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On semiconcise words

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Abstract. Let w be a group-word. For a group G, let G_w denote the set of all w-values in G and $w(G)$ the verbal subgroup of G corresponding to w. The word w is semiconcise if the subgroup $[w(G), G]$ is finite whenever G_w is finite. The group G is an FC(w)-group if the set of conjugates x^{G_w} is finite for all $x \in G$. We prove that if w is a semiconcise word and G is an FC(w)-group, then the subgroup $[w(G), G]$ is FC-embedded in G, that is, the intersection $C_G(x) \cap [w(G), G]$ has finite index in $[w(G), G]$ for all $x \in G$. A similar result holds for BFC(w)-groups, that are groups in which the sets x^{G_w} are boundedly finite. We also show that this is no longer true if w is not semiconcise.

1 Introduction

Let $w = w(x_1, \dots, x_n)$ be a group-word in the variables $x_1, \dots, x_n$. For any group G and arbitrary $g_1, \dots, g_n \in G$, the elements of the form $w(g_1, \dots, g_n)$ are called the w-values in G. The set of all w-values in G is denoted by G_w. Clearly, any conjugate of a w-value is again a w-value, and so G_w is a normal set. The verbal subgroup of G corresponding to w is the subgroup $w(G)$ generated by all w-values in G.

A word w is called concise if the verbal subgroup $w(G)$ is finite in each group G such that G_w is finite. In the sixties, P. Hall conjectured that every word is concise, but his conjecture was refuted in 1989 by S. Ivanov ([8], see also [9]). However, many words of common use are known to be concise (see, e.g., [3, 6, 10, 11]).

Let w be any group-word, and suppose that G_w is finite for a group G. It is well known that the derived group $w(G)'$ is always finite and its order is bounded by a function depending only on the order of G_w. In [6], it has been proved that the same is true for the subgroup $[w(G), G]$, when w is the n-Engel word

$$[x,_n y] = [x, \underbrace{y, \dots, y}_{n}].$$

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Motivated by this, we say that a word w is *semiconcise* if the finiteness of G_w for a group G always implies the finiteness of the subgroup $[w(G), G]$.

Of course, Engel words and concise words are semiconcise. Moreover, if w is a semiconcise word and z is any variable not appearing in w, then the word $[w, z]$ is semiconcise (see Proposition 4.2). Thus the word $[x,_n y, z]$ is semiconcise. So far, we do not know whether there exists a semiconcise word which is not concise. A suitable modification of Ivanov's example, given in [1], shows that there exists a word which is not semiconcise (see Section 4).

Further information on the subgroup $[w(G), G]$ when w is a semiconcise word can be obtained by using a verbal generalization of FC-groups, namely groups with finite conjugacy classes. For subsets X and Y of a group G, we write X^Y to denote the set of conjugates $\{x^y \mid x \in X,\ y \in Y\}$. A subgroup H of a group G is said to be FC-embedded in G if x^H is finite for all $x \in G$. The subgroup H is BFC-embedded in G if x^H is finite for all $x \in G$ and the number of elements in x^H is bounded by a constant that does not depend on the choice of x.

For any group-word w, a group G is said to be an FC(w)-group if x^{G_w} is finite for all $x \in G$, and a BFC(w)-group if x^{G_w} is finite for all $x \in G$ and the number of elements in x^{G_w} is bounded by a constant that does not depend on the choice of x. Obvious examples of FC(w)-groups (respectively, BFC(w)-groups) are provided by groups G in which the verbal subgroup $w(G)$ is FC-embedded (respectively, BFC-embedded) in G. On the other side, it has been proved in [7] (respectively, in [1]) that if w is a concise word and G is an FC(w)-group (respectively, a BFC(w)-group), then the verbal subgroup $w(G)$ is FC-embedded (respectively, BFC-embedded) in G.

In [2], for an arbitrary group-word w, we proved that if G is an FC(w)-group (respectively, a BFC-group), then $w(G)'$ is FC-embedded (respectively, BFC-embedded) in G. In this paper, we restrict our attention to semiconcise words. More precisely, we prove the following theorems.

Theorem A. *Let w be a semiconcise word, and let G be an* FC(w)*-group. Then* $[w(G), G]$ *is* FC*-embedded in G.*

Theorem B. *Let w be a semiconcise word, and let G be a* BFC(w)*-group. Then* $[w(G), G]$ *is* BFC*-embedded in G.*

We also show that, for a certain word w which is not semiconcise, there is an example of a BFC(w)-group G such that $[w(G), G]$ is not FC-embedded in G (see Proposition 4.4).

2 Proof of Theorem A

For a subset S of a group G, write $S^* = S \cup S^{-1}$. Obviously, if S is a normal set, then S^* is normal too. Moreover, if S is a finite set, we often refer to the "order of S", denoted by $|S|$, to mean "the number of elements in S".

