On $(n + \frac{1}{2})$ -Engel groups

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Communicated by Evgenii I. Khukhro

Abstract. Let n be a positive integer. We say that a group G is an $(n + \frac{1}{2})$ -Engel group if it satisfies the law $[x, {}_n y, x] = 1$. The variety of $(n + \frac{1}{2})$ -Engel groups lies between the varieties of n -Engel groups and $(n + 1)$ -Engel groups. In this paper, we study these groups, and in particular, we prove that all $(4 + \frac{1}{2})$ -Engel $\{2, 3\}$ -groups are locally nilpotent. We also show that if G is a $(4 + \frac{1}{2})$ -Engel p -group, where $p \geq 5$ is a prime, then G^p is locally nilpotent.

1 Introduction

Let G be a group and $g, h \in G$. The commutator of g and h is the element

$$[g, h] = g^{-1}h^{-1}gh \in G.$$

We recursively define $[g, {}_n h]$, where n is a positive integer, as $[g, {}_1 h] = [g, h]$ and $[g, {}_{n+1} h] = [[g, {}_n h], h]$ for $n \geq 1$. A subset $S \subseteq G$ is an Engel set of G if, for every $g, h \in S$, there is a positive integer $k = k(g, h)$ such that $[g, {}_k h] = 1$. If k is bounded above by some positive integer n , we say that S is an n -Engel subset, and if furthermore $G = S$, then G is an n -Engel group. Recall that every 2-Engel group is nilpotent of class at most 3. By a classic result of Heineken [4], every 3-Engel group is locally nilpotent, and this result was later generalized to include all 4-Engel groups [3] (see also [9]).

Recall that an element $a \in G$ is said to be left n -Engel if $[x, {}_n a] = 1$ for all $x \in G$ and right n -Engel if $[a, {}_n x] = 1$ for all $x \in G$. We denote the subset of left n -Engel elements by $L_n(G)$ and that of right n -Engel elements by $R_n(G)$.

Definition. Let G be a group and n a positive integer.

- (1) We say that $a \in G$ is a left $(n + \frac{1}{2})$ -Engel element if $[x, {}_n a, x] = 1$ for all $x \in G$.
- (2) We say that $a \in G$ is a right $(n + \frac{1}{2})$ -Engel element if $[a, {}_n x, a] = 1$ for all $x \in G$.
- (3) We say that G is an $(n + \frac{1}{2})$ -Engel group if it satisfies the law $[x, {}_n y, x] = 1$.

We denote the subset of left $(n + \frac{1}{2})$ -Engel elements by $L_{n+\frac{1}{2}}(G)$ and the right $(n + \frac{1}{2})$ -Engel elements by $R_{n+\frac{1}{2}}(G)$. Thus G is an $(n + \frac{1}{2})$ -Engel group if and only if $L_{n+\frac{1}{2}}(G) = G$ or equivalently $R_{n+\frac{1}{2}}(G) = G$. We denote the variety of m -Engel groups by $\mathcal{E}_m$.

Remark. It is not difficult to prove that

$$L_{1+\frac{1}{2}}(G) = R_2(G) \quad \text{and} \quad R_{1+\frac{1}{2}}(G) = L_2(G).$$

Thus, in particular, $\mathcal{E}_{1+\frac{1}{2}} = \mathcal{E}_2$.

Lemma 1.1. *Let G be a group and n a positive integer. We have*

$$L_n(G) \subseteq L_{n+\frac{1}{2}}(G) \subseteq L_{n+1}(G).$$

In particular, $\mathcal{E}_n \subseteq \mathcal{E}_{n+\frac{1}{2}} \subseteq \mathcal{E}_{n+1}$.

Proof. That $L_n(G) \subseteq L_{n+\frac{1}{2}}(G)$ is obvious. To see that $L_{n+\frac{1}{2}}(G) \subseteq L_{n+1}(G)$, let $a \in L_{n+\frac{1}{2}}(G)$. Then, for any $x \in G$, we have

$$1 = [ax, {}_n a, ax] = [x, {}_n a, x][x, {}_{n+1} a]^x = [x, {}_{n+1} a]^x.$$

Thus $[x, {}_{n+1} a] = 1$ and $a \in L_{n+1}(G)$. □

Our main results on $(n + \frac{1}{2})$ -Engel groups are the following.

