p -group. Let Q be a p -subgroup of G containing H and let P be a Sylow p -subgroup of G containing Q . Consider a natural homomorphism

$$\overline{} : G \rightarrow \overline{G} = G/H.$$

Then $\overline{N_K(Q)} = N_{\overline{K}}(\overline{Q}) = C_{\overline{K}}(\overline{Q})$ and $N_K(P) \leq N_K(Q)$.

Proof. $\overline{N_K(Q)} \subseteq N_{\overline{K}}(\overline{Q})$ is evident. Consider $\bar{x} \in N_{\overline{K}}(\overline{Q})$. Then every inverse image x of $\bar{x}$ normalizes the full inverse image of $\overline{Q}$. Since $H \leq Q$, the full inverse image of $\overline{Q}$ equals Q , so x normalizes Q . Consequently, $N_{\overline{K}}(\overline{Q}) \subseteq \overline{N_K(Q)}$. By the conditions of the lemma, $\overline{K} = O_{p'}(\overline{G})$, so $\overline{K} \trianglelefteq \overline{G}$ and $|\overline{K}|$ and $|\overline{Q}|$ are co-prime. Thus $N_{\overline{K}}(\overline{Q}) = C_{\overline{K}}(\overline{Q})$. Since $H \leq P$ and P is a p -group, the identity

$$\overline{N_K(P)} = N_{\overline{K}}(\overline{P}) = C_{\overline{K}}(\overline{P})$$

holds. Clearly $C_{\overline{K}}(\overline{P}) \leq C_{\overline{K}}(\overline{Q})$. It follows that $N_K(P) \leq N_K(Q)$. $\square$

In this paper we often consider subgroups of a permutation wreath product, so we fix notation for such groups. If L is a group and K is a subgroup of the symmetric group Sym_k , then $L \wr K$ denotes the permutation wreath product. By definition $L \wr K$ is isomorphic to a semidirect product $(L_1 \times \cdots \times L_k) \rtimes K$, where $L_1 \simeq \cdots \simeq L_k \simeq L$ and K permutes the L_i 's, that is, an arbitrary $\sigma \in K$ acts by

$$\sigma : (g_1, \dots, g_k) \mapsto (g_{1\sigma^{-1}}, \dots, g_{k\sigma^{-1}})$$

for $(g_1, \dots, g_k) \in L_1 \times \cdots \times L_k$. We always denote $L_1 \times \cdots \times L_k$ by $\mathbf{L}$.

Suppose that G is a subgroup of $L \wr K$. Let $\rho : G \rightarrow \text{Sym}_k$ be the homomorphism corresponding to the action of G on $\{L_1, \dots, L_k\}$ by conjugation. Each element g of G as an element from $L \wr K$ can be uniquely written as

$$g = (g_1, \dots, g_k)\rho(g),$$

where $g_i \in L_i$, and $\rho(g) \in \text{Sym}_k$ permutes the g_i 's. Given $g = (g_1, \dots, g_k)\rho(g)$ of $L \wr K$, define $\pi_i(g)$ to be equal to g_i . In particular, the restriction of π_i on $\mathbf{L}$ is the natural projection on L_i .

Now if S is a normal subgroup of L , then we can construct a normal subgroup $\mathbf{S} = S_1 \times \cdots \times S_k$ of $L \wr K$ such that

$$S_1 \simeq \cdots \simeq S_k \simeq S \quad \text{and} \quad (L \wr K)/\mathbf{S} \simeq (L/S) \wr K.$$

In order to construct $\mathbf{S}$ we need to take one representative L_j from every K -orbit on $\{L_1, \dots, L_k\}$, then in each representative we need to take $S_j \trianglelefteq L_j$ such that $S_j \simeq S$ and $L_j/S_j \simeq L/S$, and finally define $\mathbf{S}$ to be equal to the normal closure of the subgroup generated by S_j 's. From the construction it is evident that if K is transitive, then $\mathbf{S}$ is uniquely defined by the choice of $S_1 \leq L_1$.

Lemma 2.7. *Assume that T is a unique minimal normal subgroup of G . Assume also that T is nonabelian, i.e. $T = S_1 \times \cdots \times S_k$, where $S_1, \dots, S_k$ are isomorphic (moreover, G -conjugate) nonabelian simple groups. Let $\rho : G \rightarrow \text{Sym}_k$ be*

the permutation representation corresponding to the action of G on $\{S_1, \dots, S_k\}$ by conjugation. Then there exists an injective homomorphism

$$\varphi : G \rightarrow \text{Aut}_G(S_1) \wr \rho(G) = (\text{Aut}_G(S_1) \times \cdots \times \text{Aut}_G(S_k)) \rtimes \rho(G).$$

Moreover, if $H \leq G$ and $HT = G$, then φ can be chosen so that for every $H \leq X \leq G$ we have

$$\begin{aligned} \varphi(X) &\leq \text{Aut}_X(S_1) \wr \rho(X) \\ &= \text{Aut}_X(S_1) \wr \rho(G) = (\text{Aut}_X(S_1) \times \cdots \times \text{Aut}_X(S_k)) \rtimes \rho(X). \end{aligned}$$

Proof. The existence of φ is known (see, for example, [10, Lemma 2]). Hence, we only need to show that $\varphi(X) \leq \text{Aut}_X(S_1) \wr \rho(X)$. Here we provide the construction of φ and leave all technical details for the reader.

