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On metanilpotent groups satisfying the minimal condition on normal subgroups

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Abstract. A comprehensive account is given of the theory of metanilpotent groups with the minimal condition on normal subgroups. After reviewing classical material, many new results are established relating to the Fitting subgroup, the Hirsch–Plotkin radical, the Frattini subgroup, splitting and conjugacy, the Schur multiplier, Sylow structure and the maximal subgroups. Module theoretic and homological methods are used throughout.

1 Introduction

The minimal condition on subgroups, *min*, was introduced into group theory during the 1940s. Among the early investigators was S. N. Černikov, who proved that a soluble group satisfying min is an extension of a divisible abelian group of finite rank by a finite group [6]; groups with this structure are nowadays called Černikov groups. Subsequently the property min was studied for many other classes of infinite groups.

The minimal condition for normal subgroups *min-n*, which is clearly a much weaker property, was first studied by R. Baer in a paper [3] that appeared in 1949. In the same year V. S. Čarin [5] gave the first example of a metabelian group with min-n that is not a Černikov group. Then, in 1964, Baer [4] established the fundamental fact that soluble groups with min-n are locally finite. He also observed that a nilpotent group with min-n satisfies min and indeed is a Černikov group with centre of finite index – see [16, Theorem 3.14]. However, since that time, there has been relatively little progress in research on soluble groups with min-n in general.

On the other hand, the study of metabelian groups with min-n has proved to be more amenable. In 1970, D. McDougall [14] was able to show that these groups are countable. Subsequent work of McDougall, B. Hartley and H. L. Silcock in the 1970s [8, 9, 19–21] established the basis of the theory. It was soon observed that many results obtained for metabelian groups are in fact valid for metanilpotent groups, i.e., soluble groups with nilpotent length at most 2. In investigations of this class of groups, it became apparent that artinian modules over nilpotent Černikov groups must play a prominent role. In this respect, the fundamental paper [9] of Hartley and McDougall has proved invaluable.

After the burst of activity in the 1970s, little has been done on metanilpotent groups with min-n. Nevertheless, the topic remains an attractive area for investigation, calling for the use of module theoretic and homological methods, as well as established techniques from infinite soluble group theory.

Our object in this paper is twofold. First we describe the classical theory of metanilpotent groups with min-n, providing improved proofs where possible. Then we establish many new results. In particular, we consider the structure of the Fitting subgroup and the Hirsch–Plotkin radical, showing that these are respectively nilpotent and hypercentral. It is shown that the Schur multiplier of a metanilpotent group with min-n is always finite, and in many cases is zero. It turns out that metanilpotent groups with min-n have a surprisingly satisfactory Sylow theory; indeed, in every subgroup, the Sylow π-subgroups are conjugate. Necessary and sufficient conditions are found for the Frattini subgroup to be nilpotent or locally nilpotent. It is proved that the limit of the lower central series has a nilpotent Černikov supplement with finite intersection. For groups that have no proper subgroups of finite index, there is even a splitting and conjugacy theorem. The maximal subgroups of metanilpotent groups with min-n are studied, with particular attention to their cardinality. Finally, we describe recent joint work with A. Arikan and G. Cutolo [2] on countable domination of the proper subgroups of metanilpotent groups with min-n.

In much of the theory, a direct decomposition of artinian modules over locally finite groups due to Hartley and McDougall plays a prominent role in non-modular situations. In addition, we make use of a near direct decomposition established in [2], which is valid in the modular case.

It is hoped that this article will help to stimulate research in a very fertile area of infinite soluble group theory.

Notation

- G^{ab}: the abelianization
- $G^{(i)}$: a term of the derived series
- $A[n]$: the subgroup of elements with orders dividing n in an abelian group
- $\pi(G)$: the set of primes dividing orders of elements of G
- $\mathrm{Syl}_\pi(G)$: the set of Sylow π-subgroups of G
- $\gamma_i(G)$, $\gamma_\infty(G)$: terms of the lower central series
- $Z_i(G)$: a term of the upper central series

- $Z(G)$ and $\bar{Z}(G)$: the centre and hypercentre
- Fitt(G): the Fitting subgroup
- HP(G): the Hirsch–Plotkin radical
- $A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n$: the near direct sum of submodules A_i
- Der(G, A), Inn(G, A): groups of derivations and inner derivations
- A^Q, A_Q: the set of Q-fixed points and the Q-trivialization of a Q-module A

Modules are understood to be *right* modules.

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2 Basic results

We begin with a fundamental theorem of Baer, which is still the only general result known for soluble groups with min-n.

Theorem 2.1 (Baer [4]). *A soluble group with min-n is locally finite.*

The proof depends on the simple result about automorphism groups that follows; for a short proof, see [12, 1.5.3].

Lemma 2.2 (Baer [4]). *Let G be a locally finite group of automorphisms of an abelian group A. If A is simple as a G-module, then it is an elementary abelian p-group for some prime p.*

Proof of Theorem 2.1. Let G be a soluble group with min-n; we can assume the derived length d exceeds 1. Put $A = G^{(d-1)}$, and note that $Q = G/A$ is locally finite by induction on d. Since G has min-n, there is an ascending Q-composition series in A. If F is a factor of this series, it is a simple Q-module, and in addition, $Q/C_Q(F)$ is locally finite. It follows from Lemma 2.2 that F is elementary abelian, and hence A is periodic. Therefore, G is locally finite. □

We remark that H. Heineken and J. S. Wilson [10] have constructed a non-trivial locally soluble group with min-n that is torsion-free. From now on, we will deal almost exclusively with metanilpotent groups. The next result is fundamental.

Theorem 2.3. *A metanilpotent group with min-n is countable.*

This was proved in the metabelian case by McDougall [14], while the general result by Silcock appears in [19, Theorem C].

Proof of Theorem 2.3. Let G be a metanilpotent group with min-n, and write the final term of the lower central series as $N = \gamma_\infty(G)$; thus N and G/N are nilpotent by min-n. Also, G/N is a Černikov group, so it is countable and has a divisible abelian normal subgroup of finite index, say F/N. By a result of Wilson [23], the subgroup F also satisfies min-n. If N^{ab} is countable, then, by the well-known tensor product property of the lower central series (see [15]), the subgroup N, and hence G, is countable. Thus, factoring out by N' and replacing G by F, we can assume that G is metabelian.

Set $A = G'$, and observe that A is an artinian module over $Q = G/A$ by conjugation. Let $a \in A$, and put $I = \mathrm{Ann}_{\mathbb{Z}Q}(a)$, the annihilator of a in $\mathbb{Z}Q$; then $(a)\mathbb{Z}Q \overset{Q}{\simeq} \mathbb{Z}Q/I$ and $\mathbb{Z}Q/I$ is a commutative artinian ring. By a well-known theorem of Hopkins, $\mathbb{Z}Q/I$ is noetherian, and therefore $(a)\mathbb{Z}Q$ has finite $\mathbb{Q}$-composition length. From this, it follows that the length of the upper socle series of the module A is at most ω. Since Q is countable, simple Q-modules are countable. Thus A, and hence G, is countable. □

On the other hand, Hartley [8] has constructed uncountable soluble groups of derived length 3 which have min-n. This is the first indication that things can go wrong when we try to extend results proven for metanilpotent groups to soluble groups with derived length greater than 2.

The final result in this section is surely widely known, but it does not seem to appear explicitly in the literature.

Lemma 2.4. *Let G be a periodic nilpotent group, and let D be a subgroup of G which has no proper subgroups of finite index. Then D is divisible and $D \leq Z(G)$.*

Proof. Let c be the nilpotent class of G. If $c \leq 1$, then clearly D is divisible. Let $c > 1$, and proceed by induction on c. Then $DZ(G)/Z(G)$ is divisible and is contained in $Z_2(G)/Z(G)$. Let $g \in G$ and $d \in D$. If g has order n, we can write $d = d_1^n z$, where $d_1 \in D$ and $z \in Z(G)$. Since $[g, d_1] \in Z(G)$, we have $1 = [g^n, d_1] = [g, d_1]^n = [g, d_1^n] = [g, d]$. Therefore, $D \leq Z(G)$ and of course D is divisible. □

3 Module theoretic methods

Let G be a metanilpotent group with min-n, and let $N \lhd G$ with N and $Q = G/N$ nilpotent. Then N^{ab} is an artinian module over the nilpotent Černikov group Q. Since this module has a major influence on the group, it is essential to investigate the structure of artinian modules over nilpotent Černikov groups. Some initial information is given by the following simple result.

Lemma 3.1. *Let A be an artinian module over a group Q. Then the following hold.*

(i) *There is an expression $A = D + B$ with D and B submodules, where D is divisible and B is bounded as abelian groups. Moreover, $\pi(A)$ is finite.*

(ii) *If Q is a nilpotent Černikov group, then A is a countable periodic abelian group.*

Proof. (i) Since A is artinian, there is an integer $n > 0$ such that

$$n!A = (n+1)!A = \cdots = D,$$

say. Clearly, D is a divisible abelian group. If $a \in A$, we have $n!a = n!d$ for some $d \in D$. Thus $n!(a - d) = 0$ and $a - d \in A[n!]$, showing that $A = D + A[n!]$. Also, $\pi(A)$ is finite by the artinian property.