Theorem B. *Let G be a $(4 + \frac{1}{2})$ -Engel $\{2, 3\}$ -group. Then G is locally nilpotent.*

Theorem C. *Let G be a $(4 + \frac{1}{2})$ -Engel p -group, where p is a prime and $p \geq 5$. Then G^p is locally nilpotent.*

A major ingredient to the proofs is a result on Engel sets that is also of independent interest. Let $e_n(x, y)$ be the n -Engel word $[x, {}_n y]$.

Theorem A. *Let $R = \langle a, b \rangle$ be the largest 2-generator group satisfying the relations $e_3(a, b) = e_3(b, a) = e_3(a^{-1}, b^{-1}) = e_3(b^{-1}, a^{-1}) = 1$. Then R is nilpotent of class 4.*

We will see later that these relations imply that $S = \{a, b, a^{-1}, b^{-1}\}$ is a 3-Engel subset of R .

2 Proof of Theorem A

Consider the n -Engel word $e_n(x, y) = [x, {}_n y] = 1$. As we will focus in particular on the 3-Engel word, we will often use $e(x, y)$ instead of $e_3(x, y)$.

Lemma 2.1. Suppose G is a group with elements a, b , where

$$e(a, b) = e(a^{-1}, b^{-1}) = 1.$$

Then $\langle b, b^a \rangle = \langle b, [a, b] \rangle$ is nilpotent of class at most 2.

Proof. From the equations

$$\begin{aligned} 1 &= [a, b, b, b] = [b^{-a}b, b, b] = [b^{-a}, b, b]^b, \\ 1 &= [a^{-1}, b^{-1}, b^{-1}, b^{-1}] = [b^{a^{-1}}, b^{-1}, b^{-1}]^{b^{-1}}, \end{aligned}$$

we see that $1 = [b^{-a}, b, b]$ and $1 = [b^{a^{-1}}, b^{-1}, b^{-1}]^a = [b, b^{-a}, b^{-a}]$. Thus we have that $\langle b, b^a \rangle$ is nilpotent of class at most 2. $\square$

Lemma 2.2. Let G be a group with elements a, b , where

$$e(a, b) = e(a^{-1}, b^{-1}) = 1.$$

Then $[a^\epsilon, b^{\epsilon_1}, b^{\epsilon_2}, b^{\epsilon_3}] = 1$ for all $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$.

Proof. By symmetry, it suffices to deal with the case when $\epsilon = 1$. As

$$[a, b^{\epsilon_1}, b^{\epsilon_2}, b^{-1}] = [a, b^{\epsilon_1}, b^{\epsilon_2}, b]^{-b^{-1}},$$

we can also assume that $\epsilon_3 = 1$. Then, from

$$[a, b^{\epsilon_1}, b^{-1}, b] = [[a, b^{\epsilon_1}, b]^{-1}, b]^{b^{-1}} = [a, b^{\epsilon_1}, b, b]^{-[a, b^{\epsilon_1}, b]^{-1}b^{-1}},$$

we can also assume, without loss of generality, that $\epsilon_2 = 1$. We are thus only left with showing that $[a, b^{-1}, b, b] = 1$, but this follows from Lemma 2.1 and the fact that $[a, b^{-1}, b, b] = [b^a b^{-1}, b, b] \in \gamma_3(\langle b, b^a \rangle)$. $\square$

Remark. It follows from Lemma 2.2 that if

$$e(a, b) = e(a^{-1}, b^{-1}) = e(b, a) = e(b^{-1}, a^{-1}) = 1,$$

then $S = \{a, b, a^{-1}, b^{-1}\}$ is a 3-Engel subset.

Lemma 2.3. Let G be a group with elements a, b satisfying

$$e(a, b) = e(a^{-1}, b^{-1}) = 1.$$

Then $[[a, b, b]^{-1}, a]$ commutes with b^a .

