Freely indecomposable almost free groups with free abelianization

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Abstract. For certain uncountable cardinals κ , we produce a group of cardinality κ which is freely indecomposable, strongly κ -free, and whose abelianization is free abelian of rank κ . The construction takes place in Gödel's constructible universe L . This strengthens an earlier result of Eklof and Mekler.

1 Introduction

We produce examples of groups which exhibit some properties enjoyed by free groups but which, in other ways, are very far from being free. We recall some definitions before stating the main result. Given a group G and subgroup $H \leq G$, we say H is a *free factor* of G provided there exists another subgroup $K \leq G$ such that $G = H * K$ is the free product in the natural way (that is, the map $H * K \rightarrow G$ induced by the inclusions of H and K is an isomorphism). We call such a writing $G = H * K$ a *free decomposition* of G and say that G is *freely indecomposable* provided there does not exist a free decomposition of G via two nontrivial free factors.

Given a cardinal κ , we say G is κ -free if each subgroup of G generated by fewer than κ elements is a free group. Historically, a κ -free group of cardinality κ is called *almost free* [7]. By the theorem of Nielsen and Schreier, every free group is κ -free for every cardinal κ . A subgroup H of a κ -free group G is κ -pure if H is a free factor of any subgroup $\langle H \cup X \rangle$, where $X \subseteq G$ is of cardinality $< \kappa$. A κ -free group G is *strongly κ -free* provided each subset $X \subseteq G$ with $|X| < \kappa$ is included in a κ -pure subgroup of G generated by fewer than κ elements.

Let ZFC denote the Zermelo–Fraenkel axioms of set theory including the axiom of choice, and let $V = L$ denote the assertion that every set is constructible. The theory $\text{ZFC} + V = L$ is consistent provided ZFC is consistent [6]. The set-theoretic

concepts in the following statement will be reviewed in Section 3, but the reader can, for example, let κ be any uncountable successor cardinal (e.g. $\aleph_1$, $\aleph_2$, $\aleph_{\omega+1}$).

Theorem 1.1 (ZFC + V = L). *Let κ be an uncountable regular cardinal that is not weakly compact. There exists a group G of cardinality κ for which*

- (1) *G is freely indecomposable,*
- (2) *G is strongly κ -free,*
- (3) *the abelianization G/G' is free abelian of rank κ .*

The hypotheses on the cardinal κ cannot be dropped since a κ -free group of cardinality κ must be free when κ is singular or weakly compact (see respectively [15] and [3]). A group as in the conclusion seems unusual since, on a local level, it is free; on a global level, it is quite unfree and in fact indecomposable, but the abelianization is as decomposable as possible. Theorem 1.1 minus condition (3) was proved in [4], and the construction apparently does not have free abelianization; indeed, the proof that their groups are freely indecomposable involves abelianizing. A non-free $\aleph_1$ -free group of cardinality $\aleph_1$ which abelianizes to a free abelian group was produced by Bitton [1] using only ZFC, and Theorem 1.1 can be considered a constructible universe strengthening of his result. The first construction of a non-free almost free group of cardinality $\aleph_1$ was given by Higman [7] without any extra set-theoretic assumptions; a strongly $\aleph_1$ -free group of cardinality $\aleph_1$ produced only from ZFC was given by Mekler [13]. The reader can find other results related to almost free (abelian) groups in such works as [2, 5, 12].

We note that it is not possible to produce a group G of cardinality $\geq \kappa$ whose every subgroup of cardinality κ satisfies conditions (1)–(3) of Theorem 1.1. This is because, for every uncountable locally free group G , there exists a free subgroup $H \leq G$ with $|H| = |G|$ (see [14, Theorem 1.1]). Finally, we mention that the construction used in proving Theorem 1.1 allows one, as in [4], to construct 2^κ many pairwise non-isomorphic groups of this description using [16].

2 Some group-theoretic lemmas

The following appears as [7, Lemma 1].

Lemma 2.1. *If K is a free factor of G and H is a subgroup of G , then $H \cap K$ is a free factor of H .*

We will use the following construction in our proof of Theorem 1.1.

Construction 2.2. Suppose that we have a free group F_a with free decomposition $F_a = F_0 * F_1 * F_{2,a}$ with F_1 nontrivial and $F_{2,a}$ freely generated by $\{t_n\}_{n \in \omega}$. Let F_b be a free group with free decomposition $F_b = F_0 * F_1 * F_{2,b}$, where $F_{2,b}$ is freely generated by $\{z_n\}_{n \in \omega}$. Let y be an element of a free generating set for F_1 . Define $\phi: F_a \rightarrow F_b$ so that $\phi \upharpoonright F_0 * F_1$ is the identity and $\phi(t_n) = yz_{n+1}^{-1}z_nz_{n+1}$.

Property (iv) of the following lemma compares to [1, Lemma 2.9 (3) & (4)].

Lemma 2.3. Let $F_{2,a,n} = \langle t_0, \dots, t_{n-1} \rangle$. The map ϕ satisfies the following:

- (i) ϕ is a monomorphism;
- (ii) $z_n \notin \phi(F_a)$ for all $n \in \omega$;
- (iii) $\phi(F_a)$ is not a free factor of F_b , but $\phi(F_0 * F_1 * F_{2,a,n})$ is a free factor for every $n \in \omega$;
- (iv) the equality $(\phi(F_a))' = \phi(F_a) \cap F_b'$ holds, and the natural induced map $\bar{\phi}: F_a/F_a' \rightarrow F_b/F_b'$ is an isomorphism.

