Injective stability for odd-dimensional unitary K_1

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Abstract. In this paper, under the usual stable range condition, a decomposition theorem for the elementary subgroup is obtained, and the injective stability theorem for odd-dimensional unitary K_1 is proved.

1 Introduction

Stability results for classical K_1 , beginning with Bass [5] and Vaserstein [10] for linear K_1 , Bak and Tang [2] for Hermitian K_1 , Bak, Petrov and Tang [3] and Tang [9] for quadratic K_1 , and the author [11] for odd unitary K_1 , have been available in the literature for a long time and play an important role in the theory of classical groups. The stability theorems in [2, 3, 9, 11] are all proved under the Λ -stable range condition, which is weaker than its predecessors. The difficulty in the proof of stability results is injective stability, which is always concerned with a decomposition of the elementary subgroup. Does injective stability hold under the usual stable range condition which is weaker than the Λ -stable range condition? In [8], Sinchuk proved injective stability for unitary K_1 under the usual stable range condition. In [1], Ambily proved injective stability for K_1 of Roy's group under the usual stable range condition over a commutative ring.

Recently, Bak and Preusser [4] introduced odd-dimensional unitary groups, which are isomorphic to Petrov's odd hyperbolic unitary groups [7] such that $V_0 = R$ and contain as special cases the groups $\mathrm{GL}_{2n+1}(R)$, where R is any ring, the classical groups $O_{2n+1}(R)$ and $\mathrm{Sp}_{2n+1}(R)$, where R is any commutative ring, and further all even-dimensional unitary groups $U_{2n}(R, \Lambda)$, where (R, Λ) is any form ring (see [4, Example 15]). In this paper, using the usual stable range condition, we prove the injective stability theorem for odd-dimensional unitary K_1 , which generalizes similar results given in previous papers.

Let R be an associative ring with identity and (R, Δ) a Hermitian form ring. Denote by $U_{2n+1}(R, \Delta)$ the odd-dimensional unitary group and by $EU_{2n+1}(R, \Delta)$

its elementary subgroup. Under certain assumptions on R , $EU_{2n+1}(R, \Delta)$ is normal in $U_{2n+1}(R, \Delta)$ (see [4, Theorem 39], [7, Theorem 4], [11, Corollary 3.3], [12, Theorem 4.4]). Set

$$KU_{1,n}(R, \Delta) := U_{2n+1}(R, \Delta)/EU_{2n+1}(R, \Delta).$$

In this paper, we consider this object as a coset space. Under the embedding

$$\begin{aligned} \phi_{2n+1}^{2n+3}: U_{2n+1}(R, \Delta) &\rightarrow U_{2n+3}(R, \Delta), \\ \sigma &\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

we have the stabilization map $KU_{1,n}(R, \Delta) \rightarrow KU_{1,n+1}(R, \Delta)$. Then we can get the following injective stability theorem.

Theorem 1.1. *Let $n \geq \text{sr}(R) + 2$. Then the map*

$$KU_{1,n-1}(R, \Delta) \rightarrow KU_{1,n}(R, \Delta)$$

is injective.

The rest of the paper is organized as follows. In Section 2, we recall the necessary notation. In Section 3, we recall in detail the definitions of the odd-dimensional unitary group $U_{2n+1}(R, \Delta)$, elementary subgroup $EU_{2n+1}(R, \Delta)$, and the usual stable range condition. In Section 4, we establish a decomposition theorem for $EU_{2n+1}(R, \Delta)$ and prove our main result Theorem 1.1.

2 Notation

In this paper, denote by $\mathbb{N}$ the set of all positive integers. If $m, n \in \mathbb{N}$, the set of all $m \times n$ matrices with entries in the ring R is denoted by $M_{m \times n}(R)$. Let A_{ij} denote the entry at position (i, j) if $A \in M_{m \times n}(R)$. Denote by A^t the transpose of A , that is, $(A^t)_{ij} = A_{ji}$. Denote the i -th row of A by A_{i*} and the j -th column of A by A_{*j} . Set $M_n(R) := M_{n \times n}(R)$. The identity matrix in $M_n(R)$ is denoted by e or E_n . If $A \in M_n(R)$ is invertible, the entry of A^{-1} at position (i, j) is denoted by A'_{ij} . Further, we denote by R^n the set of all columns $(a_1, \dots, a_n)^t$ with entries in R .