Proof. From the Hall–Witt identity and Lemma 2.1, we have

$$\begin{aligned} 1 &= [[a, b, b], a^{-1}, b]^a [a, b^{-1}, [a, b, b]]^b [b, [a, b, b]^{-1}, a]^{[a, b, b]} \\ &= [a, b, b, a^{-1}, b]^a = [[a, b, b, a]^{-1}, b^a]. \end{aligned}$$

It follows that $[a, b, b, a]$ commutes with b^a , and thus, using Lemma 2.1 again, $[[a, b, b]^{-1}, a] = [a, b, b, a]^{-[a, b, b]^{-1}}$ commutes with b^a as well. $\square$

Proof of Theorem A. Let $R = \langle a, b \rangle$ be the largest group satisfying the relations

$$e(a, b) = e(b, a) = e(a^{-1}, b^{-1}) = e(b^{-1}, a^{-1}) = 1.$$

By the remark after Lemma 2.2, we know that $S = \{a, b, a^{-1}, b^{-1}\}$ is a 3-Engel subset of R .

In order to show that R is nilpotent of class at most 4, we need to show that $a, b \in Z_4(R)$. This is equivalent to showing that $[b, a] \in Z_3(R)$. As $\langle a, [a, b] \rangle$ and $\langle b, [a, b] \rangle$ are nilpotent of class at most 2, we see that $[a, b, a] = [b, a, a]^{-1}$ and $[b, a, b] = [a, b, b]^{-1}$. In order to show that $[b, a] \in Z_3(R)$, we need to show that $[b, a, a]$ and $[b, a, b] = [a, b, b]^{-1}$ are in $Z_2(R)$. As $[b, a, a, a] = [a, b, b, b] = 1$, it suffices to show that $[[b, a, a]^{-1}, b], [[a, b, b]^{-1}, a] \in Z(R)$. In the following calculations, we again use the fact that $\langle b, [a, b] \rangle$ and $\langle a, [a, b] \rangle$ are nilpotent of class at most 2. We have

$$\begin{aligned} [b, a, a][b, a, a, b] &= [b, a, a]^b = [[b, a]^b, a^b] = [[b, a, b][b, a], a[a, b]] \\ &= [[b, a, b][b, a], a]^{[a, b]} = [b, a, b, a]^{[b, a][a, b]}[b, a, a]^{[a, b]} \\ &= [[a, b, b]^{-1}, a][b, a, a]. \end{aligned}$$

Thus

$$[[a, b, b]^{-1}, a] = [b, a, a, b]^{[b, a, a]^{-1}} = [[b, a, a]^{-1}, b]^{-1}. \quad (2.1)$$

From (2.1), we thus see that it suffices to show that $[[a, b, b]^{-1}, a] \in Z(R)$, and in fact, it suffices to show that $[[a, b, b]^{-1}, a]$ commutes with b as then, by symmetry, the RHS of (2.1) commutes with a , and thus $[[a, b, b]^{-1}, a]$ commutes then with a as well.

From Lemma 2.2, we know that

$$e(a^\alpha, b^\beta) = e(b^\beta, a^\alpha) = 1 \quad \text{for all } \alpha, \beta \in \{1, -1\}.$$

In particular, equation (2.1) holds if we replace a by a^{-1} or b by b^{-1} . Calculating in the group $\langle a, a^b \rangle$ that is nilpotent of class at most 2, we see that $[b, a^{-1}, a^{-1}] = [a^b, a^{-1}] = [a^{-b}, a] = [b, a, a]$. From this and (2.1), it follows that $[[a, b, b]^{-1}, a]$ is invariant under replacing a by a^{-1} . Notice also that

$$[a^{-1}, b, b] = [b^{-a^{-1}}, b] = [b^{-1}, b^a]^{a^{-1}} = [b^{-a}, b]^{-a^{-1}} = [a, b, b]^{-a^{-1}}.$$

Thus

$$\begin{aligned} [[a, b, b]^{-1}, a] &= [[a^{-1}, b, b]^{-1}, a^{-1}] = [[a, b, b], a^{-1}]^{a^{-1}} \\ &= [a, b, b, a]^{-a^{-2}} = [[a, b, b]^{-1}, a]^{[a, b, b]a^{-2}}. \end{aligned}$$