Proof. Fix a possibly empty set of free generators X for F_0 and a possibly empty set Y such that the disjoint union $Y \sqcup \{y\}$ freely generates F_1 . Now $\phi(F_a)$ is the subgroup

$$\langle X \cup Y \cup \{y\} \cup \{yz_{n+1}^{-1}z_nz_{n+1}\}_{n \in \omega} \rangle = \langle X \cup Y \cup \{y\} \cup \{z_{n+1}^{-1}z_nz_{n+1}\}_{n \in \omega} \rangle,$$

and the generators $X \cup Y \cup \{y\} \cup \{z_{n+1}^{-1}z_nz_{n+1}\}_{n \in \omega}$ freely generate $\phi(F_a)$ since they satisfy the Nielsen property (see [11, I.2] and [8, Example 2.8 (iii)]). The set $X \cup Y \cup \{y\} \cup \{y^{-1}t_n\}_{n \in \omega}$ is a free generating set for F_a and maps bijectively under ϕ to the free generating set $X \cup Y \cup \{y\} \cup \{z_{n+1}^{-1}z_nz_{n+1}\}_{n \in \omega}$ for $\phi(F_a)$. Thus ϕ is a monomorphism, and we have (i). Moreover, since

$$X \cup Y \cup \{y\} \cup \{z_{n+1}^{-1}z_nz_{n+1}\}_{n \in \omega}$$

satisfies the Nielsen property, it is clear that $z_n \notin \phi(F_a)$ for all $n \in \omega$, and we have (ii).

For (iii), we notice that the smallest normal subgroup in F_b including $\phi(F_a)$, which we denote $\langle\langle \phi(F_a) \rangle\rangle$, is precisely F_b since the free generators listed for F_b are conjugate in F_b to free generators of $\phi(F_a)$. Since $\phi(F_a)$ is a proper subgroup of F_b , this means that $\phi(F_a)$ cannot be a free factor of F_b . On the other hand, we have $\phi(F_0 * F_1 * F_{2,a,n})$ generated by

$$X \cup Y \cup \{y\} \cup \{yz_1^{-1}z_0z_1, \dots, yz_n^{-1}z_{n-1}z_n\},$$

and we claim that $X \cup Y \cup \{y\} \cup \{yz_1^{-1}z_0z_1, \dots, yz_n^{-1}z_{n-1}z_n\} \cup \{z_n, z_{n+1}, \dots\}$ is a free generating set for F_b . It is plain that

$$\langle \{y\} \cup \{yz_1^{-1}z_0z_1, \dots, yz_n^{-1}z_{n-1}z_n\} \cup \{z_n\} \rangle = \langle \{y\} \cup \{z_0, \dots, z_n\} \rangle,$$

and since finitely generated free groups are Hopfian, we get that

$$\{y\} \cup \{yz_1^{-1}z_0z_1, \dots, yz_n^{-1}z_{n-1}z_n\} \cup \{z_n\}$$

is a free generating set of the free factor $\langle \{y\} \cup \{z_0, \dots, z_n\} \rangle$ of F_b . Thus, indeed, $X \cup Y \cup \{y\} \cup \{yz_1^{-1}z_0z_1, \dots, yz_n^{-1}z_{n-1}z_n\} \cup \{z_n, z_{n+1}, \dots\}$ is a free generating set for F_b , and so

$$\phi(F_0 * F_1 * F_{2,a,n}) = \langle X \cup Y \cup \{y\} \cup \{yz_1^{-1}z_0z_1, \dots, yz_n^{-1}z_{n-1}z_n\} \rangle$$

is a free factor of F_b and we have shown (iii).

For condition (iv), certainly the inclusion $(\phi(F_a))' \subseteq \phi(F_a) \cap F_b'$ holds. Moreover, a word w in $(X \cup Y \cup \{y\} \cup \{z_n\}_{n \in \omega})^{\pm 1}$ represents an element of F_b' if and only if the sum of the exponents of each element in $X \cup Y \cup \{y\} \cup \{z_n\}_{n \in \omega}$ is 0. By treating each element of $\{z_{n+1}^{-1}z_nz_{n+1}\}_{n \in \omega}$ as an unreducing letter, a word in

$$(X \cup Y \cup \{y\} \cup \{z_{n+1}^{-1}z_nz_{n+1}\}_{n \in \omega})^{\pm 1}$$

represents an element of $(\phi(F_a))'$ if and only if the sum of the exponents of each element of $X \cup Y \cup \{y\} \cup \{z_{n+1}^{-1}z_nz_{n+1}\}_{n \in \omega}$ is 0. This is clearly equivalent to having the sum of the exponents of each letter in $X \cup Y \cup \{y\} \cup \{z_n\}_{n \in \omega}$ be 0, so we have $(\phi(F_a))' = \phi(F_a) \cap F_b'$. Thus the map $\bar{\phi}: F_a/F_a' \rightarrow F_b/F_b'$ is injective. Moreover, since each element of $X \cup Y \cup \{y\} \cup \{z_n\}_{n \in \omega}$ is conjugate in F_b to an element of $X \cup Y \cup \{y\} \cup \{z_{n+1}^{-1}z_nz_{n+1}\}_{n \in \omega}$, the map $\bar{\phi}$ is onto as well. $\square$