Let G be a group and H a subset of G . Denote by $\langle H \rangle$ the subgroup of G generated by H . If $x, y \in G$, let $x^y := y^{-1}xy$ and $[x, y] := xyx^{-1}y^{-1}$. Set

$$H^x := \langle \{h^x \mid h \in H\} \rangle.$$

3 Odd-dimensional unitary groups

3.1 Hermitian form ring

Our concept closely follows [4]. Let R be an associative ring with identity 1. Recall that an anti-isomorphism $\bar{\cdot} : R \rightarrow R, a \mapsto \bar{a}$ on R is a bijective map satisfying $\overline{a+b} = \bar{a} + \bar{b}$, $\overline{ab} = \bar{b}\bar{a}$ for all $a, b \in R$ and $\bar{1} = 1$. Let $\lambda \in R$ with the property that $\bar{\bar{a}} = \lambda a \bar{\lambda}$ for all $a \in R$. It is easy to see that $\bar{\bar{\lambda}} = \lambda^{-1}$. The element λ is called a *symmetry* for $\bar{\cdot}$ and the triple $(R, \bar{\cdot}, \lambda)$ a *ring with involution and symmetry*. A *Hermitian ring* is a quadruple $(R, \bar{\cdot}, \lambda, \mu)$, where $(R, \bar{\cdot}, \lambda)$ is a ring with involution and symmetry and $\mu \in R$ such that $\mu = \bar{\mu}\lambda$.

Remark 3.1. (1) Define the map $\underline{\cdot} : R \rightarrow R, a \mapsto \underline{a} := \bar{\lambda}\bar{a}\lambda$. It is the inverse map of $\bar{\cdot}$. One checks easily that $(R, \underline{\cdot}, \lambda, \mu)$ is a Hermitian ring.

(2) For any $n \in \mathbb{N}$, $(M_n(R), *, \lambda e, \mu e)$ is a Hermitian ring, where

$$A^* := \bar{A}^t = ((\bar{A}_{ij})_{ij})^t = (\bar{A}_{ji})_{ij} \quad \text{for any } A \in M_n(R).$$

Let $R^\bullet$ be the underlying set of the ring equipped with the multiplication of the ring, but not the addition of the ring. A (right) $R^\bullet$ -module is a not necessarily abelian group $(G, \dot{+})$ equipped with a map $\circ : G \times R^\bullet \rightarrow G, (x, a) \mapsto x \circ a$, satisfying

- (1) $x \circ 0 = 0$ for all $x \in G$,
- (2) $x \circ 1 = x$ for all $x \in G$,
- (3) $(x \circ a) \circ b = x \circ (ab)$ for all $x \in G$ and $a, b \in R$,
- (4) $(x \dot{+} y) \circ a = (x \circ a) \dot{+} (y \circ a)$ for all $x, y \in G$ and $a \in R$.

We treat $\circ$ as an operator with higher priority than $\dot{+}$. An R -module is canonically an $R^\bullet$ -module, but not conversely. Let G and G' be $R^\bullet$ -modules. By a *homomorphism of $R^\bullet$ -modules*, one means a group homomorphism $f : G \rightarrow G'$ such that $f(x \circ a) = f(x) \circ a$ for all $x \in G$ and $a \in R$. By an $R^\bullet$ -submodule N , we mean a subgroup N of G which is $\circ$ -stable, that is, $x \circ a \in N$ for all $x \in N$ and $a \in R$.

Let $(R, \bar{\cdot}, \lambda, \mu)$ be a Hermitian ring. Consider the *Heisenberg group* $\mathfrak{Y}$, where $\mathfrak{Y} = R \times R$ as a set, with a composition operation “ $\dot{+}$ ”, satisfying

$$(a_1, b_1) \dot{+} (a_2, b_2) = (a_1 + a_2, b_1 + b_2 - \bar{a}_1 \mu a_2).$$

It is easy to see that the law $\dot{+}$ is associative with identity $(0, 0)$, and the inverse is given by

$$\dot{-}(a, b) = (-a, -b - \bar{a} \mu a) \quad \text{for any } (a, b) \in \mathfrak{Y}.$$

Furthermore, supply $\mathfrak{Y}$ with a right action of $R^\bullet$ through $(a, b) \circ c = (ac, \bar{c}bc)$. $\mathfrak{Y}$ becomes an $R^\bullet$ -module.

Let $(R, +)$ have the $R^\bullet$ -module structure defined by $a \circ b = \bar{b}ab$. Define the trace map $\text{tr}: \mathfrak{Y} \rightarrow R$ by $\text{tr}((a, b)) = \bar{a}\mu a + b + \bar{b}\lambda$. It is easy to see that tr is a homomorphism of $R^\bullet$ -modules.