But, from Lemma 2.3 and (2.1), we know that $[[a, b, b]^{-1}, a] = [[b, a, a]^{-1}, b]^{-1}$ commutes with b^a and a^b . Replacing a by a^{-1} for the RHS, we see that the com-

mon element also commutes with $b^{a^{-1}}$. Likewise, it commutes with $a^{b^{-1}}$. As

$$\begin{aligned} [a, b, b]a^{-2} &= [b, a]b^{-1}[a, b]ba^{-2} \\ &= a^{-b}ab^{-1}a^{-1}a^ba^{-2b^{-1}}b = a^{-b}b^{-a^{-1}}a^ba^{-2b^{-1}}b, \end{aligned}$$

it follows that $[[a, b, b]^{-1}, a]$ commutes with b .

As R is nilpotent of class at most 4, it follows that R is metabelian, and using the nilpotent quotient algorithm `nq` of Nickel [8] (which is implemented in `GAP` [10]), one can see that the class is exactly 4. It turns out that R is torsion-free with $R' \cong \mathbb{Z}^4$. $\square$

Remark. An interesting related result [1, Proposition 3.1] states that if $S = \{a, b\}$ with $[a, b, b] = [b, a, a, a] = 1$, then $\langle S \rangle$ is nilpotent of class at most 3.

The following examples show that the hypotheses of Theorem A cannot be weakened.

Example 1 ([1, Example 4.2]). Let x and y be elements of S_{12} defined by

$$\begin{aligned} x &= (1, 2)(3, 4)(5, 6)(7, 9, 10, 8)(11, 12), \\ y &= (1, 3)(2, 4, 5, 7)(6, 8)(9, 11)(10, 12). \end{aligned}$$

Then $o(x) = 4 = o(y)$, $[x, y, y, y] = 1 = [y, x, x, x]$ and $G = \langle x, y \rangle$ has order $2^5 \cdot 3^4$, so, in particular, G is not nilpotent.

Example 2. Let x and y be elements of S_{12} defined by

$$\begin{aligned} x &= (1, 2, 3, 4)(5, 6, 8, 10)(7, 9, 11, 12), \\ y &= (1, 3)(2, 4, 5, 7)(6, 9)(8, 11)(10, 12). \end{aligned}$$

Then $o(x) = 4 = o(y)$,

$$\begin{aligned} [x, y, y, y] &= 1 = [x^{-1}, y, y, y], \\ [y, x, x, x, x] &= 1 = [y^{-1}, x, x, x, x] \end{aligned}$$

and $G = \langle x, y \rangle$ has order $2^6 \cdot 3^4$, so, in particular, G is not nilpotent.

3 Proofs of Theorem B and Theorem C

Lemma 3.1. Let $G = \langle x, y \rangle$ be a group, $y \in L_{n+\frac{1}{2}}(G)$. Then $[x, {}_ny] \in Z(G)$.

Proof. That $[x, {}_ny]$ commutes with x is a direct consequence of $y \in L_{n+\frac{1}{2}}(G)$. Then $[yx, {}_ny, yx] = [x, {}_ny, yx] = [x, {}_ny, x][x_{n+1}y]^x = [x, {}_{n+1}y]^x$ shows that $[x, {}_ny]$ commutes also with y . $\square$

Lemma 3.2. *Let G be a group, and let $a, b \in G$. Suppose that, for some $n \geq 2$, we have that $\{a, b, a^{-1}, b^{-1}\}$ is an n -Engel subset of G . Then*

$$e_{n-1}(b^{-a}, b) = e_{n-1}(b, b^{-a}) = e_{n-1}(b^a, b^{-1}) = e_{n-1}(b^{-1}, b^a) = 1.$$