(ii) Form the semidirect product $G = Q \ltimes A$, observing that G is a metanilpotent group with min-n. The statement follows at once from Theorem 2.3 and Theorem 2.1. □

Module constructions

We will describe a method for constructing uniserial artinian modules over certain locally finite groups, starting from simple modules; here we follow ideas of Hartley and McDougall [9]. Let p be a prime and Q a countable centre-by-finite p'-group. Denote by $\{M_\lambda \mid \lambda \in \Lambda\}$ a complete set of non-isomorphic simple $\mathbb{Z}_p Q$-modules; thus we are in a non-modular situation. Here M_λ is an elementary abelian p-group of rank r_λ, say. Choose a divisible abelian p-group V_λ of rank r_λ, and identify M_λ with $V_\lambda[p]$, thereby endowing the latter with a Q-module structure. Since

Q is a countable locally finite p'-group, the module structure of $V_\lambda[p]$ may be extended to V_λ – see for example [9, Lemma 3.2]. The resulting $\mathbb{Z}_p Q$-module will be denoted by

$$V_\lambda(\infty).$$

It is not hard to show that the only proper submodules of $V_\lambda(\infty)$ are the

$$V_\lambda(n) = V_\lambda[p^n],$$

where $n = 0, 1, 2, \ldots$. Thus $V_\lambda(\infty)$ is a uniserial artinian Q-module, while $V_\lambda(n)$ is noetherian and artinian. In addition,

$$V_\lambda(n+1)/V_\lambda(n) \overset{Q}{\simeq} M_\lambda.$$

Keep in mind that $V_\lambda(\infty)$ is a divisible abelian p-group; in fact, it is the injective hull of M_λ, which is unique up to isomorphism. For these facts, see [9, § 2, 3].

Notice that the module $V_\lambda(\infty)$ is *p-adically irreducible*, i.e., it is a p-group that is unbounded, but every proper submodule is bounded. These observations permit the construction of uniserial artinian modules over nilpotent Černikov groups in the non-modular case since such groups are centre-by-finite.

Modules over abelian groups

In the case of abelian groups, it is easy to describe the simple modules explicitly. The following approach is well known. Let p be a prime and Q an abelian p'-group. Let $\theta\colon Q \to (\bar{\mathbb{Z}}_p)^*$ be a homomorphism, where $\bar{\mathbb{Z}}_p$ is the algebraic closure of the field of p elements and the asterisk denotes the multiplicative group. Denote by F_θ the additive subgroup of $\bar{\mathbb{Z}}_p$ which is generated by $\mathrm{Im}(\theta)$. Then F_θ is a subfield of $\bar{\mathbb{Z}}_p$, which becomes a Q-module via the action $f \cdot x = f x^\theta$, where $f \in F_\theta, x \in Q$. It is easily verified that F_θ is a simple $\mathbb{Z}_p Q$-module, and it is well known that every simple $\mathbb{Z}_p Q$-module is isomorphic with some F_θ. In addition, $F_\theta \overset{Q}{\simeq} F_\phi$ if and only if $\mathrm{Im}(\theta) = \mathrm{Im}(\phi)$ and $\theta = \phi\rho$, where $\rho \in \mathrm{Gal}(F_\theta/\mathbb{Z}_p)$ – see [9, Lemma 2.5].

Next suppose that, in addition, Q is a Černikov group. Then $G_\theta = Q \ltimes F_\theta$ is a metabelian group with min-n, and G_θ will be non-Černikov if and only if $\mathrm{Im}(\theta)$ is infinite.

For example, let π be a finite set of primes not containing the prime p. Then $\bar{\mathbb{Z}}_p$ contains primitive q^i-th roots of unity, where $i = 1, 2, \ldots$ and $q \in \pi$. Let Q be the subgroup of $(\bar{\mathbb{Z}}_p)^*$ generated by these roots; thus Q is a locally cyclic, divisible abelian π-group. Let $\theta\colon Q \to (\bar{\mathbb{Z}}_p)^*$ be inclusion. Then G_θ is a metabelian group with min-n. We will call F_θ a *Čarin* (p, π)*-module* and G_θ a *Čarin* (p, π)*-group*; in fact, when π consists of a single prime, G_θ is Čarin's original example.

The construction of uniserial artinian Q-modules given above can be applied to a simple Q-module F_θ, if Q is an abelian p'-group, to produce the modules

$$V_\theta(n), \quad V_\theta(\infty), \quad n = 0, 1, 2, \ldots.$$

Of course, $V_\theta(n+1)/V_\theta(n) \overset{Q}{\simeq} F_\theta$.

The Hartley–McDougall decomposition

The significance of the modules $V_\lambda(n)$ and $V_\lambda(\infty)$ is made clear by the following crucial result of Hartley and McDougall [9, Theorem A]. We retain the notation of the preceding discussion.

Theorem 3.2. *Let p be a prime and Q a countable centre-by-finite p'-group. Let A be an artinian Q-module which is a p-group. Then A is the direct sum of finitely many modules of types $V_\lambda(n)$, $V_\lambda(\infty)$. Moreover, the direct decomposition is unique up to an automorphism of A.*

The theorem provides information on the structure of an artinian module A over a nilpotent Černikov group Q provided that $\pi(A) \cap \pi(Q)$ is empty, i.e., we are in the non-modular case. While the Hartley–McDougall decomposition cannot be applied in modular situations, this loss is diminished by the next result.

Lemma 3.3. *If A is an artinian module over a nilpotent Černikov group Q, then $Q_p/C_{Q_p}(A_p)$ is finite for all $p \in \pi(Q)$.*

Proof. We can assume that A is a p-group and Q acts faithfully on A, so what needs to be proved is that Q_p is finite. Assume this is false, so that the maximum divisible subgroup D of Q_p satisfies $1 < D \le Z(Q)$. Suppose first of all that A is a simple Q-module, and let $d \in D$ have order p. Then

$$0 = A(d^p - 1) = A(d-1)^p$$

since A is elementary abelian. From $D \le Z(Q)$, we deduce that each $A(d-1)^i$ is a Q-submodule, and thus $A(d-1) = 0$ by the simplicity of A, a contradiction.

Reverting to the general case, we form an ascending composition series

$$\{A_\alpha \mid \alpha < \beta\} \quad \text{in the } Q\text{-module } A,$$

which is possible because A is artinian. There is a least α such that D acts non-trivially on A_α and α cannot be a limit ordinal. Hence D acts trivially on $A_{\alpha-1}$, as well as on the simple module $A_\alpha/A_{\alpha-1}$ by the previous paragraph. Therefore, there is an injective homomorphism $D/C_D(A_\alpha) \to \mathrm{Hom}(A_\alpha/A_{\alpha-1}, A_{\alpha-1})$. But the latter group is elementary abelian, while D is divisible, so it follows that $D = C_D(A_\alpha)$, a contradiction. □

Here is a first application of these methods.

Proposition 3.4. *Let A be an artinian module over a nilpotent Černikov group Q. If A is bounded as an abelian group, then it is noetherian.*

Proof. We can assume that A is a p-group and Q acts faithfully on it. Lemma 3.3 shows that $Q = P \times R$, where P is a finite p-group and R is a p'-group. Now $Q \ltimes A$ satisfies min-n, and in addition $|Q \ltimes A : R \ltimes A| = |P|$ is finite, so by Wilson's theorem $R \ltimes A$ also has min-n and A is R-artinian. Thus Theorem 3.2 is applicable and, as A is bounded, it is a direct sum of finitely many R-modules of the form $V_\lambda(n)$. Each of the latter is R-noetherian, whence it follows that A is R-noetherian and hence Q-noetherian. □

This has immediate and significant consequences.

Lemma 3.5. *Let G be a metanilpotent group with min-n. Let $N \lhd G$, where N and G/N are nilpotent. Then N' and $N/Z(N)$ have finite exponent and satisfy max-G.*

Proof. By Lemma 3.1, the abelian group N^{ab} is the direct product of a divisible subgroup and a subgroup of finite exponent. The tensor product property of the lower central series shows that $\gamma_i(N)/\gamma_{i+1}(N)$ has finite exponent for $i \geq 2$, and hence it satisfies max-G by Proposition 3.4. Since N is nilpotent, N' has finite exponent and max-G.