We recall some notions for free products of groups (see [11, Section IV.1]). Suppose that we have a free product $L_0 * L_1$. We call the nontrivial elements in $L_0 \cup L_1$ *letters*. Each element $g \in L_0 * L_1$ can be expressed uniquely as a product of letters $g = h_0h_1 \cdots h_{n-1}$ such that $h_i \in L_0$ if and only if $h_{i+1} \in L_1$ for all $0 \leq i < n-1$ (this is the *reduced* or *normal form* of the element). We call the number n the *length* of g , denoted $\text{Len}(g)$. Thus the identity element has length 0 and a nontrivial element has length 1 if and only if it is a letter. Given a writing of an element of $L_0 * L_1$ as a product of nontrivial elements $h_0 \cdots h_{n-1}$ in $L_0 \cup L_1$, it is easy to determine the normal form of the element by taking a consecutive pair h_ih_{i+1} for which h_i, h_{i+1} are both in L_0 or both in L_1 and performing the group multiplication in the appropriate group. This either gives the trivial element, in which case we remove the pair h_ih_{i+1} from the expression, or it gives a nontrivial element g_i , and we replace h_ih_{i+1} with g_i in the expression. This process reduces the number of letters in the writing by at least 1 every time, and so the process must eventually terminate, and it terminates at the normal form. We will generally consider an element of $L_0 * L_1$ as a word (the normal form) in the letters. For

words w_0 and w_1 in the letters, we will use $w_0 \equiv w_1$ to represent that w_0 and w_1 are the same word letter-for-letter when read from left to right.

We say an element of $L_0 * L_1$ is *cyclically reduced* if its reduced form is either of length 0 or 1 or begins with a letter in L_j and ends with a letter in L_{1-j} . A cyclically reduced element is of minimal length in its conjugacy class, and if two cyclically reduced elements are conjugate to each other, then the normal form of one is a cyclic shift of the other (w is a cyclic shift of u if we can write w as a concatenation $w \equiv v_0 v_1$ such that $u \equiv v_1 v_0$). Each element g of $L_0 * L_1$ is conjugate to a cyclically reduced element h , and we call $\text{Len}(h)$ the *cyclic length* of g .

Lemma 2.4. *If $\psi: F_a \rightarrow L_0 * L_1$ is a monomorphism such that*

$$\psi(F_1) = \psi(F_a) \cap L_0 \quad \text{and} \quad \psi(F_0 * F_{2,a}) = \psi(F_a) \cap L_1,$$

then there does not exist a monomorphism $\theta: F_b \rightarrow L_0 * L_1$ for which $\theta \circ \phi = \psi$.

Proof. Suppose on the contrary that such a θ exists. We will treat F_a as a subgroup of F_b since ϕ is a monomorphism and treat F_a and F_b as subgroups of $L_0 * L_1$ such that $F_1 = F_a \cap L_0$ and $F_0 * F_{2,a} = F_a \cap L_1$. We have $t_n = y z_{n+1}^{-1} z_n z_{n+1}$ for all $n \in \omega$, and so $y^{-1} t_n = z_{n+1}^{-1} z_n z_{n+1}$. Since $y^{-1} \in L_0$ and $t_n \in L_1$, we see that $y^{-1} t_n$ is cyclically reduced and $\text{Len}(y^{-1} t_n) = 2$. Therefore, for each $n \in \omega$, we know z_n is of cyclic length 2, so $\text{Len}(z_n) \geq 2$, and any cyclic reduction of z_n must be a cyclic shift of $y^{-1} t_n$. We claim that $\text{Len}(z_n) \geq \text{Len}(z_{n+1}) + 1$ for all $n \in \omega$. This immediately gives $\text{Len}(z_0) \geq \text{Len}(z_n) + n$ for all $n \in \omega$, which is a contradiction.

It remains to prove that $\text{Len}(z_n) \geq \text{Len}(z_{n+1}) + 1$. To economize on writing subscripts, we will show that $\text{Len}(z_0) \geq \text{Len}(z_1) + 1$, and the same proof will work for general n by adding n to the subscripts of t_0, z_0, t_1, z_1 . Suppose to the contrary that $\text{Len}(z_0) \leq \text{Len}(z_1)$. We have $z_1 y^{-1} t_0 z_1^{-1} = z_0$. Since z_1 is nontrivial, it must end with a letter of L_0 or a letter of L_1 .

Case A: z_1 ends with a letter of L_0 . In this case, it must be that z_1 ends with y since otherwise we readily see from the reduced form for $z_0 = z_1 y^{-1} t_0 z_1^{-1}$ that $\text{Len}(z_0) = (2\text{Len}(z_1) + 2) - 1 = 2\text{Len}(z_1) + 1 > \text{Len}(z_1)$ contrary to our assumption that $\text{Len}(z_0) \leq \text{Len}(z_1)$. Also, the second-to-last letter of z_1 must be t_0^{-1} since otherwise $\text{Len}(z_0) = (2\text{Len}(z_1) + 2) - 3 = 2\text{Len}(z_1) - 1 > \text{Len}(z_1)$ since $\text{Len}(z_1) \geq 2$. Thus we may write z_1 as a reduced word $z_1 = w(t_0^{-1}y)^k$ with $k \geq 1$ and maximal. Notice that w is nonempty, for otherwise $z_1 = (t_0^{-1}y)^k$ and $z_0 = y^{-1}t_0$, and therefore z_0 and z_1 commute instead of generating a free subgroup of rank 2. The word w must end with a letter from L_0 since $w(t_0^{-1}y)^k$ is reduced. Moreover, $\text{Len}(w) \geq 2$ since otherwise $w = g \in L_0$ and z_1 is conjugate