Set

$$\Delta_{\min} := \{(0, a - \bar{a}\lambda) \mid a \in R\} \quad \text{and} \quad \Delta_{\max} := \ker(\text{tr}).$$

An odd form parameter of $(R, -, \lambda, \mu)$ is an $R^\bullet$ -submodule Δ of $\mathfrak{Y}$ such that $\Delta_{\min} \leq \Delta \leq \Delta_{\max}$. The pair $((R, -, \lambda, \mu), \Delta)$ is called a *Hermitian form ring* and is abbreviated by (R, Δ) .

3.2 Odd-dimensional unitary groups

Let (R, Δ) be a Hermitian form ring and $n \in \mathbb{N}$. Consider the right R -module $M := R^{2n+1}$ with the fixed basis $(e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1})$, where e_i is the column vector whose i -th entry ($1 \leq i \leq -1$) is 1 and whose other entries are 0. If $u \in M$, we call $(u_1, \dots, u_n, u_{-n}, \dots, u_{-1})^t \in R^{2n}$ the *hyperbolic part* of u and denote it by u_{hb} . Define the maps

$$h: M \times M \rightarrow R,$$

$$(u, v) \mapsto u^* \begin{pmatrix} 0 & 0 & p \\ 0 & \mu & 0 \\ p\lambda & 0 & 0 \end{pmatrix} v = \sum_{i=1}^n \bar{u}_i v_{-i} + \bar{u}_0 \mu v_0 + \sum_{i=-n}^{-1} \bar{u}_i \lambda v_{-i},$$

and

$$q: M \rightarrow \mathfrak{Y},$$

$$u \mapsto (q_1(u), q_2(u)) := \left(u_0, u_{\text{hb}}^* \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} u_{\text{hb}} \right) = \left(u_0, \sum_{i=1}^n \bar{u}_i u_{-i} \right),$$

where $u^* = \bar{u}^t$, $u_{\text{hb}}^* = \bar{u}_{\text{hb}}^t$ and $p \in M_n(R)$ is the matrix with 1 on the skew diagonal and 0 elsewhere. One checks easily the following lemma.

Lemma 3.2. *The following statements hold.*

(1) *h is a λ -Hermitian form, that is,*

(a) *h is biadditive,*

(b) *$h(ua, vb) = \bar{a}h(u, v)b$ for all $u, v \in M$ and $a, b \in R$,*

(c) *$h(u, v) = \overline{h(v, u)}\lambda$ for all $u, v \in M$.*

(2) q has the following properties:

- (a) $q(ua) = q(u) \circ a$ for all $u \in M$ and $a \in R$,
- (b) $q(u + v) \equiv q(u) \dot{+} q(v) \dot{+} (0, h(u, v)) \pmod{\Delta_{\min}}$ for all $u, v \in M$,
- (c) $\text{tr}(q(u)) = h(u, u)$ for all $u \in M$.

The odd-dimensional unitary group $U_{2n+1}(R, \Delta)$ is the group consisting of all elements $\sigma \in \text{GL}_{2n+1}(R)$ satisfying

$$\begin{aligned} h(\sigma u, \sigma v) &= h(u, v), \\ q(\sigma u) &\equiv q(u) \pmod{\Delta} \end{aligned}$$

for all $u, v \in M$. These groups are isomorphic to Petrov's odd hyperbolic unitary groups such that $V_0 = R$ and contain as special cases the groups $\text{GL}_{2n+1}(R)$, where R is any ring, the classical groups $O_{2n+1}(R)$ and $\text{Sp}_{2n+1}(R)$, where R is any commutative ring, and further all even-dimensional unitary groups $U_{2n}(R, \Lambda)$, where (R, Λ) is any form ring.

Set

$$\Theta := \{1, \dots, n, 0, -n, \dots, -1\}, \quad \Theta_{\text{hb}} := \Theta \setminus \{0\}.$$

Let $\epsilon(i)$ be 1 for $i = 1, \dots, n$ and -1 for $i = -n, \dots, -1$.

Lemma 3.3. *Let $\sigma \in \text{GL}_{2n+1}(R)$. Then $\sigma \in U_{2n+1}(R, \Delta)$ if and only if conditions (1) and (2) below hold.*

(1) We have

$$\begin{aligned} \sigma'_{ij} &= \lambda^{-(\epsilon(i)+1)/2} \bar{\sigma}_{-j, -i} \lambda^{(\epsilon(j)+1)/2} && \text{for all } i, j \in \Theta_{\text{hb}}, \\ \mu \sigma'_{0j} &= \bar{\sigma}_{-j, 0} \lambda^{(\epsilon(j)+1)/2} && \text{for all } j \in \Theta_{\text{hb}}, \\ \sigma'_{i0} &= \lambda^{-(\epsilon(i)+1)/2} \bar{\sigma}_{0, -i} \mu && \text{for all } i \in \Theta_{\text{hb}}, \\ \mu \sigma'_{00} &= \bar{\sigma}_{00} \mu. \end{aligned}$$

(2) $q(\sigma_{*j}) \equiv (\delta_{0j}, 0) \pmod{\Delta}$ for all $j \in \Theta$.