Next let $D = \bigcap_{n=1,2\ldots} N^{n!}$. Since G has min-n, N/D has finite exponent, and D has no proper subgroups of finite index. It follows from Lemma 2.4 that D is divisible, and $D \leq Z(N)$. Hence $N/Z(N)$ has finite exponent and satisfies max-G. □

We will also need a decomposition that is valid in the modular case, as a replacement for the direct decomposition of Hartley and McDougall. As is usual in modular situations, some level of imprecision is inevitable; specifically, direct sums are replaced by near direct sums. Here a module A is said to be the *near direct sum* of submodules A_i, $i = 1, 2, \ldots, n$, if $A = \sum_{i=1}^{n} A_i$ and each intersection $A_i \cap \sum_{j=1, j\neq i}^{n} A_j$ is bounded as an abelian group. The notation

$$A = A_1 \dotplus A_2 \dotplus \cdots \dotplus A_n$$

will be used to denote a near direct sum. The following result provides a near direct decomposition that is valid for any artinian module over a nilpotent Černikov group. It appears in a recent work of Arikan, Cutolo and Robinson [2, Proposition 5.5].

Theorem 3.6. *Let p be a prime and A an artinian Q-module that is a p-group, where Q is a nilpotent Černikov group. Then there is a near direct decomposition*

$$A = (A_1 \dotplus A_2 \dotplus \cdots \dotplus A_n) + A[p^\ell]$$

in which the A_i are p-adically irreducible Q-modules and $\ell \geq 0$.

The theorem is proved by a variation on the averaging argument in the standard proof of Maschke's theorem. We record a straightforward consequence.

Corollary 3.7. *With the notation of Theorem* 3.6*,*

$$A/A[p^k] \overset{Q}{\simeq} A_1 \oplus A_2 \oplus \cdots \oplus A_n$$

for some $k \geq 0$. If A is a divisible group, then $A \overset{Q}{\simeq} A_1 \oplus A_2 \oplus \cdots \oplus A_n$.

Remark. In Theorem 3.6, the p-adically irreducible Q-submodules A_i are artinian as $Q_{p'}$-modules since each $Q_p/C_{Q_p}(A_i)$ is finite by Lemma 3.3. Therefore, A_i is a direct sum of finitely many isomorphic uniserial injective $Q_{p'}$-modules of some type $V_\lambda(\infty)$ by Theorem 3.2.

4 Structural applications

We will now apply the methods developed in Section 3 to investigate the structure of metanilpotent groups with min-n.

Theorem 4.1. *Let G be a soluble group with min-n. Then the following hold.*

(i) *The Fitting subgroup of G is a hypercentral group.*

(ii) *If G is metanilpotent, then the Baer radical of G is nilpotent and coincides with the Fitting subgroup.*

Proof. Let $F = \text{Fitt}(G)$, the Fitting subgroup of G. Let $d > 1$ be the derived length of G, and set $A = G^{(d-1)}$. Then F/A is hypercentral by induction on d. Next use min-n to form an ascending G-composition series in A, say $\{A_\alpha \mid \alpha < \beta\}$. If N is a nilpotent normal subgroup of G, then NA_1 is nilpotent. Thus we have $[A_1, N] < A_1$ and hence $[A_1, N] = 1$, which shows that $[A_1, F] = 1$. By this argument, F centralizes every factor of the series, so that $A \leq \bar{Z}(F)$. Therefore, F is a hypercentral group.

Now assume that G is metanilpotent. Let R denote the Baer radical of G, and put $A = \gamma_\infty(G)$; thus A is nilpotent, so $A \leq R$. If R/A' is nilpotent, a well-known theorem of P. Hall shows that R is nilpotent. Thus we may assume that A is abelian.

Furthermore, if $R/A_{p'}$ is nilpotent for all $p \in \pi(A)$, then R is nilpotent since $\pi(A)$ is finite. Hence we can also assume that A is a p-group.

Next R is locally nilpotent, so it has a primary decomposition and clearly $R_{p'}$ centralizes A. Thus it remains only to prove that R_p acts nilpotently on A since R/A is nilpotent. Now A is G-artinian, so by Lemma 3.1 it has a divisible submodule D such that A/D has finite exponent. Let $x \in R$; then $\langle x \rangle$ is subnormal in G, and therefore

$$[D, \underbrace{x, \ldots, x}_{n}] = 1 \quad \text{for some } n > 0.$$

Since x has finite order and D is divisible, it follows that $[D, x] = 1$, which shows that $[D, R] = 1$. Thus we may assume that $D = 1$ and hence that A has finite exponent. From Lemma 3.3, we know that $R_p/C_{R_p}(A)$ is a finite p-group. A straightforward argument shows that R_p acts nilpotently on A. □

We can now give a detailed description of the structure of the Fitting subgroup.

Theorem 4.2. *Let F denote the Fitting subgroup of a metanilpotent group G with min-n. Then the following hold.*

(i) *There exists $S \lhd G$ containing F' such that S has finite exponent and satisfies max-G, while F/S is the direct sum of finitely many p-adically irreducible G/F-modules for various primes p.*

(ii) *If D denotes the maximum divisible subgroup of F, then $D \leq Z(F)$ and D is the direct sum of finitely many p-adically irreducible G/F-modules for various primes p, while F/D has finite exponent and satisfies max-G.*

This follows on applying Lemmas 3.5 and 2.4, and Corollary 3.7 to the subgroup F. Next we turn to the Hirsch–Plotkin radical.

Theorem 4.3. *Let G be a metanilpotent group with min-n. The Hirsch–Plotkin radical of G is then a hypercentral group, so it coincides with the hypercentral radical.*

Proof. Let $H = \mathrm{HP}(G)$, the Hirsch–Plotkin radical of G, and write $A = \gamma_\infty(G)$; thus $A \leq H$. If H/A' is locally nilpotent, then so is H by [15, Theorem 3]. Therefore, we can assume A to be abelian. Since G/A is countable and centre-by-finite, there is an ascending series $A = H_0 \leq H_1 \leq \cdots$ such that $H_i \lhd G$, H_i/A is finite and $H = \bigcup_{i=1,2,\ldots} H_i$. Let $1 \neq a \in A$; then a^{H_i} is finite and H_i is locally nilpotent, so $a^{H_i} \cap Z(H_i) \neq 1$ and hence $1 < C_i = C_A(H_i) \lhd G$. Since $C_1 \geq C_2 \geq \cdots$ and A satisfies min-G, there is an $i \geq 0$ such that $C_i = C_{i+1} = \cdots$. Hence $C_i = C_A(H) \neq 1$. By factoring out by $C_A(H)$ and repeating the argument, we generate an ascending G-invariant, H-central series in A. Since H/A is nilpotent, it follows that H is a hypercentral group. □

Notice that the Fitting subgroup and the Hirsch–Plotkin radical are in general different even for metabelian Černikov groups, as the locally dihedral 2-group shows.

The nilpotent supplementation theorem

Our next aim is to establish the existence of a nilpotent supplement for the nilpotent residual of a metanilpotent group with min-n. First we note a useful fact about the centre.

Lemma 4.4. *Let G be a metanilpotent group with min-n, and set $A = \gamma_\infty(G)$. Then $A \cap Z(G)$ is finite.*

Proof. First we recall a result of Baer [3] (see also [16, Theorem 5.22]): *the hypercentre of a group with min-n is a Černikov group.* Thus it suffices to prove that $A \cap Z(G)$ has finite exponent. Assume for the moment that A is abelian, and suppose that it is unbounded. Then Corollary 3.7 shows that A has a bounded G/A-submodule B such that A/B is the direct sum of p-adically irreducible submodules U_i/B, $i = 1, 2, \dots, r$, for various primes p. Since $A = [A, G]$, no U_i/B can be G/A-trivial, so each $C_{U_i/B}(G)$ is bounded and hence so is $C_{A/B}(G)$. Therefore, $(A \cap Z(G))B/B$ is bounded, and $A \cap Z(G)$ has finite exponent.

Returning to the general case, we know that $(A \cap Z(G))A'/A'$ has finite exponent. Since A' has finite exponent by Lemma 3.5, it follows that $A \cap Z(G)$ has finite exponent. □

Next we establish the existence of hypercentral supplements.

Lemma 4.5. *Let G be a group with min-n, and assume G has a normal nilpotent subgroup A such that G/A is hypercentral. Then there is a hypercentral Černikov subgroup X such that $G = XA$.*

Proof. First of all suppose that A is abelian, and write $A_0 = A \cap \bar{Z}(G)$. Since G/A is hypercentral and A is an artinian G/A-module, $H^2(G/A, A/A_0) = 0$ by [17, Corollary AB]. Hence there is a subgroup X such that $G = XA$ and $X \cap A = A_0$. Now $X/A_0 \simeq G/A$, so X/A_0 is hypercentral and Černikov. Therefore, X is hypercentral. Since $A_0 \leq \bar{Z}(G)$ and $\bar{Z}(G)$ is Černikov, it follows that X is Černikov.

Now let A have nilpotent class $c > 1$. By induction on c, there exists $Y \leq G$ such that $G = YA$ and $Y/Z(A)$ is a hypercentral Černikov group. Next $Z(A)$ is an artinian Y-module since $G = YA$, so by the abelian case we have $Y = XZ(A)$, where X is a hypercentral Černikov group. Thus $G = YA = XA$, and the result is proven. □

We come now to the principal supplementation theorem.