First notice that $\rho(G)$ is transitive since T is a minimal normal subgroup. By the fundamental counting principle (see [4, Theorem 1.4]), the action of G on $\{S_1, \dots, S_k\}$ is equivalent to the action of G on right cosets of $N_G(S_1)$ by right multiplication. By [3, Theorem IV.1.4 a)], there exists an injective homomorphism

$$\psi : G \rightarrow (N_G(S_1) \times \cdots \times N_G(S_k)) \rtimes \rho(G) = N_G(S_1) \wr \rho(G),$$

defined in the following way. Fix right coset representatives $r_1, \dots, r_k$ of $N_G(S_1)$ in G . Given $g \in G$, define the element $n_i(g)$ of $N_G(S_1)$ by

$$r_i g = n_i(g) r_i \rho(g).$$

Then ψ maps g to $(n_1(g), \dots, n_k(g)) \rho(g)$.

Consider a normal subgroup $C_G(S_1)$ of $N_G(S_1)$. Since $\rho(G)$ is transitive, we have that $C_G(S_1) \times \cdots \times C_G(S_k)$ is a normal subgroup. Consider a natural homomorphism

$$\theta : N_G(S_1) \wr \rho(G) \rightarrow (N_G(S_1) \wr \rho(G)) / (C_G(S_1) \times \cdots \times C_G(S_k)).$$

Since T is the unique minimal normal subgroup of G and T is nonabelian, we have $\psi(G) \cap \text{Ker}(\theta) = 1$, while $\theta(N_G(S_1) \wr \rho(G)) = \text{Aut}_G(S_1) \wr \rho(G)$ by definition.

Now assume that $H \leq G$ is chosen so that $HT = G$. Then the right coset representatives $r_1, \dots, r_k$ of $N_G(S_1)$ can be chosen from H . Hence if $H \leq X \leq G$, then for every $x \in X$ and $i = 1, \dots, k$, we have

$$n_i(x) = r_i x r_{i\rho(x)}^{-1} \in X \cap N_G(S_1) = N_X(S_1).$$

Therefore $\psi(X) \leq N_X(S_1) \wr \rho(X)$, and so

$$\varphi(X) = \theta\psi(X) \leq \text{Aut}_X(S_1) \wr \rho(X). \quad \square$$

Lemma 2.8. *Let L be a group with a normal subgroup S such that L/S is a p -group. Let K be a transitive p -subgroup of Sym_k . Choose $S_1 \leq L_1$ with $S_1 \simeq S$ and $L_1/S_1 \simeq L/S$, then $\mathbf{S} = S_1^{L \wr K} = S_1 \times \cdots \times S_k$. Let G be a subgroup of $L \wr K$ satisfying the following conditions:*

- (a) $\mathbf{S} \leq G$;
- (b) $G\mathbf{L} = L \wr K$;
- (c) $\pi_i(N_G(S_i)) = L_i$;
- (d) for some $Q \in \text{Syl}_p(G)$, if $P_i = \pi_i(N_Q(S_i))$, then $Q \leq (P_1 \times \cdots \times P_k) \rtimes K$.

Then the restriction of π_i on $N_G(S_i)$ is a homomorphism and

$$\nu_p(G) = |S|_{p'}^{k-1} \cdot \nu_p(L).$$

Proof. Since $G\mathbf{L} = L \wr K$, we have $\rho(G) = G\mathbf{L}/\mathbf{L} \simeq G/(G \cap \mathbf{L}) \simeq K$, in particular, $G/(G \cap \mathbf{L})$ is a p -group. Hence $G = Q(G \cap \mathbf{L})$ and $L \wr K = Q\mathbf{L}$, in particular $\rho(Q) = K$. Now $g = (g_1, \dots, g_k)\rho(g) \in N_{L \wr K}(S_i)$ if and only if $i\rho(g) = i$. For every $g, h \in L \wr K$ we have $\pi_i(g) \cdot \pi_i(h) = g_i \cdot h_i$. On the other hand, if we take $g, h \in N_{L \wr K}(S_i)$, then

$$\begin{aligned} \pi_i(gh) &= \pi_i((g_1, \dots, g_k)\rho(g) \cdot (h_1, \dots, h_k)\rho(h)) \\ &= \pi_i((g_1, \dots, g_k) \cdot (h_1, \dots, h_k)^{\rho(g)^{-1}} \cdot \rho(g)\rho(h)) \\ &= \pi_i((g_1, \dots, g_k) \cdot (h_{1\rho(g)}, \dots, h_i, \dots, h_{k\rho(g)}) \cdot \rho(g)\rho(h)) \\ &= \pi_i((g_1 \cdot h_{1\rho(g)}, \dots, g_i \cdot h_i, \dots, g_k \cdot h_{k\rho(g)}) \cdot \rho(g)\rho(h)) = g_i \cdot h_i, \end{aligned}$$

thus the restriction of π_i on $N_G(S_i)$ is a homomorphism.

Since L/S is a p -group and K is a p -group, we have that $(L \wr K)/\mathbf{S}$ is a p -group. Hence $G/\mathbf{S}$ is a p -group and $Q\mathbf{S} = G$. Denote $\pi_i(Q \cap \mathbf{S})$ by Q_i . Clearly $Q \cap S_i \leq Q_i$. On the other hand, since S_i is subnormal in G , $Q \cap S_i$ is a Sylow p -subgroup of S_i , so $Q \cap S_i = Q_i$. Since $\mathbf{S}$ normalizes S_i and $G = Q\mathbf{S}$, we have $N_G(S_i) = N_Q(S_i)\mathbf{S}$, therefore $\pi_i(N_G(S_i)) = P_i S_i = L_i$. But since