Proof. We have

$$1 = [a, {}_nb] = [b^{-a}b, {}_{n-1}b] = [b^{-a}, {}_{n-1}b]^b$$

and therefore $[b^{-a}, {}_{n-1}b] = 1$. Replacing b by b^{-1} , we see that $[a, {}_nb^{-1}] = 1$ implies $[b^a, {}_{n-1}b^{-1}] = 1$. Next use $[a^{-1}, {}_nb] = 1$, which implies $[b^{-a^{-1}}, {}_{n-1}b] = 1$ and thus, after conjugation by a , that $[b^{-1}, {}_{n-1}b^a] = 1$. Replacing b by b^{-1} , we see that $[a^{-1}, {}_nb^{-1}] = 1$ implies $[b, {}_{n-1}b^{-a}] = 1$. $\square$

Proposition 3.3. *Let G be a $(4 + \frac{1}{2})$ -Engel 2-group. Then G is locally nilpotent.*

Proof. Taking the quotient of G by the Hirsch–Plotkin radical, we can assume that $\text{HP}(G) = 1$, and we want to show that $G = 1$. We argue by contradiction and suppose $G \neq 1$. As groups of exponent 4 are locally finite, there must be an element $g \in G$ of order 8. We get a contradiction by showing that $\langle g^4 \rangle^G$ is abelian and thus $g^4 \in \text{HP}(G) = 1$.

Let $h \in G$, and consider the subgroup $H = \langle g, g_1 \rangle$, where $g_1 = g^{-h}$. Let

$$\bar{H} = H/Z(G) = \langle \bar{g}, \bar{g}_1 \rangle,$$

where $\bar{g} = gZ(H)$ and $\bar{g}_1 = g_1Z(H)$. By Lemma 3.2, we know that

$$e(g, g_1) = e(g_1, g) = e(g^{-1}, g_1^{-1}) = e(g_1^{-1}, g^{-1}) = 1.$$

By Theorem A, we then know that $\bar{H}$ and therefore H is finite. Using GAP or MAGMA, one can then check that $[g^{-4h}, g^4] = [g_1^4, g^4] = 1$, and thus we have shown that $\langle g^4 \rangle^G$ is abelian. $\square$

Proposition 3.4. *Let G be a $(4 + \frac{1}{2})$ -Engel 3-group. Then G is locally nilpotent.*

Proof. As before, we can assume that $\text{HP}(G) = 1$, and the aim is then to show that $G = 1$. We argue by contradiction and suppose that $G \neq 1$. As groups of exponent 3 are locally finite, there must be an element $g \in G$ of order 9. Let $h \in G$ and $g_1 = g^{-h}$. As in the proof of Proposition 3.3, one sees that H is finite and then, with the help of GAP or MAGMA, that $[g_1^3, g^3, g^3] = 1$. Thus $[h, g^3, g^3, g^3] = 1$ for all $h \in G$, and thus g^3 is a left 3-Engel element of G . By the main result of [5], we then know that $g^3 \in \text{HP}(G) = 1$, which contradicts the fact that $o(g) = 9$. $\square$

Lemma 3.5. *Let G be a group, and let $a, b \in G$ be two elements of finite order such that $S = \{a, b, a^{-1}, b^{-1}\}$ is a 4-Engel set. Then every prime divisor of $o([a, b])$ is a divisor of $o(a)$ and $o(b)$. In particular, if a and b are of coprime order, then $[a, b] = 1$.*

Proof. By Lemma 3.2 together with Lemma 2.2, we know that

$$S_1 = \{b^a, b, b^{-a}, b^{-1}\} \quad \text{and} \quad S_2 = \{a^b, a, a^{-1}, a^{-b}\}$$

are 3-Engel subsets of G . By Theorem A, we know that

$$H_1 = \langle a, a^b \rangle \quad \text{and} \quad H_2 = \langle b, b^a \rangle$$

are nilpotent. As these groups are nilpotent, we know that every prime divisor of $|H_1|$ divides $o(a)$ and every prime divisor of $|H_2|$ divides $o(b)$. Now we have $[a, b] \in H_1 \cap H_2$, and thus $o([a, b])$ divides $|H_1|$ and $|H_2|$ and thus also $o(a)$ and $o(b)$ from the discussion above. $\square$