Theorem 4.6. *Let G be a metanilpotent group with min-n, and write $A = \gamma_\infty(G)$. Then there is a nilpotent Černikov subgroup X such that $G = XA$ and $X \cap A$ is finite.*

Proof. By Lemma 4.5, there is a hypercentral Černikov subgroup Y such that $G = YA$. Let D denote the maximum divisible subgroup of $Y \cap A$, so $(Y \cap A)/D$ is finite. Now $D \lhd Y$ because D is characteristic in $Y \cap A$. Since Y satisfies min, there is an $r > 0$ such that $[D,_r Y] = [D,_{r+1} Y] = \cdots = C$ say. Then Y/C is nilpotent since Y is hypercentral, $(Y \cap A)/D$ is finite and $Y/Y \cap A$ is nilpotent. Also, $C = [C, Y]$ is divisible. Therefore, by [17, Corollary CD], the subgroup Y nearly splits over C, i.e., $Y = XC$, where $X \cap C$ is finite. Hence $G = YA = XA$ since $C \leq A$. In addition, $X \cap C \leq Z_s(X)$ for some $s > 0$ since $X \cap C \lhd X$ and X is hypercentral. Finally, $X/X \cap C \simeq Y/C$, which is nilpotent. Therefore, X is nilpotent; of course, it is also a Černikov group.

It remains to prove that $X \cap A$ is finite. Let E denote the maximum divisible subgroup of $X \cap A$. Then $E \leq Z(X) \cap Z(A)$ by Lemma 2.4 since X and A are nilpotent. Consequently, $E \leq A \cap Z(G)$. Since the latter is finite by Lemma 4.4, it follows that $E = 1$ and $X \cap A$ is finite. □

In the following section, we will see that there are sharper versions of Theorem 4.6 for groups that have no proper subgroups of finite index – see Theorem 5.7.

The behaviour of the subnormal subgroups of metanilpotent groups has been studied by McCaughan and McDougal [13]. The following result is a consequence of their work.

Theorem 4.7. *The subnormal subgroups of a metanilpotent group with min-n have bounded defects.*

A corollary of this result is that *the subnormal subgroups of a metanilpotent group with min-n form a complete lattice*. Indeed, if a group has bounded subnormal defects, it is easy to see that any intersection of subnormal subgroups is subnormal and also that the union of an ascending chain of subnormals is subnormal. The latter property is known to imply that the join of an arbitrary set of subnormals is subnormal – for detailed proofs, see [18, 13.1].

Theorem 4.7 will follow from [13, Theorem B] once the following result has been established.

Lemma 4.8. *Let G be a metanilpotent group with min-n, and let D be the maximum divisible subgroup of $A = \gamma_\infty(G)$. If $D \leq E \lhd G$, then the Sylow p-subgroups of G/E are nilpotent for all $p \in \pi(G)$.*

Proof. By [7, Lemma 2.3.9], it is sufficient to prove the result for G/D; thus we can assume that $D = 1$, so that A has finite exponent. Let $P \in \mathrm{Syl}_p(G)$; then $A_p \leq P$. If P/A_p' is nilpotent, then P will be nilpotent. Thus we can assume that A_p is abelian. By Lemma 3.3, the group $P/C_P(A_p)$ is finite, so P acts nilpotently on A_p. Finally, $P/A_p \simeq PA/A$ is nilpotent, which implies that P is nilpotent. □

5 Groups without proper subgroups of finite index

If G is a metanilpotent group with min-n, it has a unique smallest subgroup of finite index, say G_0. Then $G_0 \lhd G$, and G_0 has no proper subgroups of finite index; moreover, G_0 satisfies min-n by Wilson's theorem. The subgroup G_0 is evidently of importance for the structure of G and, remarkably, it has much simpler structure than G does. First we note a simple criterion.

Lemma 5.1. *A metanilpotent group G with min-n has no proper subgroups of finite index if and only if $G/\gamma_\infty(G)$ is a divisible abelian group.*

Proof. Let $A = \gamma_\infty(G)$, and assume that G/A is divisible abelian. Let $H \lhd G$ have finite index, and put $B = A'(A \cap H)$. Then A/B is finite, so $G/C_G(A/B)$ is finite and divisible. Therefore, A/B is central in G. Since $A = [A, G]$, it follows that $A = B$, and hence $A \leq H$. Thus $H = G$, which proves the sufficiency of the condition, while necessity is evident. □

The next result provides some basic information on the structure of the groups under consideration.

Lemma 5.2. *Let G be a metanilpotent group with min-n which has no proper subgroups of finite index. Then*

(i) $Z(G) = \bar{Z}(G)$*;*

(ii) $\gamma_\infty(G) = G'$*, and $G/C_G((G'/G'')_p)$ is a divisible abelian p'-group for all $p \in \pi(G)$.*

Proof. (i) Put $Y = Z_2(G)$ and $Z = Z(G)$, noting that Y is a Černikov group. Then $\bar{G} = G/C_G(Y)$ is isomorphic with a periodic subgroup of

$$H = \mathrm{Hom}(Y/Z, Z).$$

Let D/Z denote the maximum divisible subgroup of Y/Z. If $\theta \in H$ has finite order m, then $(D/Z)^\theta = (D/Z)^{m\theta} = 1$. Hence $\bar{G}$ is isomorphic with a subgroup of $\mathrm{Hom}(Y/D, Z)$, a group which is finite since Y/D is finite and Z is Černikov. Hence $\bar{G}$ is finite, and consequently $G = C_G(Y)$. Therefore,

$$Y = Z \quad \text{and} \quad \bar{Z}(G) = Z(G).$$

(ii) Let $A = \gamma_\infty(G)$; then G/A is abelian, so $A = G'$. Since $G/C_G((A^{ab})_p)$ is divisible, it is a p'-group by Lemma 3.3. □

Next we present some module theoretic results which are essential for further progress. The first of these is surely well known.

Lemma 5.3. *Let p be a prime, Q a centre-by-finite p'-group and A a $\mathbb{Z}_p Q$-module. Then $A_Q = 0$ if and only if $A^Q = 0$.*

Proof. Assume first that $A^Q = 0$. Now we have $A = [A, Q] \oplus B$ for some Q-submodule B by a result of Kovács and Newman [11]. Thus $B \overset{Q}{\simeq} A/[A, Q]$, so that $B \leq A^Q = 0$ and $A = [A, Q]$, i.e., $A_Q = 0$. Conversely, let $A_Q = 0$; then, for some submodule C, $A = A^Q \oplus C$. Hence $A = [A, Q] \leq C$, whence $A = C$ and $A^Q = 0$. □

Of greater consequence for our purposes here is the next result. In this, all tensor products are modules via the diagonal action of the group. We follow the notation of Section 3.

Proposition 5.4. *Let p be a prime and Q a divisible abelian p'-group, and let $\theta, \phi\colon Q \to (\bar{\mathbb{Z}}_p)^*$ be homomorphisms which are not both zero. Then*

$$(V_\theta(m) \otimes V_\phi(n))_Q = 0 = (V_\theta(m) \otimes V_\phi(n))^Q$$

for all $m, n \in \mathbb{N} \cup \{\infty\}$.

Proof. Assume that $\theta \neq 0$. Write $A = V_\theta(m)$ and $B = V_\phi(n)$. Then A and B have ascending composition series $0 = A_0 \leq A_1 \leq \cdots$ and $0 = B_0 \leq B_1 \leq \cdots$ with factors that are Q-isomorphic with F_θ and F_ϕ respectively. Moreover, F_θ is not a trivial Q-module since $\theta \neq 0$. Now it is straightforward to show that the Q-module $A \otimes B$ has an ascending series whose factors are images of the modules $(A_{r+1}/A_r) \otimes (B_{s+1}/B_s)$. Therefore, in view of Lemma 5.3, it is sufficient to prove that $(F_\theta \otimes_{\mathbb{Z}_p} F_\phi)^Q = 0$.

Assume that $0 \neq z \in (F_\theta \otimes_{\mathbb{Z}_p} F_\phi)^Q$. Then z is a $\mathbb{Z}_p$-linear combination of elements of the form $f^\theta \otimes g^\phi$, where $f, g \in Q$. (Recall that F_θ is the additive subgroup of $\bar{\mathbb{Z}}_p$ generated by $\mathrm{Im}(\theta)$.) All these f, g are contained in a finite subgroup H of Q such that $H^\theta \neq 1$. Now H^θ is cyclic, say $H^\theta = \langle u^\theta \rangle$, where

$u \in H$. It follows that $f^\theta \in \langle u^\theta \rangle$ for all f involved in z, showing that z is a linear combination of elements of the form $(u^\theta)^r \otimes (g^\phi)$, where $0 \le r < |u^\theta|$, $g \in H$.

Choose a prime q which divides $m = |u^\theta| = |H^\theta|$. Hence $q \in \pi(Q)$ and $q \neq p$. Next choose elements $w_i \in Q$ such that

$$w_1^q = u, \quad w_{i+1}^q = w_i, \quad i = 1, 2, \ldots,$$

which is possible since Q is divisible. Observe that $|w_i^\theta| = mq^i$.