to $(t_0^{-1}y)^{k-1}t_0^{-1}yg$, and cyclically reducing this word cannot produce a cyclic shift of $y^{-1}t_1$. Then the second-to-last letter of w is an element from L_1 . If the last letter of w is y , then the second-to-last letter of w is not t_0^{-1} by maximality of k , and we get

$$\text{Len}(z_0) = (2 \text{Len}(z_1) + 2) - 4k - 2 - 1 = 2 \text{Len}(z_1) - 1 - 4k.$$

If, on the other hand, the last letter of w is $g \in L_0 \setminus \{1, y\}$, then we see that

$$\text{Len}(z_0) = (2 \text{Len}(z_1) + 2) - 4k - 1 = 2 \text{Len}(z_1) - 4k + 1,$$

so, in either case, we know $\text{Len}(z_0) \geq 2 \text{Len}(z_1) - 1 - 4k$. Since we are assuming $\text{Len}(z_1) \geq \text{Len}(z_0)$, we have $\text{Len}(z_1) \leq 4k + 1$.

Write $z_1 = (y^{-1}t_0)^m u (t_0^{-1}y)^k$ with $m \geq 0$ maximal. Certainly, u is nontrivial since otherwise z_1 would commute with z_0 . We also claim that u must begin and end with a letter from L_0 . It is clear, since z_1 is written in reduced form, that u must end with a letter from L_0 since $t_0^{-1} \in L_1$. Also, if $m > 0$, we have u beginning with a letter from L_0 for a similar reason. But if $m = 0$, then we have $z_1 = u(t_0^{-1}y)^k$, and were it the case that u started with a letter in L_1 , then this writing would already be cyclically reduced and z_1 would have cyclic length greater than 2 since u is nontrivial. Thus, in all cases, u must begin and end with a letter from L_0 .

If $m > k$, then $\text{Len}(z_1) = 2m + 2k + \text{Len}(u) > 4k + 1$. If $k = m$, then, since $4k + 1 \geq \text{Len}(z_1) = 2m + 2k + \text{Len}(u) = 4k + \text{Len}(u)$, we get $\text{Len}(u) = 1$. Then z_1 is conjugate to an element of length 1, contradicting the fact that z_1 is of cyclic length 2. Thus $k > m$.

If $k > m + 1$, then, by maximality of m , we conjugate z_1 to a word

$$u(t_0^{-1}y)^{k-m-1}t_0^{-1}y,$$

where u might or might not start with y^{-1} , but if it starts with y^{-1} , the second letter of u would not be t_0 . Thus, whether or not u starts with y^{-1} , we know that $u(t_0^{-1}y)^{k-m-1}t_0^{-1}y$ has cyclic length at least 3 despite being conjugate to z_1 , a contradiction. Thus we know precisely that $k = m + 1$, and so z_1 is conjugate to $ut_0^{-1}y$. If u does start with y^{-1} , say $u \equiv y^{-1}v$ for some word v , then we get z_1 conjugate to vt_0^{-1} . Since the cyclic length of z_1 is 2, we know that v is nonempty, must start with a letter from L_1 and that letter must not be t_0 by the maximality of m . Then $v \equiv tv_0$ with $t \in L_1 \setminus \{1, t_0^{-1}\}$, and v_0 begins and ends with a letter in L_0 . Thus z_1 is conjugate to $v_0(t_0^{-1}t)$ which is a cyclically reduced word. Since z_1 has cyclic length 2 and cyclic reduction $y^{-1}t_1$, we obtain that $v_0 = y^{-1}$ and $t_0^{-1}t = t_1$. But now

$$z_1 = (y^{-1}t_0)^{k-1}u(t_0^{-1}y)^k = (y^{-1}t_0)^{k-1}y^{-1}(t_0t_1)y^{-1}(t_0^{-1}y)^k$$

so that z_1 is expressed as a product of elements in F_a , contradicting Lemma 2.3 part (ii). Therefore, u must start with some $g \in L_0 \setminus \{1, y^{-1}\}$, say $y \equiv gv_0$. Now we conjugate $ut_0^{-1}y \equiv gv_0t_0^{-1}y$ to $v_0t_0^{-1}(yg)$ which is cyclically reduced (by considering (yg) as a single letter in L_0). But $v_0t_0^{-1}(yg)$ cannot possibly be a cyclic shift of $y^{-1}t_1$, regardless of whether v_0 is empty or not. This finishes the proof of Case A.