$$\begin{aligned} |P_i| &= |P_i/(P_i \cap S_i)| \cdot |P_i \cap S_i| \\ &= |P_i/(P_i \cap S_i)| \cdot |\pi_i(N_Q(S_i)) \cap S_i| \\ &= |P_i/(P_i \cap S_i)| \cdot |\pi_i(N_Q(S_i) \cap S_i)| \\ &= |P_i/(P_i \cap S_i)| \cdot |\pi_i(Q \cap S_i)| \\ &= |L_i/S_i| \cdot |Q_i| = |L_i|_p, \end{aligned}$$

we obtain $P_i \in \text{Syl}_p(L_i)$. Moreover, since Q as a subgroup of G permutes S_i 's, we have that Q permutes $N_Q(S_i)$'s, and so Q permutes P_i 's. This implies that Q normalizes $P_1 \times \cdots \times P_k$. Consequently, $\rho(Q) = K$ normalizes $P_1 \times \cdots \times P_k$ and the subgroup $P = (P_1 \times \cdots \times P_k) \rtimes K$ in condition (d) of the lemma is correctly defined. Since $P_i \in \text{Syl}_p(L_i)$ and K is a p -group, we also have that $P \in \text{Syl}_p(L \wr S)$. Then by condition (d), $Q \leq P$, and so $Q = P \cap G$.

Since G/S is a p -group, Lemma 2.1 implies that

$$\nu_p(G) = |\mathbf{S} : N_{\mathbf{S}}(Q)|$$

so

$$\nu_p(G) = \frac{|\mathbf{S}|_{p'}}{|N_{\mathbf{S}}(Q)|_{p'}}.$$

Since $L_1 = P_1 S_1$, by Lemma 2.1 again, we have

$$\nu_p(L_1) = |S_1 : N_{S_1}(P_1)| = \frac{|S_1|_{p'}}{|N_{S_1}(P_1)|_{p'}}. \quad (2.1)$$

If we can show that

$$|N_{\mathbf{S}}(Q)|_{p'} = |N_{S_1}(P_1)|_{p'}, \quad (2.2)$$

then (2.1) and (2.2) would imply

$$\nu_p(G) = \frac{|\mathbf{S}|_{p'}}{|N_{\mathbf{S}}(Q)|_{p'}} = \frac{|\mathbf{S}|_{p'}}{|S_1|_{p'}} \cdot \frac{|S_1|_{p'}}{|N_{S_1}(P_1)|_{p'}} = |S_1|_{p'}^{k-1} \cdot \nu_p(L_1)$$

and so the lemma follows.

Thus we only have to prove (2.2).

Denote $N_{S_i}(P_i)$ by X_i and let $\mathbf{X} = X_1 \times \cdots \times X_k$. Clearly

$$Q_i = P_i \cap S_i \trianglelefteq X_i,$$

and since Q_i is a Sylow p -subgroup of S_i , Q_i is a (normal) Sylow p -subgroup of X_i . Therefore $Q_1 \times \cdots \times Q_k = O_p(\mathbf{X}) \in \text{Syl}_p(\mathbf{X})$ and $\mathbf{X}$ is an extension of a p -group by a p' -group. We show that P normalizes $\mathbf{X}$ and that $N_{\mathbf{S}}(Q)$ lies in $\mathbf{X}$. By construction, P permutes P_i 's, so P permutes X_i 's, therefore P normalizes $X_1 \times \cdots \times X_k = \mathbf{X}$. In particular, $\rho(P) = \rho(Q) = K$ normalizes $\mathbf{X}$ and $\mathbf{X}K = X_1 \wr K$. In order to show that $N_{\mathbf{S}}(Q) \leq \mathbf{X}$, consider $x = (x_1, \dots, x_k) \in N_{\mathbf{S}}(Q)$. Since x normalizes Q and S_i , we obtain that x normalizes $N_Q(S_i)$ for $i = 1, \dots, k$. Take $y = (y_1, \dots, y_k)\rho(y) \in N_Q(S_i)$. Then $i\rho(y) = i$. Recall that

by condition (d) of the lemma $Q \leq P$, so $y_i \in P_i$ for $i = 1, \dots, k$. Since x_j centralizes y_t for $j \neq t$, we obtain

$$\begin{aligned} y^x &= (y_1^{x_1}, \dots, y_k^{x_k}) \cdot \rho(y)^x \\ &= (y_1^{x_1}, \dots, y_k^{x_k}) \cdot (x_1^{-1}, \dots, x_k^{-1}) \cdot (x_1, \dots, x_k)^{\rho(y)^{-1}} \cdot \rho(y) \\ &= (y_1^{x_1}, \dots, y_k^{x_k}) \cdot (x_1^{-1}, \dots, x_i^{-1}, \dots, x_k^{-1}) \\ &\quad \cdot (x_{1\rho(y)}, \dots, x_i, \dots, x_{k\rho(y)}) \cdot \rho(y) \\ &= (y_1^{x_1} \cdot x_1^{-1} \cdot x_{1\rho(y)}, \dots, y_i^{x_i}, \dots, y_k^{x_k} \cdot x_k^{-1} \cdot x_{k\rho(y)}) \cdot \rho(y). \end{aligned}$$

Thus $\pi_i(y^x) = y_i^{x_i} \in P_i$, i.e. x_i lies in X_i and so $N_S(Q)$ lies in $\mathbf{X}$.