Let e_i denote the exponent of p modulo $|{w_i}^\theta|$; thus

$$(\mathbb{Z}_p(w_i^\theta) : \mathbb{Z}_p) = e_i \quad \text{and} \quad p^{e_i} \equiv 1 \bmod mq^i.$$

Clearly, $e_i \mid e_{i+1}$ and $e_{i+1} \mid qe_i$, so that $e_{i+1} = e_i$ or qe_i. Since $\lim_{i\to\infty} e_i = \infty$, there must exist an i such that $e_{i+1} = qe_i$; fix such an i. The elements $(w_{i+1}^\theta)^r$, $0 \le r < e_{i+1}$, are linearly independent, so they form part of a $\mathbb{Z}_p$-basis of $\bar{\mathbb{Z}}_p$, say $\mathcal{B} = \{x_s \mid s \in S\}$. Next if $0 \le r < e_i$, then

$$qr + 1 \le q(e_i - 1) + 1 = qe_i - (q-1) < qe_i = e_{i+1}.$$

Now the tensors $x_s \otimes x_t$, $s, t \in S$, form a basis $\bar{\mathcal{B}}$ of $\bar{\mathbb{Z}}_p \otimes \bar{\mathbb{Z}}_p$, and thus the $(w_{i+1}^\theta)^{qr+1} \otimes x_s$ are distinct elements of $\bar{\mathcal{B}}$ provided that $0 \le r < e_i$.

Since $\langle u^\theta \rangle \le \langle w_i^\theta \rangle$, we can write

$$z = \sum_{r,g} \ell_{r,g} ((w_i^\theta)^r \otimes (g^\phi)),$$

where $\ell_{r,g} \in \mathbb{Z}_p$, $0 \le r < e_i$ and $g \in H$. Hence

$$\begin{aligned} z \cdot w_{i+1} &= \sum_{r,g} \ell_{r,g} \big((w_i^\theta)^r w_{i+1}^\theta \otimes (g^\phi) w_{i+1}^\phi\big) \\ &= \sum_{r,g} \ell_{r,g} \big((w_{i+1}^\theta)^{qr+1} \otimes (g w_{i+1})^\phi\big) \end{aligned}$$

since $w_i^\theta = (w_{i+1}^\theta)^q$.

Consequently, $z \cdot w_{i+1}$ is a linear combination of tensors of the form

$$(w_{i+1}^\theta)^{qr+1} \otimes x_t, \quad \text{where } 0 \le r < e_i,\ t \in S.$$

These elements belong to the basis $\bar{\mathcal{B}}$. On the other hand, we can also write

$$z = \sum_{r,t} a_{r,t} \big((w_{i+1}^\theta)^{qr} \otimes x_t\big)$$

with $a_{r,t} \in \mathbb{Z}_p$, $0 \le r < e_i$, $t \in S$. Moreover, $a_{r,t} \neq 0$ for some (r,t) since $z \neq 0$. However, the element

$$(w_{i+1}^\theta)^{qr} \otimes x_t \in \bar{\mathcal{B}}$$

cannot occur in the expression for $z \cdot w_{i+1}$; for if it did, we would have

$$(w_{i+1}^{\theta})^{qr} = (w_{i+1}^{\theta})^{qr'+1}$$

for some r', whence $(w_{i+1}^{\theta})^{q(r'-r)+1} = 1$, and therefore $(u^{\theta})^{q(r'-r)+1} = 1$. But then $q \mid m \mid q(r'-r)+1$, which is impossible. As a consequence, $(w_{i+1}^{\theta})^{qr} \otimes x_t$ does not occur in $z \cdot w_{i+1}$, and thus $z \neq z \cdot w_{i+1}$, a final contradiction. □

On the basis of Proposition 5.4, we obtain an important piece of structural information.

Lemma 5.5. *Let G be a metanilpotent group with min-n, and assume that G has no proper subgroups of finite index. Write $A = \gamma_{\infty}(G) = G'$ and $Q = G/A$. Then*

$$(\gamma_i(A)/\gamma_{i+1}(A))_Q = 0 = (\gamma_i(A)/\gamma_{i+1}(A))^Q \quad \textit{for all } i.$$

Proof. We may assume that A is a p-group. Then $\bar{Q} = Q/C_Q(A^{ab})$ is a divisible abelian p'-group by Lemma 3.3. Put $F_i = \gamma_i(A)/\gamma_{i+1}(A)$, which is a $\bar{Q}$-module. Since $A = [A, G]$, we have $(A^{ab})_{\bar{Q}} = 0$. Assume inductively that $(F_i)_{\bar{Q}} = 0$. The assignment $x\gamma_{i+1}(A) \otimes yA' \mapsto [x, y]\gamma_{i+2}(A)$, where $x \in \gamma_i(A)$, $y \in A$, determines a well-defined surjective $\bar{Q}$-homomorphism from $F_i \otimes A^{ab}$ to F_{i+1}. Now Theorem 3.2 shows that A^{ab} and F_i are direct sums of finitely many uniserial artinian $\bar{Q}$-modules of the form $V_{\theta}(j)$ with non-zero θ since $\bar{Q}$ is a p'-group. Therefore, $(F_i \otimes A^{ab})_{\bar{Q}} = 0$ by Proposition 5.4, and finally $(F_{i+1})_{\bar{Q}} = 0 = (F_{i+1})^{\bar{Q}}$ by Lemma 5.3. □

Corollary 5.6. $A \cap Z(G) = 1$.

It is now possible to establish an important splitting property of metanilpotent groups with min-n which have no proper subgroups of finite index.

Theorem 5.7. *Let G be a metanilpotent group with min-n, and assume that G has no proper subgroups of finite index. Let $A = \gamma_{\infty}(G) = G'$. Then there is a divisible abelian Černikov subgroup D such that $G = DA$ and $D \cap A = 1$. Moreover, all complements of A in G are conjugate to D.*

Proof. Let A have nilpotent class $c > 0$, and write $Q = G/A$. Put $B = \gamma_c(A)$, and assume inductively that G/B splits conjugately over A/B, say $G = XA$, where $X \cap A = B \lhd G$. Thus $X/B \simeq Q$, and B is an artinian X-module. By Lemma 5.5, we have $B^X = B^Q = 0$, so that $H^1(X/B, B) = 0 = H^2(X/B, B)$ by [17, Corollary AB]. Therefore, we have $X = D \ltimes B$, where $D \simeq Q$. Also, $G = XA = DA$, and clearly $D \cap A = 1$.

Suppose next that E is another complement of A in G. Now

$$G/B = EB/B \ltimes A/B \quad \text{and also} \quad G/B = X/B \ltimes A/B.$$

Hence, by conjugacy in G/B, we conclude that EB is conjugate to X, and we may assume that in fact $X = EB$. Hence $X = E \ltimes B = D \ltimes B$, and we can conclude that E is conjugate to D since $H^1(X/B, B) = 0$. □

The foregoing theorem is a generalization of results of McDougall [14, Theorem 5.6] and Silcock [19, Theorem B]. In fact, McDougall established the result for metabelian groups, a case of particular interest since G' is the direct sum of finitely many $G/C_G(A_p)$-modules of types $V_\lambda(n)$, $V_\lambda(\infty)$, each one arising from a non-trivial simple module since $A = [A, G]$. This is effectively a classification of metabelian groups with min-n having no proper subgroups of finite index.

6 The Schur multiplier

There is a nice application of the results of the last section to Schur multipliers.

Theorem 6.1. *Let G be a metanilpotent group with min-n, and let R denote its finite residual. Then the Schur multiplier $M(G) = H_2(G)$ is finite and has exponent dividing $|G : R|^2$. In particular, if G has no proper subgroups of finite index, then $M(G) = 0$.*

In the proof, we will make use of the Lyndon–Hochschild–Serre spectral sequence for homology which is associated with a group extension.

Proof of Theorem 6.1. First of all, assume that $G = R$, i.e., G has no proper subgroups of finite index. Write $A = \gamma_\infty(G)$, $Z = Z(A)$, $Q = G/A$. Let c denote the nilpotent class of A. If $c = 0$, then G a divisible abelian Černikov group, and thus $M(G) \simeq G \wedge G = 0$. We proceed by induction on $c > 0$, so that $M(G/Z) = 0$. Note that Z_p is an artinian module over $Q(p) = Q/C_Q(Z_p)$ and that $Q(p)$ is a divisible abelian p'-group for $p \in \pi(Z)$.

Consider the homology spectral sequence for the extension $Z \hookrightarrow G \twoheadrightarrow G/Z$ with coefficients in the trivial module $\mathbb{Z}$. The relevant terms for $M(G)$ on the E^2-page are E_{20}, E_{11} and E_{02}. In the first place, $E_{20} = M(G/Z) = 0$. We have $Z^Q = 0$ by Lemma 5.5, which means that we can write $Z_p = \bigoplus_{i=1}^{r_p} L_i$, where L_i a uniserial artinian $Q(p)$-module of some type $V_\theta(n)$ with $\theta\colon Q(p) \to (\bar{\mathbb{Z}}_p)^*$ non-zero.