Case B: z_1 ends with a letter of L_1 . The reasoning in this case follows that in the other case more or less, and we give the sketch. Arguing as before, we see that z_1 must end with t_0 , and the second-to-last letter must be y^{-1} . Write $z_1 \equiv w(y^{-1}t_0)^k$ with $k \geq 1$ maximal. As before, w is nonempty. Again, we have $\text{Len}(z_0) \geq 2\text{Len}(z_1) - 1 - 4k$ from which $\text{Len}(z_1) \leq 4k + 1$. Writing

$$z_1 \equiv (t_0^{-1}y)^m u (y^{-1}t_0)^k$$

with $m \geq 0$ maximal, we again see that u is nonempty and that $k = m + 1$. Also, u must begin and end with letters from L_1 . Therefore, z_1 is conjugate to $uy^{-1}t_0$. If $u \equiv t_0^{-1}v$ for some v , then we get z_1 conjugate to vy^{-1} , and as m was maximal, the word v starts with an element of L_0 which is not y . Then $v \equiv gv_0$ with $g \in L_0 \setminus \{1, y\}$ and v_0 nontrivial since u , and therefore v , must end in a letter from L_1 . Conjugating vy^{-1} to the cyclically reduced word $v_0(y^{-1}g)$, we must have that this word is a cyclic shift of $y^{-1}t_1$. Then $v_0 = t_1$, but $y^{-1}g$ cannot be y^{-1} since g was nontrivial. Therefore, it must be that $u \equiv tv$ for some $t \in L_1 \setminus \{1, t_0\}$. Conjugating $uy^{-1}t_0 = tvy^{-1}t_0$ to the cyclically reduced word $vy^{-1}(t_0t)$, it must be that this word is a cyclic shift of $y^{-1}t_1$. Then v is empty and $t_0t = t_1$. Therefore,

$$z_1 = (t_0^{-1}y)^{k-1} u (y^{-1}t_0)^k = (t_0^{-1}y)^{k-1} (t_0^{-1}t_1) (y^{-1}t_0)^k,$$

and we have $z_1 \in F_a$, which contradicts Lemma 2.3 part (ii). This completes the proof of Case B and of the lemma. $\square$

3 Proof of the main theorem

The groups that we produce for Theorem 1.1 will follow the induction used in [4, Theorem 2.2], using Construction 2.2 at the key stages. We will use combinatorial principles which follow from $\text{ZFC} + \text{V} = \text{L}$ to rule out any possible free decomposition while ensuring strong κ -freeness and free abelianization. We first review some concepts from set theory.

Definitions 3.1. (see [9]) Recall that a cardinal number is naturally considered as an ordinal number which cannot be injected into a proper initial subinterval of itself. A subset E of an ordinal α is *bounded in α* if there exists $\beta < \alpha$ which

is an upper bound on E . The *cofinality* of an ordinal α is the least cardinal κ for which there exists an unbounded $E \subseteq \alpha$ of cardinality κ . An infinite cardinal κ is *regular* if the cofinality of κ is κ . A subset C of ordinal α is *club* if it is unbounded in α and closed under the order topology in α . The intersection of two club sets in an uncountable regular cardinal is again a club set. A subset E of ordinal α is *stationary* if it has nonempty intersection with every club subset of α . The intersection of a club set and a stationary set in a regular cardinal is again stationary. We mention that weakly compact cardinals are inaccessible, and it is therefore consistent to assume that a universe of set theory does not contain any (see [9, Chapters 9, 17]).

The following is a theorem of Solovay (see [16] or [9, Theorem 8.10]).

Theorem 3.2. *If κ is an uncountable regular cardinal, then each stationary subset of κ can be decomposed as the disjoint union of κ many stationary subsets of κ .*

We quote Jensen's $\diamond_\kappa(E)$ principle ([10, Lemma 6.5] or [9, Assertion 27.16]), remark an easy consequence (see [4, page 97]) and quote one more result of Jensen (see [10, Theorem 5.1], [4, Theorem 1.3]).

Theorem 3.3 (ZFC + $V = L$). *If κ is an uncountable regular cardinal and $E \subseteq \kappa$ is stationary in κ , there exists a sequence $\{S_\alpha\}_{\alpha \in E}$ such that $S_\alpha \subseteq \alpha$ and, for any $J \subseteq \kappa$, the set $\{\alpha \in E \mid J \cap \alpha = S_\alpha\}$ is stationary in κ .*

Remark 3.4. From a sequence given by Theorem 3.3, one obtains a sequence of ordered pairs $\{(T_\alpha^0, T_\alpha^1)\}_{\alpha \in E}$ for which $T_\alpha^0, T_\alpha^1 \subseteq \alpha$ and, given any subsets $A, B \subseteq \kappa$, the set $\{\alpha \in E \mid A \cap \alpha = T_\alpha^0 \text{ and } B \cap \alpha = T_\alpha^1\}$ is stationary in κ . To see this, we give the product $\{0, 1\} \times \kappa$ the lexicographic order. There is an order isomorphism $f: \{0, 1\} \times \kappa \rightarrow \kappa$ under which $f(0, \alpha) = \alpha$ for every limit ordinal $\alpha < \kappa$. Let $f_i: \kappa \rightarrow \kappa$ be given by $f_i(\alpha) = f(i, \alpha)$ for each $i \in \{0, 1\}$, so f_0 has disjoint image from f_1 . Notice that, for each limit ordinal $\alpha < \kappa$ and $X \subseteq \kappa$, we have $f_i(X \cap \alpha) = f_i(X) \cap \alpha$. Letting E_0 be the intersection of E with the set of limit ordinals below κ , we have that E_0 is stationary as the intersection of a stationary with a club. For $\alpha \in E_0$, define $T_\alpha^0 = f_0^{-1}(S_\alpha)$ and $T_\alpha^1 = f_1^{-1}(S_\alpha)$. Given $A, B \subseteq \kappa$, we let $J = f_0(A) \sqcup f_1(B)$ and notice that