Consider $\mathbf{X}P \leq L \wr K$. Since $Q_1 \times \dots \times Q_k = O_p(\mathbf{X}) \in \text{Syl}_p(\mathbf{X})$, the group $\mathbf{X}P$ has a normal series

$$1 \triangleleft Q_1 \times \dots \times Q_k = O_p(\mathbf{X}) \trianglelefteq \mathbf{X} \trianglelefteq \mathbf{X}P.$$

By Lemma 2.6, $\overline{N_{\mathbf{X}}(Q)} = N_{\overline{\mathbf{X}}}(\overline{Q}) = C_{\overline{\mathbf{X}}}(\overline{Q})$, where

$$\overline{} : P\mathbf{X} \rightarrow \overline{P\mathbf{X}} = P\mathbf{X}/(Q_1 \times \dots \times Q_k)$$

is the natural homomorphism.

Now consider the corresponding natural homomorphisms

$$\overline{} : (X_i P_i) \rightarrow \overline{X_i P_i} = (X_i P_i)/Q_i$$

(we simply restrict the above homomorphism on $X_i P_i$) for $i = 1, \dots, k$. Then each element $x = (x_1, \dots, x_k) \in \mathbf{X}$ is mapped to $\bar{x} = (\bar{x}_1, \dots, \bar{x}_k)$ and each element $y = (y_1, \dots, y_k)\rho(y) \in Q$ is mapped to $(\bar{y}_1, \dots, \bar{y}_k)\rho(y)$. Since X_i normalizes P_i , we obtain that $\bar{X}_i$ centralizes $\bar{P}_i$, i.e. $(\bar{x}_1, \dots, \bar{x}_k)$ centralizes $(\bar{y}_1, \dots, \bar{y}_k)$. So

$$\begin{aligned} \bar{x}\bar{y} &= (\bar{x}_1, \dots, \bar{x}_k)^{(\bar{y}_1, \dots, \bar{y}_k)\rho(y)} = (\bar{x}_1^{\bar{y}_1}, \dots, \bar{x}_k^{\bar{y}_k})^{\rho(y)} \\ &= (\bar{x}_1, \dots, \bar{x}_k)^{\rho(y)} = (\bar{x}_{1\rho(y)^{-1}}, \dots, \bar{x}_{k\rho(y)^{-1}}). \end{aligned}$$

Since $\rho(Q) = K \leq \text{Sym}_k$ is transitive, we obtain that

$$C_{\overline{\mathbf{X}}}(\overline{Q}) = C_{\overline{\mathbf{X}}}(\rho(Q)) = \{(\bar{x}_1, \dots, \bar{x}_k) \mid \bar{x}_1 = \dots = \bar{x}_k\}.$$

Thus

$$|N_S(Q)|_{p'} = |\overline{N_{\mathbf{X}}(Q)}| = C_{\overline{\mathbf{X}}}(\overline{Q}) = |\overline{X_1}| = |X_1|_{p'} = |N_{S_1}(P_1)|_{p'}$$

and (2.2) holds. $\square$

Lemma 2.9. Assume T is a normal subgroup of G such that $T = G_1 \times \cdots \times G_k$, and G/T is a p -group. Assume also that G acts on $\{G_1, \dots, G_k\}$ by conjugation, i.e. for every $x \in G$ and $i = 1, \dots, k$ there exists $j \in \{1, \dots, k\}$ with $G_i^x = G_j$. Denote by π_i the natural projection $\pi_i : T \rightarrow G_i$. Let H be a subgroup of G such that for every $i = 1, \dots, k$ we have that $\pi_i(H \cap T)$ equals either G_i or 1, that is, $H \cap T$ is a subdirect product of some of G_i 's. Then $v_p(H)$ divides $v_p(G)$.

Proof. Suppose that this lemma is false and consider a counterexample (G, H) with $|G|$ minimal. Then $H \leq G$ is such that $H \cap T$ is a subdirect product of some of G_i 's and $v_p(H)$ does not divide $v_p(G)$.

Let $\Omega_1, \dots, \Omega_s$ be G -orbits of $\{G_1, \dots, G_k\}$ and let G_{i_j} be a fixed element of Ω_j . In order to simplify notation, we assume $G_{i_1} = G_1$. Assume that L_1 is a minimal characteristic subgroup of G_1 , then L_1 is a direct product of isomorphic simple groups, and either L_1 is elementary abelian or L_1 is a direct product of isomorphic nonabelian simple groups. For every $i = 2, \dots, s$, fix the corresponding characteristic subgroup L_{i_j} of G_{i_j} so that $L_1 \simeq L_{i_j}$ and $G_1/L_1 \simeq G_{i_j}/L_{i_j}$. Denote by L the normal closure of $\langle L_1, \dots, L_{i_s} \rangle$ in G , that is,

$$L = \langle L_{i_j}^x \mid x \in G, j = 1, \dots, s \rangle.$$

Evidently L is a direct product of some groups isomorphic to L_1 . In particular, L is abelian if and only if L_1 is abelian and L is a direct product of isomorphic nonabelian simple groups if and only if L_1 is a direct product of isomorphic nonabelian simple groups. Since G acts transitively by conjugation on each G -orbit, we obtain that $L = \pi_1(L) \times \cdots \times \pi_k(L)$, $\pi_1(L) \simeq \cdots \simeq \pi_k(L)$, and G also acts by conjugation on $\{\pi_1(L), \dots, \pi_k(L)\}$. Hence, if we consider the natural homomorphism

$$\overline{} : G \rightarrow \overline{G} = G/L,$$

then $\overline{T} = \overline{G}_1 \times \cdots \times \overline{G}_k = (G_1/\pi_1(L)) \times \cdots \times (G_k/\pi_k(L))$. Clearly $\overline{G}$ acts by conjugation on $\{\overline{G}_1, \dots, \overline{G}_k\}$ and $\overline{G}/\overline{T} \simeq G/T$ is a p -group. This shows that $\overline{G}$ satisfies the conditions of the lemma. We also have

$$\pi_i(\overline{H} \cap \overline{T}) = \pi_i(\overline{HL \cap T}),$$

so $\pi_i(\overline{H} \cap \overline{T}) = \overline{G}_i$ if $\pi(H \cap T) = G_i$ and $\pi_i(\overline{H} \cap \overline{T}) = 1$ if $\pi_i(H \cap T) = 1$. Thus $\overline{H}$ also satisfies the conditions of the lemma. By induction, $v_p(\overline{H})$ divides $v_p(\overline{G})$.