Next, by the Künneth formula [22, Section II, 5],

$$M(Z_p) \simeq \bigoplus_{i=1}^{r_p} M(L_i) \oplus \bigoplus_{i<j=1}^{r_p} (L_i \otimes L_j).$$

Now $M(L_i) \simeq L_i \wedge L_i$, which is a quotient of $L_i \otimes L_i$. Therefore, Proposition 5.4

applies to show that $(M(Z_p))_Q = 0$ for all $p \in \pi(Z)$, and since

$$M(Z) \simeq \bigoplus_{p\in\pi(Z)} M(Z_p),$$

it follows that $E_{02} = (M(Z))_Q = 0$.

Finally, consider $E_{11} = H_1(G/Z, Z)$. Using the spectral sequence for the extension $A/Z \hookrightarrow G/Z \twoheadrightarrow Q$, we find from the E^2-page for this extension that $E_{10} = H_1(Q, Z) = 0$ since $Z^Q = 0$ and Q is nilpotent. Moreover,

$$E_{01} = (H_1(A/Z, Z))_Q \simeq (Z \otimes (A/Z)^{ab})_Q.$$

By Lemma 5.5, both $(A/Z)^{ab}$ and Z are direct sums of finitely many $V_\theta(n)$'s with non-zero θ. Thus, arguing as above, using Proposition 5.4, we obtain $E_{01} = 0$. Therefore, for the first extension, we have $E_{11} = 0$ and hence $M(G) = 0$.

We return to the general case and apply the spectral sequence associated with the extension $R \hookrightarrow G \twoheadrightarrow G/R$. We have $E_{02} = 0$ because $M(R) = 0$ by the first part of the proof. Further, $E_{20} = M(G/R)$ is finite with exponent dividing $e = |G : R|$. Finally, consider $E_{11} = H_1(G/R, R^{ab})$; this has finite exponent dividing e. But, in addition, R^{ab} is a Černikov group since R has min-n, and it is a routine argument to show that $H_1(G/R, R^{ab})$ is actually finite. It follows that $M(G)$ is finite with exponent dividing e^2. □

For example, the original example of Čarin has zero multiplier, as does the group obtained by using the corresponding injective module.

Corollary 6.2. *A metanilpotent group G with min-n has finitely many covering groups.*

For the isomorphism classes of covering groups of G correspond bijectively to the elements of Ext$(G^{ab}, M(G))$ by [22, V, Proposition 5.3], and this is clearly finite.

7 Sylow properties

Metanilpotent groups with min-n, unlike periodic metabelian groups in general, have excellent Sylow properties. The standard reference for the Sylow theory of locally finite groups is Dixon [7], and we follow the terminology used there. If π is a non-empty set of primes, a *Sylow π-subgroup* of a group is a maximal π-subgroup. A group is said to be *Sylow π-connected* if all its Sylow π-subgroups are conjugate and *Sylow π-integrated* if every subgroup is Sylow π-connected. The following result was established by McDougall [14] in the case of metabelian groups.

Theorem 7.1. *A metanilpotent group with min-n is Sylow π-integrated for every non-empty set of primes π.*

Proof. Let G be a metanilpotent group with min-n, and write $A = \gamma_\infty(G)$. It is sufficient to prove that G/A_π is Sylow π-integrated. For suppose this is true. Let $S \leq G$, and let $P, Q \in \mathrm{Syl}_\pi(S)$. Then $S \cap A_\pi \leq P \cap Q$ since $S \cap A_\pi \lhd S$. By assumption, $S/S \cap A_\pi$ is Sylow π-connected. Hence $P/S \cap A_\pi$ and $Q/S \cap A_\pi$ are conjugate in S, whence so are P and Q. From now on, we will assume that $A_\pi = 1$, so A is a π'-group.

Since G/A is a Černikov group, there exists $A \leq H \lhd G$ such that G/H is finite and H/A is abelian. Assume it has been shown that H is Sylow π-integrated, and let $S \leq G$. Then $S \cap H$ is Sylow π-connected, while $S/S \cap H$ is a finite soluble group and hence is Sylow π-connected by the well-known theorem of P. Hall. If $P \in \mathrm{Syl}_\pi(S \cap H)$, then $P \cap A = 1$, so $P \simeq PA/A$ and P is nilpotent. It now follows via [7, Theorem 2.4.4] that S is Sylow π-connected; hence G is Sylow π-integrated. This observation allows us to assume that G/A is abelian.

Let $c > 0$ be the nilpotent class of A, and write $Z = Z(A)$; then G/Z is Sylow π-integrated by induction on c. Let $S \leq G$, and write $Y = S \cap Z$; then S/Y is Sylow π-connected. Choose a Sylow π-subgroup of S/Y; by [7, Lemma 2.3.9], this has the form PY/Y for some $P \in \mathrm{Syl}_\pi(S)$, which will be fixed from now on. Let Q be any Sylow π-subgroup of S. Now $(QY/Y)^s \leq PY/Y$ for some $s \in S$, and hence $Q^s \leq PY$. On replacing Q by Q^s, we obtain $Q \leq PY$ and $QY = (P \cap (QY))Y$.

Since G/A is countable and abelian, there is an ascending chain

$$A = F_0 \leq F_1 \leq \cdots$$

of normal subgroups of G such that $QA = \bigcup_{i=1,2,\ldots} F_i$ and F_i/A is finite. Then $C_Z(F_i) \lhd G$. Now Z is an artinian G-module, so there exists $i \geq 0$ such that $C_Z(F_i) = C_Z(F_{i+1}) = \cdots$, and hence $C_Z(F_i) = C_Z(QA) = C_Z(Q)$. Moreover, $F_i = (F_i \cap Q)A$, which shows that $F_i = FA$, where F is some finite subgroup of $F_i \cap Q$. It follows that $C_Z(Q) = C_Z(F)$ and, on intersecting with S, we obtain $C_Y(Q) = C_Y(F)$. Apply [7, Lemma 2.4.6] to the semidirect product

$$QY = (P \cap (QY))Y,$$

keeping in mind that A is a π'-group. Hence all complements of Y in QY are conjugate, so $Q^x = P \cap (QY) \leq P$ for some $x \in QY \leq S$. Since $Q^x \in \mathrm{Syl}_\pi(S)$, we deduce that $Q^x = P$. Hence S is Sylow π-connected and G is Sylow π-integrated. □

On the other hand, as Dixon [7, Example 4.5.2] has observed, there are soluble groups with min-n of derived length 3 which are not Sylow p-connected for some prime p.

Corollary 7.2. *A metanilpotent group with min-n has countably many Sylow π-subgroups for all π.*

This is a consequence of Theorem 7.1 and the countability of the groups concerned. The next result, which generalizes work of Silcock [19, Theorem A], gives information on the structure of the Sylow p-subgroups in situations where there are no proper subgroups of finite index.

Corollary 7.3. *Let G be a metanilpotent group with min-n which has no proper subgroups of finite index. Then the following hold.*

(i) *Each Sylow p-subgroup of G is the direct product of a divisible abelian p-group of finite rank and $(G')_p$.*

(ii) *For $p \in \pi(G')$, the Sylow p-subgroups of G are nilpotent with the same class as $(G')_p$.*

Proof. By Theorem 5.7, we can write $G = D \ltimes A$, where $A = G'$ and D is a divisible abelian group of finite rank. Clearly, $P = D_p A_p$ is a Sylow p-subgroup of G. By Lemma 3.3, the subgroup D_p centralizes $(A_p)^{ab}$, so $P/(A_p)'$, and hence P, is nilpotent. Therefore, $D_p \leq Z(P)$ by Lemma 2.4; also, $P = D_p \times A_p$ and $\gamma_i(P) = \gamma_i(A_p)$ for $i \geq 2$. Hence P and A_p have the same nilpotent class for $p \in \pi(A)$. All the Sylow p-subgroups have this form since they are conjugate in G. □

In particular, the Sylow p-subgroups of a metabelian group with min-n and no proper subgroups of finite index are abelian, which is a result of McDougall [14, Theorem 3.2].

8 Frattini subgroups

It is easy to find metanilpotent groups with min-n which have non-nilpotent Frattini subgroups, even in the Černikov case.

(i) Let $G = \langle x \rangle \ltimes (A \oplus A)$, where A is a 2^∞-group, $x^4 = 1$ and

$$(a_1, a_2)x = (a_2, -a_1), \quad a_i \in A;$$

here $\phi(G) = \langle x^2, A \oplus A \rangle$, which is not nilpotent.

(ii) Let p, q be distinct primes, and let A be an injective uniserial Q-module arising from a simple module of Čarin type (p, q), where Q is a q^∞-group. Let $G = Q \ltimes A$; in this case, $\phi(G) = G$, which is not even locally nilpotent.

These examples suggest that the obstruction to nilpotence of the Frattini subgroup of a metanilpotent group G with min-n may lie in the divisible part of $(\gamma_\infty(G))^{ab}$. This is confirmed by the following theorem.