$$\begin{aligned} & \{\alpha \in E_0 \mid A \cap \alpha = T_\alpha^0 \text{ and } B \cap \alpha = T_\alpha^1\} \\ &= \{\alpha \in E_0 \mid f_0(A \cap \alpha) \sqcup f_1(B \cap \alpha) = S_\alpha\} \\ &= \{\alpha \in E_0 \mid (f_0(A) \cap \alpha) \sqcup (f_1(B) \cap \alpha) = S_\alpha\} \\ &= \{\alpha \in E_0 \mid J \cap \alpha = S_\alpha\} \end{aligned}$$

is stationary in κ . Letting $T_\alpha^0 = \emptyset = T_\alpha^1$ for $\alpha \in E \setminus E_0$ gives the desired sequence of ordered pairs.

Theorem 3.5 (ZFC+ V = L). *If κ is an uncountable regular cardinal which is not weakly compact, then there is a stationary subset $E \subseteq \kappa$ for which each element of E is a limit ordinal of cofinality ω and, for each limit ordinal $\alpha < \kappa$, the set $E \cap \alpha$ is not stationary in α .*

Proof of Theorem 1.1. Let κ be as in the hypotheses of Theorem 1.1. We inductively define a group structure on κ , and this will serve as our group G . Let $\gamma_0 = \omega$, $\gamma_{\alpha+1} = \gamma_\alpha + \gamma_\alpha$ and $\gamma_\beta = \bigcup_{\alpha < \beta} \gamma_\alpha$ for limit ordinal $\beta < \kappa$. The set $\{\gamma_\alpha\}_{\alpha < \kappa}$ is obviously a club set in κ . We will define a group structure on each γ_α so that γ_α will be a subgroup of γ_β whenever $\alpha < \beta$, and thus the group structure on $\kappa = \bigcup_{\alpha < \kappa} \gamma_\alpha$ will be well defined.

Select stationary $E_0 \subseteq \kappa$ satisfying the conclusion of Theorem 3.5 and such that $\kappa \setminus E_0$ is stationary in κ (using Theorem 3.2). It is easy to check that $E = \{\gamma_\alpha\}_{\alpha \in E_0}$ is also stationary in κ . Let $\{(T_{\gamma_\alpha}^0, T_{\gamma_\alpha}^1)\}_{\gamma_\alpha \in E}$ be a sequence as in Remark 3.4.

The following properties will hold on the groups for all $\alpha < \beta < \kappa$:

- (i) γ_β is free of infinite rank;
- (ii) γ_α is a proper subgroup of γ_β ;
- (iii) γ_α is a free factor of γ_β if and only if $\gamma_\alpha \notin E$;
- (iv) $\gamma'_\alpha = \gamma'_\beta \cap \gamma_\alpha$.

We let γ_0 be free of countably infinite rank. Suppose that we have defined the group structure on γ_α for all $\alpha < \beta < \kappa$ such that the above conditions hold. If $\beta = \delta + 1$ and $\gamma_\delta \notin E$, then we give γ_β the group structure obtained by any bijection $f: \gamma_\beta \rightarrow \gamma_\delta * \mathbb{Z}$ such that $f \upharpoonright \gamma_\alpha$ is the identity map. Such a bijection exists since $|\gamma_\beta \setminus \gamma_\delta| = |\gamma_\delta|$. Conditions (i), (ii) and (iv) obviously hold. Notice also that, for $\alpha < \beta$, we have γ_α a free factor of γ_β if and only if γ_α is a free factor of γ_δ (this uses Lemma 2.1), and so (iii) also holds.

If $\beta = \delta + 1$ and $\gamma_\delta \in E$, then we consider two subcases. Firstly, suppose that there exists a strictly increasing sequence $\{\alpha_n\}_{n \in \omega}$ for which

- $\gamma_{\alpha_n} \notin E$,
- $\bigcup_{n \in \omega} \alpha_n = \delta$,
- $\gamma_{\alpha_n} = (\gamma_{\alpha_n} \cap T_{\gamma_\delta}^0) * (\gamma_{\alpha_n} \cap T_{\gamma_\delta}^1)$ with both free factors nontrivial for all $n \in \omega$.

From this, it follows immediately that $\gamma_\delta = (\gamma_\delta \cap T_{\gamma_\delta}^0) * (\gamma_\delta \cap T_{\gamma_\delta}^1) = T_{\gamma_\delta}^0 * T_{\gamma_\delta}^1$. Since $\{\gamma_{\alpha_n}\}_{n \in \omega}$ is a strictly increasing sequence of sets, we know for each $n \in \omega$

that it is either the case that

$$\gamma_{\alpha_{n+1}} \cap T_{\gamma_\delta}^0 \not\supseteq \gamma_{\alpha_n} \cap T_{\gamma_\delta}^0 \quad \text{or that} \quad \gamma_{\alpha_{n+1}} \cap T_{\gamma_\delta}^1 \not\supseteq \gamma_{\alpha_n} \cap T_{\gamma_\delta}^1.$$

By choosing a subsequence, we can assume without loss of generality that

$$\gamma_{\alpha_{n+1}} \cap T_{\gamma_\delta}^1 \not\supseteq \gamma_{\alpha_n} \cap T_{\gamma_\delta}^1 \quad \text{for all } n \in \omega$$