Let Q be a Sylow p -subgroup of H and let P be a Sylow p -subgroup of G containing Q . By Lemma 2.1,

$$v_p(H) = v_p(Q(H \cap L)) \cdot v_p(H/(H \cap L)) = v_p(Q(H \cap L)) \cdot v_p(\overline{H})$$

and

$$v_p(G) = v_p(PL) \cdot v_p(G/L) = v_p(PL) \cdot v_p(\overline{G}).$$

Thus we need to show that $v_p(Q(H \cap L))$ divides $v_p(PL)$ in order to obtain a contradiction with the choice of (G, H) .

Assume first that L is abelian. Then PL is solvable, hence $v_p(Q(H \cap L))$ divides $v_p(PL)$ by Lemma 2.3. Thus we may assume that G_1 has no characteristic abelian subgroups and so G_1 has no normal abelian subgroups.

Now assume that L is nonabelian. As we note above, L_1 is a direct product of isomorphic nonabelian simple groups. So L is also a direct product of isomorphic nonabelian simple groups, say $L = S_1 \times \cdots \times S_m$. Then G acts by conjugation on $\{S_1, \dots, S_m\}$. Denote by $\psi_j : L \rightarrow S_j$ the natural projection. If $\pi_i(H \cap T) = 1$, then clearly $\psi_j(H \cap L) = 1$ for every $S_j \leq G_i$. Assume that $\pi_i(H \cap T) = G_i$. Since $H \cap L \trianglelefteq H$, we obtain that $\pi_i(H \cap L)$ is a normal subgroup of $\pi_i(H \cap T) = G_i$. Moreover, $\pi_i(H \cap L)$ is a normal subgroup of $\pi_i(L) = \prod_{S_j \leq L_i} S_j$, so $\pi_i(H \cap L)$ is a product of some of S_j 's. Therefore, $H \cap L$ is a subdirect product of S_j 's. This shows that $(PL, Q(H \cap L))$ satisfies the conditions of the theorem. If $PL \neq G$, then by induction $v_p(Q(H \cap L))$ divides $v_p(PL)$. Thus we may assume that $PL = G$.

Assume that L is not a minimal normal subgroup in PL . Then there exists $K \trianglelefteq PL$ such that $K < L$. Since $L = S_1 \times \cdots \times S_m$ is a direct product of isomorphic simple groups, we obtain $K = S_{i_1} \times \cdots \times S_{i_t}$ for some $\{i_1, \dots, i_t\} \subseteq \{1, \dots, m\}$. Consider a natural homomorphism $\varphi : PL \rightarrow PL/K$. Clearly $(\varphi(PL), \varphi(Q(H \cap L)))$ satisfies the conditions of the lemma, so by induction $v_p(\varphi(Q(H \cap L)))$ divides $v_p(\varphi(PL))$. It is also clear that $(PK, Q(H \cap K))$ also satisfies the conditions of the lemma, so by induction $v_p(Q(H \cap K))$ divides $v_p(PK)$. By Lemma 2.1,

$$v_p(Q(H \cap L)) = v_p(\varphi(Q(H \cap L))) \cdot v_p(Q(H \cap K))$$

and

$$v_p(PL) = v_p(\varphi(PL)) \cdot v_p(PK),$$

so $v_p(Q(H \cap L))$ divides $v_p(PL)$. Hence, we may assume that L is a minimal normal subgroup of PL .

Now we have that $G = PL$, where $P \in \text{Syl}_p(G)$, $L = S_1 \times \cdots \times S_m$ is a minimal normal subgroup of G , and $S_1, \dots, S_m$ are isomorphic (even G -conjugate) nonabelian simple groups. By Lemma 2.7, there exists an embedding $\varphi : G \rightarrow \text{Aut}_G(S_1) \wr \rho(G)$, where $\rho : G \rightarrow \text{Sym}_m$ is the permutation representation corresponding to the action of G on $\{S_1, \dots, S_m\}$ by conjugation. Moreover, since $G = PL$, φ can be chosen so that for every $P \leq X \leq G$ we have

$$\varphi(X) \leq \text{Aut}_X(S_1) \wr \rho(X) = \text{Aut}_X(S_1) \wr \rho(G). \quad (2.3)$$

We show that $\varphi(G)$ as a subgroup of $\text{Aut}_G(S_1) \wr \rho(G)$ satisfies the conditions of Lemma 2.8.

Since $S_1 \times \cdots \times S_m \leq \text{Ker } \rho$, we obtain that $\rho(G) = \rho(P)$ is a transitive p -subgroup of Sym_m . Now $S_2 \times \cdots \times S_m$ centralizes S_1 , so $\text{Aut}_G(S_1)/S_1$ is a p -group. Thus $\text{Aut}_G(S_1)$ with normal subgroup S_1 plays the role of the group L with the notation of Lemma 2.8, while $\rho(G)$ is the group K with the notation of Lemma 2.8.