Proof. See [4, Lemma 17]. □

Define the embedding $\Phi: \text{GL}_n(R) \rightarrow U_{2n+1}(R, \Delta)$ by

$$\Phi(\sigma) = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p(\sigma^*)^{-1}p \end{pmatrix}.$$

3.3 The elementary subgroup

Let $(R, \bar{\cdot}, \lambda, \mu)$ be a Hermitian ring. We defined an $R^\bullet$ -module structure on $\mathcal{Y}$. Let $(R, \underline{\cdot}, \underline{\lambda}, \underline{\mu})$ be the Hermitian ring defined in Remark 3.1 (1), and let $\underline{\mathcal{Y}}$ be the Heisenberg group corresponding to $(R, \underline{\cdot}, \underline{\lambda}, \underline{\mu})$. Since the underlying set of both $(R, \bar{\cdot}, \lambda, \mu)$ and $(R, \underline{\cdot}, \underline{\lambda}, \underline{\mu})$ is $R \times R$, we get another $R^\bullet$ -module structure on $R \times R$ if we replace the Hermitian ring $(R, \bar{\cdot}, \lambda, \mu)$ by $(R, \underline{\cdot}, \underline{\lambda}, \underline{\mu})$. We denote the group operation (resp. scalar multiplication) defined by $(R, \bar{\cdot}, \lambda, \mu)$ and $(R, \underline{\cdot}, \underline{\lambda}, \underline{\mu})$ on $R \times R$ by $\dot{+}_1$ and $\dot{+}_{-1}$ (resp. $\circ_1$ and $\circ_{-1}$), respectively. Set

$$\Delta^1 := \Delta \quad \text{and} \quad \Delta^{-1} := \{(a, b) \in R \times R \mid (a, \bar{b}) \in \Delta\}.$$

It is easy to see that $((R, \underline{\cdot}, \underline{\lambda}, \underline{\mu}), \Delta^{-1})$ is a Hermitian form ring.

If $i, j \in \Theta$, let e^{ij} denote the matrix in $M_{2n+1}(R)$ with 1 in the (i, j) -th position and 0 elsewhere. If $i, j \in \Theta_{\text{hb}}$, $i \neq \pm j$ and $a \in R$, the element

$$T_{ij}(a) := e + ae^{ij} - \lambda^{(\epsilon(j)-1)/2} \bar{a} \lambda^{(1-\epsilon(i))/2} e^{-j, -i} \quad \text{of } U_{2n+1}(R, \Delta)$$

is called an (*elementary*) *short root matrix*. If $i \in \Theta_{\text{hb}}$ and $(a, b) \in \Delta^{-\epsilon(i)}$, the element

$$T_i(a, b) := e + ae^{0, -i} - \lambda^{-(1+\epsilon(i))/2} \bar{a} \mu e^{i0} + be^{i, -i} \quad \text{of } U_{2n+1}(R, \Delta)$$

is called an (*elementary*) *extra short root matrix*. The extra short root matrices of the kind $T_i(0, b) = e + be^{i, -i}$ are called (*elementary*) *long root matrices*. An element of $U_{2n+1}(R, \Delta)$ is called *elementary matrix* if it is a short or extra short root matrix. The *elementary subgroup* $EU_{2n+1}(R, \Delta)$ is the subgroup of $U_{2n+1}(R, \Delta)$ generated by all elementary matrices. The coset space

$$KU_{1,n}(R, \Delta) := U_{2n+1}(R, \Delta) / EU_{2n+1}(R, \Delta)$$

is called the *odd-dimensional unitary K_1 -functor*. Under the embedding

$$\phi_{2n+1}^{2n+3}: U_{2n+1}(R, \Delta) \rightarrow U_{2n+3}(R, \Delta),$$

$$\sigma \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

it is easy to see that $\phi_{2n+1}^{2n+3}(EU_{2n+1}(R, \Delta)) \subseteq EU_{2n+3}(R, \Delta)$. Therefore, we have the stabilization map $KU_{1,n}(R, \Delta) \rightarrow KU_{1,n+1}(R, \Delta)$.