Proof of Theorem B. Let G be a $\{2, 3\}$ -group that is $(4 + \frac{1}{2})$ -Engel. Let H_2 be the set consisting of all elements in G whose order is a power of 2 and H_3 that of those elements whose order is a power of 3. In view of Propositions 3.3 and 3.4, it suffices to show that H_2 and H_3 are subgroups and that G is a direct product of H_2 and H_3 . Now take any two elements $a, b \in G$ of coprime orders, and let $T = \langle a, b \rangle$. By Lemma 3.1, we know that $e(a^\alpha, b^\beta) = e(b^\beta, a^\alpha) \in Z(T)$ for all $\alpha, \beta \in \{1, -1\}$. By Lemma 3.5, it follows that $[a, b] \in Z(T)$. Thus T is nilpotent, and as a and b are of coprime order, it follows that $[a, b] = 1$. Now let $a \in H_2$, and let $b \in \langle H_2 \rangle$ be an element that has odd order. By the argument above, we know that $[a, b] = 1$, and as $a \in H_2$ was arbitrary, we see that $b \in Z(\langle H_2 \rangle)$. Thus $\langle H_2 \rangle / Z(\langle H_2 \rangle)$ is a 2-group, and by Proposition 3.3, it is locally nilpotent and thus so is $\langle H_2 \rangle$. As $\langle H_2 \rangle$ is generated by 2-elements, it is then a 2-group, and thus $\langle H_2 \rangle = H_2$, and thus H_2 is a subgroup. The proof that H_3 is a subgroup is similar, using Proposition 3.4. Now let $a \in H_2$ and $b \in H_3$. Then $[a, b] \in H_2 \cap H_3$ and is thus trivial. Hence G is a direct product of H_2 and H_3 and thus locally nilpotent. $\square$

Lemma 3.6. *Let $p \geq 5$ be a prime, and consider the group $G = \langle x, y \rangle$, where $x^{p^2} = y^{p^2} = 1$ and $\{x, y, x^{-1}, y^{-1}\}$ is a 3-Engel set. Then G has exponent p^2 , and G^p is abelian.*

Proof. By Theorem A, we know that G is nilpotent of class at most 4. Then G is regular, and it follows easily that $G^{p^2} = 1$ and then that $[G^p, G^p] = 1$ (for definition and properties of regular p -groups, see [2, Section 12.4], in particular Theorem 12.4.3). $\square$

Proof of Theorem C. Let $p \geq 5$ be a prime, and let G be a $(4 + \frac{1}{2})$ -Engel p -group. Consider $H = G/\text{HP}(G)$, where $\text{HP}(G)$ is the Hirsch–Plotkin radical of G . The aim is to show that H is of exponent p . Passing from G to H , we can thus without loss of generality assume that the Hirsch–Plotkin radical of G is trivial, and the aim is to show that G is then of exponent p . We argue by contradiction and suppose that G has an element g of order p^2 . Let $h \in G$, and consider the subgroup $H = \langle g, g_1 \rangle$, where $g_1 = g^{-h}$. Let $\overline{H} = H/Z(H) = \langle \bar{g}, \bar{g}_1 \rangle$, where $\bar{g} = gZ(H)$ and $\bar{g}_1 = g_1Z(H)$. By Lemma 3.2, we know that

$$e(g, g_1) = e(g_1, g) = e(g^{-1}, g_1^{-1}) = e(g_1^{-1}, g^{-1}) = 1.$$

By Theorem A, we then know that $\overline{H}$ is finite, and thus also H is finite. By Lemma 3.6, we know that $[\bar{g}_1^p, \bar{g}^p] = 1$, that is, $[g_1^p, g^p] \in Z(H)$, and thus, in particular, $[h, g^p, g^p, g^p] = [g_1^p, g^p, g^p] = 1$. Thus g^p is a left 3-Engel element of odd order in G . By the main result of [5], it follows that $g^p \in \text{HP}(G) = 1$, which contradicts our assumption that $o(g) = p^2$. $\square$

Remark. The variety $\mathcal{E}_{n+\frac{1}{2}}$ seems to be the “engelization” of the variety of groups satisfying the law $[y, x_1, x_2, \dots, x_n, y] = 1$ studied by Macdonald in [6, 7].

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Received July 17, 2019; revised November 29, 2019.

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