Since $\varphi(S_1 \times \cdots \times S_m) = S_1 \times \cdots \times S_m$, we obtain that $S_1 \times \cdots \times S_m \leq \varphi(G)$ and condition (a) of Lemma 2.8 is satisfied. Clearly

$$\varphi(G)(\text{Aut}_G(S_1) \times \cdots \times \text{Aut}_G(S_m)) = \text{Aut}_G(S_1) \wr \rho(G),$$

so condition (b) of Lemma 2.8 is satisfied. By definition, $\pi_i(N_G(S_i)) = \text{Aut}_G(S_i)$, thus condition (c) is true. Finally, by (2.3) we have

$$\varphi(P) \leq (\text{Aut}_P(S_1) \times \cdots \times \text{Aut}_P(S_m)) \rtimes \rho(P)$$

and by definition $\text{Aut}_P(S_i) = \pi_i(N_P(S_i))$, so condition (d) is also true. Thus

$$v_p(G) = |S_1|_{p'}^{m-1} \cdot v_p(\text{Aut}_G(S_1)). \quad (2.4)$$

On the other hand, by Lemma 2.1,

$$v_p(PL) = |L : N_L(P)|,$$

$$v_p(Q(H \cap L)) = |(H \cap L) : N_{H \cap L}(Q)|.$$

Assume that $H \cap L \neq L$. Since L is a direct product of nonabelian simple groups and $H \cap L$ is a subdirect product of some simple factors of L , we obtain that $|H \cap L| = |S_1|^s$ for some $s < m$. Clearly $|(H \cap L) : N_{H \cap L}(Q)|$ divides $|S_1|_{p'}^s$, since $|S_1|_{p'}^s$ is the index of a Sylow p -subgroup of $H \cap L$. Since $s \leq m - 1$, equation (2.4) implies that $v_p(H)$ divides $v_p(G)$.

Assume finally that $H \cap L = L$. Then from $Q \leq P$ we obtain $Q \cap L = P \cap L \in \text{Syl}_p(L)$. Hence the group $PN_L(P \cap L)$ has the normal series

$$1 \triangleleft (P \cap L) \triangleleft N_L(P \cap L) \triangleleft PN_L(P \cap L),$$

where $P \cap L$ is a p -group, $N_L(P \cap L)/(P \cap L)$ is a p' -group, and

$$PN_L(P \cap L)/N_L(P \cap L) \simeq P/(P \cap N_L(P \cap L))$$

is a p -group. By Lemma 2.6,

$$N_{N_L(P \cap L)}(Q) \geq N_{N_L(P \cap L)}(P).$$

But, clearly, $N_{H \cap L}(Q) = N_L(Q)$ normalizes $Q \cap L$, so lies in $N_L(P \cap L)$. Hence

$$N_{H \cap L}(Q) = N_L(Q) = N_{N_L(P \cap L)}(Q) \geq N_{N_L(P \cap L)}(P) = N_L(P),$$

and so $v_p(Q(H \cap L))$ divides $v_p(PL)$. $\square$

3 Proof of the main theorem

Assume Theorem 1.1 is false and let G be a counterexample of minimal order. We proceed in a series of steps to get a contradiction.

Step 1. If $1 \neq A \trianglelefteq G$, then G/A satisfies $\text{DivSyl}(p)$.

Indeed, by Lemma 2.5, the group of induced automorphisms does not depend on the choice of an (rc) -series. Hence we may assume that

$$1 = G_0 < G_1 < \cdots < G_n = G$$

goes through A , i.e. $A = G_i$ for some i . Consider a natural homomorphism

$$\bar{} : G \rightarrow \bar{G} = G/A.$$

Clearly

$$\bar{1} = \bar{G}_i < \bar{G}_{i+1} < \cdots < \bar{G}_n = \bar{G}$$

is an (rc) -series of $\bar{G}$. By [8, Lemma 1.2],

$$\text{Aut}_{\bar{G}}(\bar{G}_j/\bar{G}_{j-1}) \simeq \text{Aut}_G(G_j/G_{j-1})$$

for $j = i + 1, \dots, n$, so $\bar{G}$ satisfies $\text{DivSyl}(p)$ by minimal counterexample.

Step 2. The solvable radical $S(G)$ (that is the largest solvable normal subgroup of G) is trivial.

Suppose this is false. Then $G/S(G)$ satisfies $\text{DivSyl}(p)$ by Step 1. Moreover, for given $P \in \text{Syl}_p(G)$, the group $PS(G)$ satisfies $\text{DivSyl}(p)$ by Lemma 2.3. Hence G satisfies $\text{DivSyl}(p)$ by Lemma 2.2, a contradiction.

Step 3. Let $T = S_1 \times \cdots \times S_k$ be a minimal normal subgroup of G . Then G/T is a p -group. Consequently, T is the unique minimal normal subgroup of G , and S_i is a nonabelian simple group for $i = 1, \dots, k$.

Indeed, let P be a Sylow p -subgroup of G . If G/T is not a p -group, then PT is a proper subgroup of G . By Lemma 2.5, we may assume that

$$1 = G_0 < G_1 < \cdots < G_n = G$$

goes through T , that is, $T = G_k$ for some $k \in \{1, \dots, n-1\}$. Denote the image of $P \cap N_G(G_i/G_{i-1})$ in $\text{Aut}_G(G_i/G_{i-1})$ by P_i for $i = 1, \dots, k$. By definition, $P_i(G_i/G_{i-1}) = \text{Aut}_{PT}(G_i/G_{i-1}) \leq \text{Aut}_G(G_i/G_{i-1})$, and the conditions of the theorem say that for every p -subgroup Q of $P_i(G_i/G_{i-1})$ the group $Q(G_i/G_{i-1})$ satisfies $\text{DivSyl}(p)$. Hence PT satisfies the conditions of the theorem, so by minimality, PT satisfies $\text{DivSyl}(p)$. Then Step 1 and Lemma 2.2 imply that G satisfies $\text{DivSyl}(p)$, a contradiction.