The elementary matrices satisfy the following properties:

$$T_{ij}(a) = T_{-j,-i}(-\lambda^{(\epsilon(j)-1)/2} \bar{a} \lambda^{(1-\epsilon(i))/2}), \quad (\text{S1})$$

$$T_{ij}(a)T_{ij}(b) = T_{ij}(a+b), \quad (\text{S2})$$

$$[T_{ij}(a), T_{kl}(b)] = e \quad \text{if } k \neq -i, j \text{ and } l \neq i, -j, \quad (\text{S3})$$

$$[T_{ij}(a), T_{jk}(b)] = T_{ik}(ab) \quad \text{if } k \neq \pm i, \quad (\text{S4})$$

$$[T_{ij}(a), T_{j,-i}(b)] = T_i(0, ab - \lambda^{-(1+\epsilon(i))/2} \bar{b} \bar{a} \lambda^{(1-\epsilon(i))/2}), \quad (\text{S5})$$

$$T_i(a_1, b_1)T_i(a_2, b_2) = T_i((a_1, b_1) \dot{+}_{-\epsilon(i)} (a_2, b_2)), \quad (\text{E1})$$

$$[T_i(a_1, b_1), T_j(a_2, b_2)] = T_{i,-j}(-\lambda^{-(1+\epsilon(i))/2} \bar{a}_1 \mu a_2) \quad \text{if } j \neq \pm i, \quad (\text{E2})$$

$$[T_i(a_1, b_1), T_i(a_2, b_2)] = T_i(0, -\lambda^{-(1+\epsilon(i))/2} (\bar{a}_1 \mu a_2 - \bar{a}_2 \mu a_1)), \quad (\text{E3})$$

$$[T_{ij}(a), T_k(b, c)] = e \quad \text{if } k \neq j, -i, \quad (\text{SE1})$$

$$\begin{aligned} [T_{ij}(a), T_j(b, c)] &= T_{j,-i}(c \lambda^{(\epsilon(j)-1)/2} \bar{a} \lambda^{(1-\epsilon(i))/2}) \\ &\quad \cdot T_i(b \lambda^{(\epsilon(j)-1)/2} \bar{a} \lambda^{(1-\epsilon(i))/2}, \\ &\quad \quad \quad ac \lambda^{(\epsilon(j)-1)/2} \bar{a} \lambda^{(1-\epsilon(i))/2}). \end{aligned} \quad (\text{SE2})$$

Let $i, j \in \Theta_{\text{hb}}$ such that $i \neq \pm j$. In the following, the usual ordering of the numbers $-n, \dots, -1, 0, 1, \dots, n$ is used. Denote by T_{ij} the subgroup consisting of all $T_{ij}(a)$ with $a \in R$ and by T_i the subgroup consisting of all $T_i(a, b)$ with $(a, b) \in \Delta^{-\epsilon(i)}$. Let $n \geq 3$. We define the following subgroups of $EU_{2n+1}(R, \Delta)$ which will be used to establish a decomposition theorem in next section.

$$U_n = \langle T_{ij}, T_i \mid j < 0 < i \rangle,$$

$$U_n^- = \langle T_{ij}, T_i \mid i < 0 < j \rangle,$$

$$L_n = \langle T_{ij} \mid i, j > 0 \rangle,$$

$$F_n = \langle T_{ij}, T_i \mid i \neq -1, j \neq 1 \rangle,$$

$$Y^+ = \langle T_n, T_{n-1}, T_{n-1,-n} \rangle \leq U_n,$$

$$Y^- = \langle T_{-n}, T_{-(n-1)}, T_{-(n-1),n} \rangle \leq U_n^-,$$

$$V^+ = \langle T_{i,n}, T_{i,n-1} \mid i = 1, \dots, n-2 \rangle \leq L_n,$$

$$V^- = \langle T_{n,i}, T_{n-1,i} \mid i = 1, \dots, n-2 \rangle \leq L_n.$$

Denote by $\rtimes$ the semidirect product. Set

$$U^+ = U_n \rtimes V^+, \quad U^- = U_n^- \rtimes V^-, \quad G_n = U_n \rtimes L_n.$$

3.4 Stable range condition

Let R be an associative ring with 1. A vector $(a_1, \dots, a_n)^t$ with $a_i \in R$ is called (left) *unimodular* if there exist elements $b_1, \dots, b_n \in R$ such that

$$b_1 a_1 + \dots + b_n a_n = 1.$$

The *stable range condition* S_n says that, for every unimodular vector

$$(a_1, \dots, a_{n+1})^t \in R^{n+1},$$

there exist elements $b_1, \dots, b_n \in R$ such that

$$(a_1 + b_1 a_{n+1}, \dots, a_n + b_n a_{n+1})^t \in R^n$$

is unimodular. The *stable rank* of R , denoted by $\text{sr}(R)$, is the smallest number n such that S_n holds.