Theorem 8.1. *Let G be a metanilpotent group with min-n, and put $A = \gamma_\infty(G)$. Let D/A' denote the maximum divisible subgroup of A^{ab}, and let N be a normal subgroup of G containing $\phi(G)$. Then*

(i) *N is locally nilpotent if and only if $N/\phi(G)$ is locally nilpotent and N/D acts hypercentrally on D/A',*

(ii) *N is nilpotent if and only if $N/\phi(G)$ is nilpotent and N/D centralizes D/A'.*

Since A is nilpotent, $A' \leq \phi(G)$. Also, if M is a maximal subgroup of G that does not contain A, then $A/A \cap M$ is a simple G-module and hence is elementary abelian. Thus $D \leq A \cap M$, so that we have $A' \leq D \leq \phi(G) \leq N$ and $\phi(G/D) = \phi(G)/D$. We can take N to be $\phi(G)$ in Theorem 8.1, thus obtaining the next result.

Corollary 8.2. *The following statements hold.*

(i) *$\phi(G)$ is locally nilpotent if and only if $\phi(G/D)$ acts hypercentrally on D/A'.*

(ii) *$\phi(G)$ is nilpotent if and only if $\phi(G/D)$ centralizes D/A'.*

Recall the well-known theorem of W. Gaschütz that, for any finite group H,

$$\mathrm{Fitt}(H/\phi(H)) = \mathrm{Fitt}(H)/\phi(H).$$

There is an analogue of this result for metanilpotent groups with min-n; in what follows, we maintain the notation of the theorem.

Corollary 8.3. *The following statements hold.*

(i) *Assume that $\phi(G)$ is locally nilpotent, and put $L/\phi(G) = \mathrm{HP}(G/\phi(G))$. Then $\mathrm{HP}(G/\phi(G)) = \mathrm{HP}(G)/\phi(G)$ if and only if L/D acts hypercentrally on D/A'.*

(ii) *Assume that $\phi(G)$ is nilpotent, and put $M/\phi(G) = \mathrm{Fitt}(G/\phi(G))$. Then $\mathrm{Fitt}(G/\phi(G)) = \mathrm{Fitt}(G)/\phi(G)$ if and only if M/D centralizes D/A'.*

To prove Corollary 8.3, take N to be successively L and M in Theorem 8.1.

Proof of Theorem 8.1. (i) In the first place, if N is locally nilpotent, it is hypercentral by Theorem 4.3. Thus the conditions in (i) are necessary. Assume that N satisfies the conditions, and observe that these are inherited by quotients of G. If the result has been proved for G/A', then NA'/A' is locally nilpotent, which implies that NA/A' is locally nilpotent by the Hirsch–Plotkin theorem; hence N is locally nilpotent ([15, Theorem 3]). Therefore, we can assume that A is abelian. We may also assume that A is a p-group; for if $NA_{p'}/A_{p'}$ is locally nilpotent for all $p \in \pi(A)$, then N is locally nilpotent.

Put $A_0 = A \cap \bar{Z}(G)$; if NA_0/A_0 is locally nilpotent, then N is locally nilpotent, an observation that allows us to assume that $A \cap \bar{Z}(G) = 1$. Thus, by [17, Corollary AB], $H^2(G/A, A) = 0$, and hence G splits over A, say $G = Q \ltimes A$, where Q is a nilpotent Černikov group. Write $Q = P \times R$, where P and R are the p- and p'-components of Q. Using bars to denote quotients modulo $C_Q(A)$, we have $\bar{Q} = \bar{P} \times \bar{R}$, and $\bar{P}$ is a finite p-group by Lemma 3.3.

For the moment, suppose that $A^p = 1$. The $\mathbb{Z}_p\bar{R}$-module $A/[A, P]$ is artinian, and hence it is the direct sum of finitely many simple $\mathbb{Z}_p\bar{R}$-modules $B_i/[A, P]$, $i = 1, 2, \ldots, r$ (by Theorem 3.2, for example). Put $C_i = \sum_{j=1,2\ldots,r, j\neq i} B_j$. Then A/C_i is a simple module, and thus QC_i is a maximal subgroup of G. Therefore,

$$\phi(G) \leq \bigcap_{i=1}^{r} QC_i = Q \bigcap_{i=1}^{r} C_i = Q[A, P]$$

and $\phi(G) \cap A \leq [A, P]$.

Now let U be a finitely generated subgroup of N; then $U\phi(G)/\phi(G)$ is nilpotent, and $\gamma_n(U) \leq \phi(G) \cap A \leq [A, P]$ for some $n > 0$. Hence $U[A, P]/[A, P]$ is nilpotent, and $N[A, P]/[A, P]$ is locally nilpotent. Moreover, PA is nilpotent since $A^p = 1$, PA/A is nilpotent and $P/C_P(A) \simeq \bar{P}$ is a finite p-group. Since $PA \lhd QA = G$, it follows via the Hirsch–Plotkin theorem that $NPA/[A, P]$ is locally nilpotent. Also, $(PA)' = P'[A, P]$, so that $NPA/(PA)'$ is locally nilpotent, which implies that N is locally nilpotent.

Returning to the general case, from the previous discussion, we know that NA^p/A^p is locally nilpotent, and hence so is NA/A^p. This means that N acts hypercentrally on A/A^p, so it acts hypercentrally on each factor $A^{p^i}/A^{p^{i+1}}$. However, $A^{p^n} = A^{p^{n+1}} = D$ for some $n > 0$, so N acts hypercentrally on A/D. By hypothesis, N acts hypercentrally on D and hence on A. Since NA/A is nilpotent, NA is a hypercentral group, as is N, which establishes (i).

(ii) First of all, assume that N is nilpotent. Then NA is nilpotent. Since D/A' is divisible, it follows from Lemma 2.4 that N centralizes D/A'. Conversely, assume that the conditions on N in (ii) hold. As before, we may assume that A is abelian. By (i), the subgroup N – hence NA – is hypercentral, so N acts hypercentrally on A/D. Proposition 3.4 shows that the G/A-module A/D is a noetherian, from which it follows that N acts nilpotently on A/D. Since, by hypothesis, N centralizes D, it follows that N acts nilpotently on A. Therefore, N is nilpotent. □

9 Maximal subgroups

The maximal subgroups of metanilpotent groups satisfying min-*n* are objects that are worthy of study, as is suggested by results in Section 8. In particular, questions about their cardinality merit attention; one notable fact that seems to have escaped notice by earlier researchers is that there are just countably many of them.

Theorem 9.1. *A metanilpotent group with min-n has countably many maximal subgroups.*

A generalization of this result appears in [1, Theorem 7], where many other classes of groups with a countable number of maximal subgroups are identified. We include a short proof of the theorem.

Proof of Theorem 9.1. Let G be a metanilpotent group satisfying min-n, and put $A = \gamma_\infty(G)$; then A is nilpotent, so $A' \leq \phi(G)$, and we can assume that A is abelian. Since G/A is a Černikov group, it has finitely many maximal subgroups.

Assume the result is false; then there are uncountably many maximal subgroups M not containing A. For such an M, we have $G = MA$ and $M \cap A \lhd G$, while $M \cap A$ is a maximal G-submodule of A. Let B denote the intersection of all the maximal submodules of A. Since A is artinian, B is the intersection of finitely many maximal submodules, and thus A/B is noetherian. We can assume that $B = 1$. Since A is noetherian, it has only countably many submodules, so there must exist uncountably many maximal subgroups M such that $G = MA$ and $M \cap A = A_0$ is fixed. By passing to G/A_0, we may assume that $M \cap A = 1$ for all such M. Thus $G = M \ltimes A$, and A is a simple G-module.

Next $C_A(G) = 1$ since A is simple and $1 < A = [A, G]$. By [17, Corollary AB], we have $H^1(G/A, A) = 0$ and hence $\mathrm{Der}(G/A, A) = \mathrm{Inn}(G/A, A)$. Since A is countable, so is $\mathrm{Inn}(G/A, A)$. It follows that there are only countably many complements of A in G and hence countably many subgroups M. This is a contradiction. □

We consider next the stronger property – for countable groups at least – that there are finitely many conjugacy classes of maximal subgroups. As it turns out, not all metanilpotent groups with min-*n* possess this property.

Theorem 9.2. *Let G be a metanilpotent group with min-n, and write $A = \gamma_\infty(G)$ and $Q = G/A$. Then the following statements are equivalent.*

(i) *The maximal subgroups of G fall into finitely many conjugacy classes.*

(ii) *No two infinite simple Q-quotients of A^{ab} are Q-isomorphic.*

Proof. If D/A' denotes the maximum divisible subgroup of A^{ab}, then $D \leq \phi(G)$. Therefore, we can factor out by D and assume that A is an abelian group of finite exponent. Hence A has finite Q-composition length by Proposition 3.4.