Step 4. If H is a subgroup of G containing T , then $v_p(H)$ divides $v_p(G)$.

Assume that $T \leq H$. Let Q be a Sylow p -subgroup of H . Since G/T is a p -group, so is H/T , in particular, $H = QT$. Let $P \in \text{Syl}_p(G)$ be chosen so that $P \cap H = Q$. Then $P \cap T = Q \cap T \in \text{Syl}_p(T)$ is a normal subgroup of P and of Q . Denote $N_T(P \cap T)$ by K . Since $P \cap T \in \text{Syl}_p(T)$, we have $P \cap T \in \text{Syl}_p(K)$, so $K/(P \cap T)$ is a p' -group, and PK satisfies the conditions of Lemma 2.6. Clearly both $N_T(Q)$ and $N_T(P)$ lie in K , so $N_T(Q) = N_K(Q)$ and $N_T(P) = N_K(P)$. By Lemma 2.6, $N_T(Q) = N_K(Q) \geq N_K(P) = N_T(P)$. Finally, by Lemma 2.1,

$$v_p(H) = v_p(QT) = |T : N_T(Q)|,$$

$$v_p(G) = v_p(PT) = |T : N_T(P)|.$$

Thus $v_p(H)$ divides $v_p(G)$.

Denote $\text{Aut}_G(S_i)$ by L_i . Since T is a unique minimal normal subgroup and T is nonabelian, by Lemma 2.7, there exists an injective homomorphism $\varphi : G \rightarrow L_1 \wr \rho(G)$, where $\rho : G \rightarrow \text{Sym}_k$ is the permutation representation corresponding to the action of G on $\{S_1, \dots, S_k\}$ by conjugation. Moreover, since S_i is a unique minimal normal subgroup of L_i and $\rho(G)$ is transitive, we have $\mathbf{S} = S_1 \times \dots \times S_k$ is a unique minimal normal subgroup of $L \wr \rho(G)$. In particular, for every such inclusion φ the identity $\varphi(T) = \mathbf{S}$ holds. Below we identify G with its image $\varphi(G) \leq L_1 \wr \rho(G)$ and T with $\mathbf{S}$. For every $H \leq G$ denote $\pi_i(H \cap T)$ by H_i , where π_i is defined before Lemma 2.7. Clearly $H \cap T$ is a subdirect product of $H_1, \dots, H_k$.

Step 5. If $H \leq G$ is chosen so that $HT = G$, then $|H_1| = \dots = |H_k|$ and H normalizes $H_1 \times \dots \times H_k$.

Since T is a minimal normal subgroup of G and T is nonabelian, G acts transitively by conjugation on $\{S_1, \dots, S_k\}$. Since $HT = G$ and $T \leq \text{Ker}(\rho)$, we obtain $\rho(H) = \rho(G)$, in particular, for every $i = 1, \dots, k$, there exists $x_i \in H$ such that $S_1^{x_i} = S_i$.

Let $h \in H \cap T$. Then we have $h = h_1 \cdot \dots \cdot h_k$, where $h_i = \pi_i(h) \in L_i$. Since $H \cap T \trianglelefteq H$, it follows that $h^{x_i} \in H \cap T$ for $i = 1, \dots, k$. On the other hand, $h^{x_i} = h_1^{x_i} \cdot \dots \cdot h_k^{x_i}$ and this identity with the fact that x_i permutes S_i 's implies $\pi_i(h^{x_i}) = h_1^{x_i}$. Whence $H_1^{x_i} \subseteq H_i$. The same arguments show that

$$H_i^{x_i^{-1}} \subseteq H_1.$$

Hence $H_1^{x_i} = H_i$ and $|H_1| = |H_i|$ for $i = 1, \dots, k$. These arguments also show that an arbitrary $x \in H$ permutes H_i 's, so x normalizes $H_1 \times \dots \times H_k$.

Step 6. Assume $H \leq G$ is chosen so that $HT = G$.

- If $H \cap T = H_1 \times \cdots \times H_k$, then $v_p(H) = v_p(\text{Aut}_H(S_1)) \cdot |H_1|_{p'}^{k-1}$ divides $v_p(G)$.
- If $H \cap T \neq H_1 \times \cdots \times H_k$, then $v_p(H)$ divides $v_p(\text{Aut}_H(S_1)) \cdot |H_1|_{p'}^{k-1}$, and so $v_p(H)$ also divides $v_p(G)$.

Let Q be a Sylow p -subgroup of H and let $P \in \text{Syl}_p(G)$ be chosen so that $P \cap H = Q$. By Step 3, G/T is a p -group. So $H = Q(H \cap T)$ and $G = QT$. By Lemma 2.7, there exists an embedding $\varphi : G \rightarrow \text{Aut}_G(S_1) \wr \rho(G)$ such that for every $Q \leq X \leq G$ we have $\varphi(X) \leq \text{Aut}_X(S_1) \wr \rho(Q)$, so we choose the embedding φ with this property. Clearly G as a subgroup of $\text{Aut}_G(S_1) \wr \rho(G)$ with Sylow p -subgroup $P \leq \text{Aut}_P(S_1) \wr \rho(G)$ satisfies the conditions of Lemma 2.8, so $v_p(G) = |S_1|_{p'}^{k-1} \cdot v_p(\text{Aut}_G(S_1))$. If $H \cap T = H_1 \times \cdots \times H_k$, then H as a subgroup of $\text{Aut}_H(S_1) \wr \rho(H)$ with Sylow p -subgroup $Q \leq \text{Aut}_Q(S_1) \wr \rho(G)$ satisfies the conditions of Lemma 2.8, so $v_p(H) = |H_1|_{p'}^{k-1} \cdot v_p(\text{Aut}_H(S_1))$. Clearly $|\text{Aut}_H(S_1)|$ divides $|\text{Aut}_G(S_1)|$, while $v_p(\text{Aut}_H(S_1))$ divides $v_p(\text{Aut}_G(S_1))$ by the conditions of the theorem. Hence $v_p(H)$ divides $v_p(G)$.