Lemma 3.4 ([8, Lemma 9]). *Let $m < n$ and $\text{sr}(R) \leq n - m$. Let v be a unimodular column of height n . Then there exists a matrix $B \in M_{(n-m) \times m}(R)$ such that $(E_{n-m}, B)v \in R^{n-m}$ is unimodular.*

4 Injective stability

In this section, we get a decomposition theorem for $EU_{2n+1}(R, \Delta)$ and prove an injective stability theorem for odd-dimensional unitary K_1 under stable range condition.

Lemma 4.1. *Let $n \geq 3$. Then $EU_{2n+1}(R, \Delta)$ is generated by G_n and Y^- .*

Proof. Denote by H the subgroup of $EU_{2n+1}(R, \Delta)$ generated by G_n and Y^- . We only need to prove that $U_n^- \subseteq H$. Let $i = 1, \dots, n-1$, $j = 1, \dots, n-2$, $a \in R$ and $(b, c) \in \Delta$. It follows from (S1), (S4) and (SE2) that

$$\begin{aligned} T_{-n,j}(a) &= [T_{-n,n-1}(a), T_{n-1,j}(1)] \in H, \\ T_{-j,i}(a) &= [T_{-j,-n}(1), T_{-n,i}(a)] \in H, \\ T_{-j}(b, c) &= T_{-n,j}^{-1}(c)[T_{nj}(-1), T_{-n}(b, c)] \in H. \end{aligned}$$

Hence H contains all the generators of $EU_{2n+1}(R, \Delta)$. □

Let $H_1, \dots, H_s$ be arbitrary subsets of a group G . Define the Minkowski product $H_1 \dots H_s := \{h_1 \dots h_s \mid h_i \in H_i, i = 1, \dots, s\}$.

Lemma 4.2. *Let $n \geq 3$. We have the following inclusions:*

- (i) $Y^-U^+ \subseteq U^+Y^-Y^+$,
- (ii) $Y^+U^- \subseteq U^-Y^+Y^-$,
- (iii) $Y^-U^+U^- \subseteq U^+U^-Y^+Y^-$.

Proof. (i) Let $A = T_n \cup T_{n-1} \cup T_{n-1,-n}$ and B be the union of all T_{ij} and T_i , where $i \in \{1, \dots, n-2\}$ and either $j < 0$ or $j \in \{n-1, n\}$. Clearly, we have $U^+ = \langle A \cup B \rangle$ and $Y^+ = \langle A \rangle$. If $a \in A$ and $b \in B$, then $ab = [a, b]ba$.

Case 1: $a \in T_n$. If $b \in T_{ij}$, it follows from (SE1) that $[a, b] = e$. If $b \in T_i$, it follows from (E2) that $[a, b] \in T_{i,-n}$.

Case 2: $a \in T_{n-1}$. If $b \in T_{ij}$, it follows from (SE1) that $[a, b] = e$. If $b \in T_i$, it follows from (E2) that $[a, b] \in T_{i,-(n-1)}$.

Case 3: $a \in T_{n-1,-n}$. Assume that $b \in T_{ij}$. If $j < 0$, it follows from (S3) that $[a, b] = e$. If $j = n-1$, it follows from (S4) that $[a, b] \in T_{i,-n}$. If $j = n$, it follows from (S4) that $[a, b] \in T_{i,-(n-1)}$. Assume that $b \in T_i$. It follows from (SE1) that $[a, b] = e$. Hence $ab = b'a$ for some $b' \in B$ and $U^+ = \langle B \rangle \langle A \rangle = \langle B \rangle Y^+$.

Let $\alpha = \beta\gamma \in U^+$, where $\beta \in \langle B \rangle$ and $\gamma \in Y^+$. Since $\sigma\beta\sigma^{-1}$ lies in U^+ for any $\sigma \in Y^-$, we have $\sigma\alpha = \sigma\beta\sigma^{-1} \cdot \sigma\gamma \in U^+Y^-Y^+$.

(ii) Similar proof as (i).

(iii) Follows from (i) and (ii). □

Set $S := \{\alpha \in L_n \mid \alpha_{n-1,1} = \alpha_{n1} = 0\}$.