(else we exchange the roles of $T_{\gamma_\delta}^0$ and $T_{\gamma_\delta}^1$ in the construction that follows). Since each $\gamma_{\alpha_n} \notin E$, we know by our induction that γ_{α_n} is a free factor of γ_δ and also of $\gamma_{\alpha_{n+1}}$. Then, by Lemma 2.1, we know that $\gamma_{\alpha_n} \cap T_{\gamma_\delta}^1$ is a free factor of $\gamma_\delta \cap T_{\gamma_\delta}^1$ and of $\gamma_{\alpha_{n+1}} \cap T_{\gamma_\delta}^1$. Inductively select a free basis Q of $\gamma_\delta \cap T_{\gamma_\delta}^1$ such that $Q \cap \gamma_{\alpha_n} \cap T_{\gamma_\delta}^1$ is a free generating set for $\gamma_{\alpha_n} \cap T_{\gamma_\delta}^1$ for each $n \in \omega$. For each $n \in \omega$, select $t_n \in Q \cap \gamma_{\alpha_{n+1}} \cap T_{\gamma_\delta}^1 \setminus \gamma_{\alpha_n}$, and let $X = Q \setminus \{t_n\}_{n \in \omega}$. Let $\{y\} \sqcup Y$ be a free generating set for $\gamma_\delta \cap T_{\gamma_\delta}^0$. By how the set $\{\gamma_\alpha\}_{\alpha < \kappa}$ was defined, we know $|\gamma_\beta| = |\gamma_\beta \setminus \gamma_\delta| = |\gamma_\delta|$. Hence we let γ_β be given a group structure using a bijection $f: \gamma_\beta \rightarrow F(X) * F(\{y\} \cup Y) * F(\{z_n\}_{n \in \omega})$ such that $f \upharpoonright \gamma_\delta$ is the isomorphism $\gamma_\delta \simeq F(X) * F(\{y\} \cup Y) * F(\{t_n\}_{n \in \omega})$ noted above and the inclusion map $\iota: \gamma_\delta \rightarrow \gamma_\beta$ satisfies $\iota = f^{-1} \circ \phi \circ f \upharpoonright \gamma_\delta$, where ϕ is the map from Construction 2.2.

Certainly, conditions (i), (ii) hold, and condition (iv) holds by Lemma 2.3 (iv). Also, we know γ_δ is not a free factor of γ_β by Lemma 2.3 (iii). Notice that, for each $n \in \omega$, we have that $\langle X \cup \{t_0, \dots, t_{n-1}\} \cup \{y\} \cup Y \rangle$ is a free factor of γ_β by Lemma 2.3 (iii). For each $n \in \omega$, we know γ_{α_n} is a free factor of

$$\langle X \cup \{t_0, \dots, t_{n-1}\} \cup \{y\} \cup Y \rangle,$$

and for each $\alpha < \delta$, there exists $n \in \omega$ for which $\alpha < \alpha_n$. It follows by Lemma 2.1 and by our induction that, for $\alpha < \delta$, we have γ_α a free factor of γ_β if and only if $\gamma_\alpha \notin E$, and condition (iii) holds.

On the other hand, suppose $\beta = \delta + 1$ and $\gamma_\delta \in E$ and no such increasing sequence $\{\alpha_n\}_{n \in \omega}$ exists. Since $\delta \in E_0$ is of cofinality ω and $E_0 \cap \delta$ is not stationary in δ , we may select a strictly increasing sequence $\alpha_n \notin E_0$ such that $\bigcup_{n \in \omega} \alpha_n = \delta$. Then $\gamma_{\alpha_n} \notin E$. As each γ_{α_n} is a free factor of γ_δ and the α_n are strictly increasing, we may select by induction a free generating set Q for γ_δ such that $Q \cap \gamma_{\alpha_n}$ is a free generating set for γ_{α_n} . Pick $y \in \gamma_{\alpha_0} \cap Q$ and $t_n \in \gamma_{\alpha_{n+1}} \cap Q \setminus \gamma_{\alpha_n}$, and letting $X = \emptyset$ and $Y = Q \setminus (\{t_n\}_{n \in \omega} \cup \{y\})$, we can let γ_β be a group freely generated by $X \cup \{z_n\}_{n \in \omega} \cup \{y\} \cup Y$ such that the inclusion map $\iota: \gamma_\delta \rightarrow \gamma_\beta$ is the map ϕ from Construction 2.2. The check that the induction conditions still hold is as in the other subcase.

When $\beta < \kappa$ is a limit ordinal, the binary operation on $\gamma_\beta = \bigcup_{\alpha < \beta} \gamma_\alpha$ is defined by that on the γ_α for $\alpha < \beta$. By how E_0 was chosen, we know $E_0 \cap \beta$ is

not stationary in β , and so we select a club set $C \subseteq \beta$ for which $C \cap E_0 = \emptyset$. By induction, we know that, for $\alpha, \delta \in C$ with $\alpha < \delta$, we have that γ_α is a proper free factor of γ_δ , and since C is closed, we have $\gamma_\delta = \bigcup_{\alpha < \delta, \alpha \in C} \gamma_\alpha$ for any $\delta \in C$ which is a limit under the ordering of κ restricted to C . Then, by induction on C , we can select a free generating set Q for $\gamma_\beta = \bigcup_{\alpha \in C} \gamma_\alpha$ for which $Q \cap \gamma_\alpha$ is a free generating set for γ_α for each $\alpha \in C$. Then (i) and (ii) hold. For $\alpha < \beta$, it is clear by Lemma 2.1 that γ_α is a free factor of γ_β if and only if γ_α is a free factor of γ_δ for some $\delta \in C$ with $\delta > \alpha$, and since γ_δ is a free factor of γ_β for each $\delta \in C$, condition (iii) holds. Condition (iv) follows by induction since $\gamma'_\beta = \bigcup_{\alpha < \beta} \gamma'_\alpha$. This completes the construction of the group structure on κ .