If $H \cap T \neq H_1 \times \cdots \times H_k$, then H as a subgroup of $H(H_1 \times \cdots \times H_k)$ satisfies the conditions of Lemma 2.9. So $v_p(H)$ divides $v_p(H(H_1 \times \cdots \times H_k))$, and $v_p(H(H_1 \times \cdots \times H_k)) = |H_1|_{p'}^{k-1} \cdot v_p(\text{Aut}_H(S_1))$ divides $v_p(G)$ in view of the previous paragraph. Hence $v_p(H)$ divides $v_p(G)$.

Step 7. If $H \leq G$ and $HT \neq G$, then $v_p(H)$ divides $v_p(G)$.

Since the nonabelian composition factors of HT and G are the same, like in the proof of Step 3, we can show that HT satisfies the conditions of the theorem, so HT satisfies $\text{DivSyl}(p)$ by induction. Therefore $v_p(H)$ divides $v_p(HT)$. But by Step 4, $v_p(HT)$ divides $v_p(G)$. Consequently, $v_p(H)$ divides $v_p(G)$.

In view of Steps 4, 6 and 7, we obtain that in any case $v_p(H)$ divides $v_p(G)$, which contradicts the choice of G . This completes the proof of Theorem 1.1.

4 Remarks and questions

In this section we discuss natural questions arising in connection with the main theorem. We now can prove the following result.

Proposition 4.1. *Let G be an almost simple group with simple socle S such that p does not divide $|S|$. Then G satisfies $\text{DivSyl}(p)$.*

Proof. By Corollary 2.4, we only need to verify that PS satisfies $\text{DivSyl}(p)$ for $P \in \text{Syl}_p(G)$. Let H be a subgroup of PS . We may assume that $P \cap H$ is a Sylow p -subgroup of H . Since p does not divide $|S|$, we have $PS = S \rtimes P$, and so $H = (H \cap S) \rtimes (H \cap P)$. The known properties of coprime action (see, e.g., [4, Exercise 3E.4]) imply that

$$v_p((H \cap S) \rtimes (H \cap P)) = |(H \cap S) : C_{H \cap S}(H \cap P)|$$

divides the index $|S : C_S(H \cap P)|$, while $|S : C_S(H \cap P)|$ clearly divides $|S : C_S(P)| = v_p(PS)$. Therefore $v_p(H)$ divides $v_p(PS)$. $\square$

From Proposition 4.1 we immediately get the following remark.

Remark 4.2. The main theorem is a direct generalization of Navarro's theorem since all nonabelian composition factors of a p -solvable group are p' -groups.

Remark 4.3. In the conditions of the main theorem, we assume that $P(G_i/G_{i-1})$ satisfies $\text{DivSyl}(p)$ for every p -subgroup P of $\text{Aut}_G(G_i/G_{i-1})$. Corollary 2.4 implies that under the conditions of the main theorem every group L with

$$G_i/G_{i-1} \leq L \leq \text{Aut}_G(G_i/G_{i-1})$$

satisfies $\text{DivSyl}(p)$. However the authors do not know any example of an almost simple group L with simple socle S such that L satisfies $\text{DivSyl}(p)$, while S does not. So it is natural to assert the following:

Conjecture 4.4. *Let S be a simple group satisfying $\text{DivSyl}(p)$. Then every L such that $S \leq L \leq \text{Aut}(S)$ satisfies $\text{DivSyl}(p)$.*

Remark 4.5. The structure of the normalizers of Sylow 2-subgroups in finite simple groups (see [5]) implies that in (infinitely) many almost simple groups the Sylow 2-subgroups are self-normalizing, so these groups satisfy $\text{DivSyl}(2)$. Also in [12] the complete description of simple groups $\text{PSL}_2(q)$ satisfying $\text{DivSyl}(p)$ for every prime p is obtained. In particular, in [12] an infinite series of simple groups, satisfying $\text{DivSyl}(p)$ for every prime p is obtained, so, if a group satisfies $\text{DivSyl}(p)$ for every prime p , then it is not necessarily solvable.

Of course, other examples exist. We provide these examples just to show that there are many almost simple groups satisfying $\text{DivSyl}(p)$ that are not p' -groups.

In connection with Conjecture 4.4 and Remark 4.5, the following questions arise naturally:

Problem 4.6. How often do almost simple groups satisfy $\text{DivSyl}(p)$?

Problem 4.7. What almost simple groups satisfy $\text{DivSyl}(p)$ for every prime p ?

Remark 4.8. Other possible questions arise if we try to consider π -Hall subgroups instead of Sylow p -subgroups. In [7], A. Turull proved that if G is π -separable, then $v_\pi(H)$ divides $v_\pi(G)$ for every $H \leq G$. It is not hard to see that Lemmas 2.1–2.3 hold if we consider π -separable groups and π -Hall subgroups, so an alternative proof of Turull's theorem can be also obtained by the same arguments. But in order to extend Turull's theorem, one needs to introduce an appropriate class of finite groups so that each proper subgroup satisfies E_π at least. We do not go into details in this case, we only mention that such a possibility exists and could be considered later.

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