Lemma 4.3. *Let $n \geq \text{sr}(R) + 2$. Then, for every $\alpha \in L_n$, there exist $\beta_\alpha \in V^+$ and $\gamma_\alpha \in V^-$ such that $\gamma_\alpha\beta_\alpha\alpha \in S$.*

Proof. Let $\alpha \in L_n$, and let v be the first column of α . It follows from Lemma 3.4 that there exists a matrix $B \in M_{(n-2) \times 2}(R)$ such that the first $n-2$ coordinates of

$$v' = \Phi \left(\begin{pmatrix} E_{n-2} & B \\ 0 & E_2 \end{pmatrix} \right) v$$

form a unimodular column. Now, there exists another matrix $C \in M_{2 \times (n-2)}(R)$ such that both $(n-1)$ -th and n -th coordinates of

$$\Phi \left(\begin{pmatrix} E_{n-2} & 0 \\ C & E_2 \end{pmatrix} \right) v'$$

are 0. Let

$$\beta_\alpha = \Phi \left(\begin{pmatrix} E_{n-2} & B \\ 0 & E_2 \end{pmatrix} \right) \in V^+, \quad \gamma_\alpha = \Phi \left(\begin{pmatrix} E_{n-2} & 0 \\ C & E_2 \end{pmatrix} \right) \in V^-.$$

Then we have $\gamma_\alpha \beta_\alpha \alpha \in S$. □

Corollary 4.4. *Let $n \geq \text{sr}(R) + 2$. Then $U_n U_n^- L_n \subseteq U^+ U^- S$.*

Proof. Let $\alpha \in L_n$. Since β_α normalizes U_n^- , it follows from Lemma 4.3 that

$$U_n U_n^- \alpha = (U_n \beta_\alpha^{-1})(\beta_\alpha U_n^- \beta_\alpha^{-1} \gamma_\alpha^{-1})(\gamma_\alpha \beta_\alpha \alpha) \in U^+ U^- S. \quad \square$$

Lemma 4.5. *Let $n \geq \text{sr}(R) + 2$. Then $Y^- U^+ U^- S \subseteq U^+ U^- L_n F_n$.*

Proof. Let $\alpha \in S$ and $\beta \in (Y^-)^\alpha$. Since $(Y^-)^\alpha \subseteq U_n^-$, we have

$$\beta = \begin{pmatrix} E_n & 0 & 0 \\ A & 1 & 0 \\ C & B & E_n \end{pmatrix}$$

for some $A \in M_{1 \times n}(R)$, $B \in M_{n \times 1}(R)$ and $C \in M_{n \times n}(R)$. By the definition of S , the first column of α remains unchanged if we multiply α on the left by an element of Y^- . Hence the first column of β coincides with that of the identity matrix. Then $A_{11} = 0$ and $C_{*1} = 0$. It follows from Lemma 3.3 that $C_{n*} = 0$ and $B_{n1} = 0$. Therefore, we have $\beta \in U_n^- \cap F_n \subseteq F_n$. Since $(Y^+)^\alpha \subseteq F_n$, by Lemma 4.2, we have

$$\begin{aligned} Y^- U^+ U^- \alpha &\subseteq U^+ U^- Y^+ Y^- \alpha \\ &= U^+ U^- \alpha (Y^+)^\alpha (Y^-)^\alpha \subseteq U^+ U^- L_n F_n. \end{aligned} \quad \square$$

Lemma 4.6. *Let $\sigma \in U_{2n+1}(R, \Delta)$ have the form*

$$\begin{pmatrix} E_n & 0 & 0 \\ A & 1 & 0 \\ C & B & E_n \end{pmatrix},$$

where $A \in M_{1 \times n}(R)$, $B \in M_{n \times 1}(R)$, $C \in M_{n \times n}(R)$. Then $\sigma \in EU_{2n+1}(R, \Delta)$.

Proof. σ^{-1} has the form

$$\begin{pmatrix} E_n & 0 & 0 \\ -A & 1 & 0 \\ BA - C & -B & E_n \end{pmatrix}.$$

Let $i < 0 < j$. It follows from Lemma 3.3 that

$$B_{n-j+1,1} = -\bar{A}_{1j}\mu, \quad B_{n+i+1,1}A_{1j} - C_{n+i+1,j} = \bar{C}_{n-j+1,-i}\lambda$$

and $(A_{1j}, C_{n-j+1,j}) \equiv (0, 0) \pmod{\Delta}$. Then $(A_{1j}, C_{n-j+1,j}) \in \Delta$ and

$$\sigma = \prod_{j=1}^n T_j(A_{1j}, C_{n-j+1,j}) \prod_{j-i < n+1} T_{ij}(C_{n-j+1,-i}).$$

Hence $\sigma \in EU_{2n+1}(R, \Delta)$. □

Now we have the following decomposition theorem.