(i) ⇒ (ii) Assume that (i) is valid, but nevertheless there exist distinct isomorphic infinite simple Q-quotients A/A_1 and A/A_2. Then $A = A_2 + A_1$ and $A/A_1 \cap A_2 \overset{Q}{\simeq} (A/A_1) \oplus (A/A_2)$. Thus, in order to reach a contradiction, we may assume that $A = B \oplus C$, where B, C are isomorphic infinite simple Q-modules. Observe that B and C are non-trivial as Q-modules since $A = [A, G]$. Consequently, $C_A(G) = 1$ and $H^2(Q, A) = 0$; thus $G \simeq Q \ltimes A$.

Now fix B, and consider the Q-module complements of B in A. These correspond to elements of $\mathrm{Hom}_Q(A/B, B) \simeq \mathrm{End}_Q(B)$. Let E denote the maximum divisible subgroup of the Černikov group Q. If $[B, E] = 1$, then $[A, E] = 1$, and hence B and C are finite since they are simple Q/E-modules. By this contradiction, $[B, E] \neq 1$, and hence $E/C_E(B)$ is infinite. Let $e \in E \backslash C_E(B)$; then, since $E \leq Z(Q)$, the mapping τ_e defined by $b \mapsto [b, e]$, $(b \in B)$, belongs to $\mathrm{End}_Q(B)$. Also $\tau_{e_1} = \tau_{e_2}$ implies that $e_1 e_2^{-1} \in C_E(B)$. It follows that $\mathrm{End}_Q(B)$ is infinite, which means that there are infinitely many Q-complements of B in A. If C_1 and C_2 are two such complements, then QC_i is maximal in G. Moreover, QC_1 and QC_2 are clearly not conjugate in G, which contradicts (i).

(ii) ⇒ (i) Assume that condition (ii) holds, yet G has infinitely many pairwise non-conjugate, maximal subgroups. Let A_0 denote the minimum Q-submodule with finite index in A. Then A/A_0 is finite, and G/A_0 is a Černikov group. Hence only finitely many maximal subgroups of G can contain A_0. Consequently, there is an infinite set $\mathcal{M}$ of mutually non-conjugate, maximal subgroups of G that fail to contain A_0. If $M \in \mathcal{M}$, then $G = MA$ and $A/M \cap A$ is an infinite simple Q-module. Each $A/M \cap A$ is isomorphic with a factor in a fixed Q-composition series of A, so by condition (ii) there is an infinite subset $\mathcal{M}_0$ of $\mathcal{M}$ such that $M \cap A = L$ is fixed for all $M \in \mathcal{M}_0$. Notice that A/L is a non-trivial simple Q-module; therefore, $H^1(G/A, A/L) = 0$, showing that all the complements of A/L in $G/L = M/L \ltimes A/L$ are conjugate and hence that all $M \in \mathcal{M}_0$ are conjugate; this is a contradiction. □

The second condition in Theorem 9.2 is clarified by a simple observation. In what follows, $\phi_Q(A)$ denotes the intersection of all the maximal submodules of the Q-module A.

Lemma 9.3. *Let A be an artinian module over a nilpotent Černikov group Q. Then the following conditions are equivalent.*

(i) *No two infinite simple quotient modules of A are isomorphic.*

(ii) *$A/\phi_Q(A)$ is the direct sum of finitely many non-isomorphic simple modules.*

Proof. There is no loss in assuming that $\phi_Q(A) = 0$. Also, it is straightforward to show that we can assume A to be a p-group. Thus A is an elementary abelian p-group.

(i) $\Rightarrow$ (ii) Let $\bar{Q} = Q/C_Q(A)$; then we can write $\bar{Q} = \bar{P} \times \bar{R}$, where $\bar{P}$ is a finite p-group and $\bar{R}$ is a p'-group. Let A/B be a simple $\bar{Q}$-quotient; then $\bar{P} \ltimes (A/B)$ is nilpotent, so $[A, \bar{P}] \leq B$ for all such B. Therefore, $[A, \bar{P}] = 0$ and $\bar{Q} = \bar{R}$. By Theorem 3.2, the module A is the direct sum of finitely many simple $\bar{Q}$-modules. By condition (i), the isomorphism types of the factors in the direct decomposition are different, so (ii) holds.

(ii) $\Rightarrow$ (i) Let $A = \bigoplus_{i=1}^{k} A_i$ be the given decomposition, with A_i a simple module of type λ_i. Consider a simple Q-quotient A/B of A. Then $A_j \not\leq B$ for some j and $A = A_j \oplus B$. Hence A/B is a simple module of type λ_j. Since the λ_i are all different, we must have $A_i \leq B$ for all $i \neq j$, and thus $B = \bigoplus_{i \neq j, i=1}^{k} A_i$. Hence B is determined by the isomorphism type of A/B, which establishes the validity of (i). □

A still more stringent group theoretic property is that of having finitely many maximal subgroups. It is easy to see that *a group has finitely many maximal subgroups if and only if the Frattini subgroup has finite index*. For groups in our class, the situation is clarified by the following result.

Theorem 9.4. *Let G be a metanilpotent group with min-n. Then the following conditions are equivalent.*

(i) *G has finitely many maximal subgroups.*

(ii) *$(\gamma_\infty(G))^{ab}$ has a divisible subgroup of finite index.*

(iii) *G is an extension of a divisible abelian normal subgroup by a Černikov group.*

Proof. Write $A = \gamma_\infty(G)$, noting that $A' \leq \phi(G)$.

(i) $\Rightarrow$ (ii) We may assume that A is abelian; write D for its maximum divisible subgroup. Then $D \leq \phi(G)$ and $\phi(G/D) = \phi(G)/D$. Hence $\phi(G)/D$ is nilpotent by Corollary 8.2. Moreover, as $G/\phi(G)$ is finite, $\phi(G)$ has min-n, so G/D is Černikov. Since A/D has finite exponent, it must be finite.

(ii) $\Rightarrow$ (iii) Define $E = \bigcap_{n=1,2,\ldots} A^{n!}$. Then, by Lemma 2.4, $E \leq Z(A)$, and E is divisible, while A/E has finite exponent. Thus E is the maximum divisible subgroup of A, and it follows from (ii) that A/E is finite. Therefore, G/E is a Černikov group.

(iii) $\Rightarrow$ (i) This is clear since a maximal subgroup contains the divisible abelian normal subgroup. □

Corollary 9.5. *Let G be a metanilpotent group with min-n. Then G has no maximal subgroups if and only if G has the form $D \ltimes A$, where D and A are divisible abelian groups and D is a Černikov group.*

Proof. Assume that G has no maximal subgroups. Then G has no proper subgroups of finite index, and thus $A = \gamma_\infty(G) = G'$ by Lemma 5.2. Let A_0 denote the minimum G-invariant subgroup with finite index in A. Thus G/A_0 is a Černikov group, so it is abelian, which implies that $A_0 = A$. Since A is nilpotent, it follows from Lemma 2.4 that A is abelian and divisible. The assertion is now a consequence of Theorem 5.7. The converse is obvious. □

10 Countable domination

Quite recently, there has been interest in metanilpotent groups with min-n that was motivated by research on countability restrictions on subgroup lattices. One such restriction is that of countable domination.

A group G is said to be *countably dominated* (or CD) if there is a countable set $\mathcal{S}$ of proper subgroups such that every proper subgroup of G is contained in some member of $\mathcal{S}$. If G is a CD-group, clearly it can have only countably many maximal subgroups. Thus, in the light of Theorem 9.1, it is natural to ask whether metanilpotent groups with min-n are countably dominated. This turns out to be a subtle question whose answer is also module theoretic in character. The following definitive result is established in [2, Theorem 5.1].

Theorem 10.1. *Let G be a metanilpotent group with min-n, and put $A = \gamma_\infty(G)$, $Q = G/A$. Then G is countably dominated if and only if A^{ab} has only countably many Q-submodules and the finite residual of Q is locally cyclic.*

The condition on A^{ab} in the theorem can be expressed in terms of the Q-module structure.

Theorem 10.2 ([2, Theorem 5.7]). *Let A be an artinian module over a nilpotent Černikov group Q. Then the following statements are equivalent.*

(i) *A has countably many submodules.*

(ii) *$A = (A_1 \dotplus A_2 \dotplus \cdots \dotplus A_n) + S$, where the A_i are pairwise non-near isomorphic, p-adically irreducible submodules for various primes p and S is a bounded submodule.*

Here the modules A_i, A_j are said to be *near isomorphic* if $A_i/B_i \simeq A_j/B_j$ for some bounded submodules B_i and B_j.

We conclude by illustrating these results with some examples. Let p and q be distinct primes, and let Q be a q^∞-group. Let A be a simple Q-module of Čarin type (p, q). Then the Čarin (p, q)-group $G_1 = Q \ltimes A$ is a CD-group since $\gamma_\infty(G_1) = A$. Next let B denote the injective hull of the module A, and put $G_2 = Q \ltimes B$; then $\gamma_\infty(G_2) = B$, and G_2 is also a CD-group. On the other hand, $G_3 = Q \ltimes (B \oplus B)$ is not a CD-group because $\gamma_\infty(G_3) = B \oplus B$, which is the direct sum of two isomorphic p-adically irreducible Q-modules.

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