We verify that conditions (1)–(3) of the statement of Theorem 1.1 hold. Imagine for contradiction that $\kappa = A * B$ with A, B nontrivial subgroups of κ . Letting α' be such that $\gamma_{\alpha'} \cap A$ and $\gamma_{\alpha'} \cap B$ are both nontrivial and

$$C = \{\alpha' \leq \alpha < \kappa \mid \gamma_\alpha = (\gamma_\alpha \cap A) * (\gamma_\alpha \cap B)\},$$

it is straightforward to verify that C is club in κ . Since the set $\kappa \setminus E_0$ is stationary in κ , we know $D = C \setminus E_0$ is stationary and therefore unbounded in κ . Then the closure $\overline{D}$ is club in κ , and so is $\{\gamma_\alpha\}_{\alpha \in \overline{D}}$. Then there exists $\gamma_\delta \in E$ with $\delta \in \overline{D}$ and $A \cap \gamma_\delta = T_{\gamma_\delta}^0$ and $B \cap \gamma_\delta = T_{\gamma_\delta}^1$. Then $\delta \in E_0$, and so $\delta \in E_0 \cap \overline{D}$. As $\delta \in E_0$, we know that δ has cofinality ω in κ . Certainly, $\delta \notin D = C \setminus E_0$, and so there exists a strictly increasing sequence $\{\alpha_n\}_{n \in \omega}$ with $\alpha_n \in C \setminus E_0$ such that $\bigcup_{n \in \omega} \alpha_n = \delta$. Then

$$\gamma_{\alpha_n} = (\gamma_{\alpha_n} \cap A) * (\gamma_{\alpha_n} \cap B) = (\gamma_{\alpha_n} \cap T_{\gamma_\delta}^0) * (\gamma_{\alpha_n} \cap T_{\gamma_\delta}^1).$$

By our construction, we know that γ_δ includes into $\gamma_{\delta+1}$ in such a way that $\gamma_{\delta+1}$ is not a subgroup of $A * B$ (using Lemma 2.4 and letting $(A, B) = (L_0, L_1)$ or $(A, B) = (L_1, L_0)$ depending on whether we selected $\{t_n\}_{n \in \omega} \subseteq T_{\gamma_\delta}^1$ or, respectively, $\{t_n\}_{n \in \omega} \subseteq T_{\gamma_\delta}^0$ in the construction), and this is a contradiction. Thus κ is freely indecomposable, and we have part (1).

For part (2), we let $X \subseteq \kappa$ with $|X| < \kappa$. By the regularity of κ , select $\alpha < \kappa$ large enough that $\gamma_\alpha \supseteq X$. Notice that $\gamma_{\alpha+1} \notin E$. Any subgroup H of κ with $|H| < \kappa$ and $H \geq \gamma_{\alpha+1}$ satisfies $H \leq \gamma_\beta$ for some $\beta > \alpha$ by the regularity of κ . Since $\gamma_{\alpha+1}$ is a free factor of γ_β by our construction, we have by Lemma 2.1 that $\gamma_{\alpha+1}$ is a free factor of H . Thus $\gamma_{\alpha+1}$ is κ -pure, and the group κ is strongly κ -free, and we have verified (2).

For part (3), we notice that, since

$$\gamma'_\alpha = \gamma_\alpha \cap \gamma'_\beta \quad \text{for each } \alpha < \beta, \quad \text{and since} \quad \kappa' = \bigcup_{\beta < \kappa} \gamma'_\beta,$$

the equality $\gamma'_\alpha = \gamma_\alpha \cap \kappa'$ holds for all $\alpha < \kappa$. In particular, the homomorphism induced by the inclusion map $\gamma_\alpha/\gamma'_\alpha \rightarrow \kappa/\kappa'$ is injective, and so the abelianization of κ is the increasing union of free abelian subgroups $\gamma_\alpha/\gamma'_\alpha$. In our construction, when $\beta = \delta + 1$ and $\gamma_\delta \in E$, the map induced by inclusion $\gamma_\delta/\gamma'_\delta \rightarrow \gamma_\beta/\gamma'_\beta$ is an isomorphism by Lemma 2.3 (iv). When $\beta = \delta + 1$ and $\gamma_\delta \notin E$, we have $\gamma_\delta/\gamma'_\delta$ as a proper direct summand of $\gamma_\beta/\gamma'_\beta$. As well, we have $\gamma_\beta/\gamma'_\beta = \bigcup_{\alpha < \beta} \gamma_\alpha/\gamma'_\alpha$ for limit $\beta < \kappa$. Thus we may inductively select a free abelian basis for the abelianization of κ , and the free basis will be of cardinality κ since there are κ many β for which $\beta = \delta + 1$ with $\gamma_\delta \notin E$. We have verified (3) and finished the proof of the theorem. $\square$

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