Theorem 4.7 (Decomposition theorem). *Let $n \geq \text{sr}(R) + 2$. Then every element of $EU_{2n+1}(R, \Delta)$ has a $G_n U_n^- F_n$ -decomposition, that is,*

$$EU_{2n+1}(R, \Delta) = G_n U_n^- F_n.$$

Proof. It suffices to prove that $G_n U_n^- F_n$ is stable under left multiplications by the generators of $EU_{2n+1}(R, \Delta)$. By Lemma 4.1, it is enough to show that

$$Y^- G_n U_n^- F_n \subseteq G_n U_n^- F_n.$$

Since L_n normalizes U_n and U_n^- , we have

$$G_n U_n^- F_n = U_n L_n U_n^- F_n = U_n U_n^- L_n F_n.$$

It follows from Corollary 4.4 and Lemma 4.5 that

$$\begin{aligned} Y^- G_n U_n^- F_n &= Y^- U_n U_n^- L_n F_n \\ &\subseteq Y^- U^+ U^- S F_n \subseteq U^+ U^- L_n F_n \subseteq G_n U_n^- F_n. \end{aligned} \quad \square$$

Proof of Theorem 1.1. Let

$$\sigma \in U_{2(n-1)+1}(R, \Delta) \cap EU_{2n+1}(R, \Delta).$$

By Theorem 4.7, we have $\sigma = \alpha\beta\gamma$, where

$$\alpha = \begin{pmatrix} A^{11} & A^{12} & A^{13} \\ 0 & 1 & A^{23} \\ 0 & 0 & A^{33} \end{pmatrix} \in G_n, \quad \beta = \begin{pmatrix} E_n & 0 & 0 \\ B^{21} & 1 & 0 \\ B^{31} & B^{32} & E_n \end{pmatrix} \in U_n^-, \quad \gamma \in F_n.$$

Since the first columns of both σ and γ coincide with that of the identity matrix, we get $A^{33}(B^{31})_{*1} = 0$ and $B_{11}^{21} + A^{23}(B^{31})_{*1} = 0$. Since A^{33} is invertible, we have $(B^{31})_{*1} = 0$ and $B_{11}^{21} = 0$. Hence $\beta \in F_n$. Thus we can assume that $\sigma = \alpha\gamma$. Since the first columns of both σ and γ coincide with that of the identity matrix, we get that $(A^{11})_{*1}$ equals the first column of the identity matrix and $(A^{33})_{n*}$ equals the last row of the identity matrix. Hence we can express α as $\alpha = \alpha_1\alpha_2$, where $\alpha_1 \in U_{2(n-1)+1}(R, \Delta)$ and $\alpha_2 \in F_n$. Thus we can assume that $\sigma = \alpha\gamma$ with $\alpha, \gamma \in U_{2(n-1)+1}(R, \Delta)$.

Since

$$\alpha = \begin{pmatrix} E_n & A^{12} & A^{13}(A^{33})^{-1} \\ 0 & 1 & A^{23}(A^{33})^{-1} \\ 0 & 0 & E_n \end{pmatrix} \begin{pmatrix} A^{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A^{33} \end{pmatrix},$$

where the second matrix lies in L_n , we get that $A^{11} \in E_n(R)$. Let $A^{11} = \begin{pmatrix} 1 & 0 \\ 0 & A_{*}^{11} \end{pmatrix}$, we have $A_{*}^{11} \in E_n(R) \cap \text{GL}_{n-1}(R)$. Since $n \geq \text{sr}(R) + 2$, it follows from the stability of the K_1 -functor under the stable range condition S_n [6, Theorem 5.4.2] that $A_{*}^{11} \in E_{n-1}(R)$. Then $\alpha \in EU_{2(n-1)+1}(R, \Delta)$.

Since $\gamma \in F_n$, we can express γ as

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & C^{12} & C^{13} \\ 0 & E_{2(n-1)+1} & C^{23} \\ 0 & 0 & 1 \end{pmatrix},$$

where $C \in EU_{2(n-1)+1}(R, \Delta)$. It follows from $\gamma \in U_{2(n-1)+1}(R, \Delta)$ that

$$C^{12} = 0, \quad C^{13} = 0 \quad \text{and} \quad C^{23} = 0.$$

Then $\gamma \in EU_{2(n-1)+1}(R, \Delta)$. Hence $\sigma \in EU_{2(n-1)+1}(R, \Delta)$. □

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