Lemma 2.1. *Let w be a group-word. If G is an* FC(w)*-group, then the conjugacy class $x^{G_w^*}$ is finite for all $x \in G$.*

Proof. By the first statement in [2, Proposition 2.9], a group G is an FC(w)-group if and only if it is an FC(w^{-1})-group. For all $g \in G$, we have $g \in G_w$ if and only if $g^{-1} \in G_{w^{-1}}$. Thus $G_{w^{-1}} = (G_w)^{-1}$, and the result follows. □

Lemma 2.2. *Let w be a group-word, and let G be an* FC(w)*-group. Choose $x \in G$, and denote by A a finite subset of G_w^* such that $x^{G_w^*} = x^A$. Then, for any $j \geq 1$ and $y_1, \ldots, y_j \in G_w^*$, there exist $a_1, \ldots, a_j \in A$ such that $x^{y_1 \ldots y_j} = x^{a_1 \ldots a_j}$.*

Proof. We argue by induction on j. The case $j = 1$ is clear. Let $j > 1$, and assume that $x^{y_1 \ldots y_{j-1}} = x^{a_1 \ldots a_{j-1}}$ with $a_1, \ldots, a_{j-1} \in A$. Then

$$x^{y_1 \ldots y_j} = x^{a_1 \ldots a_{j-1} y_j} = x^{{y_j}^b a_1 \ldots a_{j-1}},$$

where $b = (a_1 \ldots a_{j-1})^{-1}$. Since ${y_j}^b \in G_w^*$, we have $x^{{y_j}^b} = x^{a_j}$ for some $a_j \in A$, and so $x^{y_1 \cdots y_j} = x^{a_j a_1 \ldots a_{j-1}}$. After renumbering the elements $a_i \in A$, we obtain the required result. □

Lemma 2.3. *Let $w = w(x_1, \ldots, x_n)$ be a group-word, and set*

$$v = [w(x_1, \ldots, x_n), x_{n+1}].$$

If G is an FC(w)*-group, then it is an* FC(v)*-group.*

Proof. Let $y \in G_v$. Then there exist $g_1, \ldots, g_n, g_{n+1} \in G$ such that $y = zt$, with $z = w(g_1, \ldots, g_n)^{-1} \in (G_w)^{-1} = G_{w^{-1}}$ and $t = w(g_1, \ldots, g_n)^{g_{n+1}} \in G_w$. By Lemma 2.2, for any $x \in G$, the conjugate x^y can only take finitely many values. This proves the result. □

Lemma 2.4. *Let $w = w(x_1, \ldots, x_n)$ be a semiconcise word, and set*

$$v = [w(x_1, \ldots, x_n), x_{n+1}].$$

Let G be an FC(w)*-group and B a finite subset of G_v^*. Then, for any $x \in G$, there exists a positive integer e such that $b^e \in Z(\langle x, B \rangle)$ for all $b \in B$.*

Proof. Write $B = \{b_1, \dots, b_r\}$, and let x be an arbitrary element of G. For any $b_i \in B$, there exist elements $g_{i_1}, \dots, g_{i_n}, g_{i_{n+1}} \in G$ such that either

$$b_i = [w(g_{i_1}, \dots, g_{i_n}), g_{i_{n+1}}] \quad \text{or} \quad b_i = [w(g_{i_1}, \dots, g_{i_n}), g_{i_{n+1}}]^{-1}.$$

Put

$$J = \langle x, g_{i_j} \mid 1 \le i \le r, 1 \le j \le n+1 \rangle.$$

By [2, Lemma 2.7 (i)], the set $(J/Z(J))_w$ is finite. As w is semiconcise, the subgroup $[w(J/Z(J)), J/Z(J)]$ is finite. Thus $v(J)$ has finite order modulo $Z(J)$, say e. Since $B \subseteq v(J)$, it follows that ${b_i}^e \in Z(J)$ for all i. As $\langle x, B \rangle \le J$, the result follows. □

Proof of Theorem A. Set $v = [w(x_1, \dots, x_n), x_{n+1}]$. Then G is an FC(v)-group by Lemma 2.3. Let x be an arbitrary element of G. By Lemma 2.1, we can choose $b_1, \dots, b_r \in G_v^*$ such that $x^{G_v^*} = \{x^{b_1}, \dots, x^{b_r}\}$. Write $B = \{b_1, \dots, b_r\}$. Define the order $<$ on the set of all (formal) products of the form $b_{i_1} \dots b_{i_j}$, with $1 \le i_k \le r$ and $j \ge 1$, as follows. Put

$$b_{i_1} \dots b_{i_j} < b_{i'_1} \dots b_{i'_{j'}} \tag{2.1}$$

if and only if one of the following conditions is satisfied: $j < j'$, or $j = j'$ and there is a positive integer $l \le j$ such that $i_l < i'_l$ and $i_k = i'_k$ for all $k > l$.

Let y be an arbitrary element of $v(G)$. Then $y = y_1 \dots y_j$, where each $y_i \in G_v^*$. By Lemma 2.2, for all $k \in \{1, \dots, j\}$, there exists an integer $i_k \in \{1, \dots, r\}$ such that $x^y = x^{b_{i_1} \dots b_{i_j}}$. Clearly, we can choose $b_{i_1} \dots b_{i_j}$ to be the smallest (in the sense of the order $<$) product of elements from B such that $x^y = x^{b_{i_1} \dots b_{i_j}}$. Let us now show that $i_1 \ge i_2 \ge \dots \ge i_j$. Suppose to the contrary that $i_k < i_{k+1}$ for some k. Then

$$x^y = x^{b_{i_1} \dots b_{i_{k-1}} b_{i_k} b_{i_{k+1}} b_{i_{k+2}} \dots b_{i_j}} = x^{b_{i_1} \dots b_{i_{k-1}} c b_{i_k} b_{i_{k+2}} \dots b_{i_j}},$$

where $c = b_{i_k} b_{i_{k+1}} b_{i_k}^{-1} \in G_v^*$. In view of Lemma 2.2, we have

$$x^{b_{i_1} \dots b_{i_{k-1}} c} = x^{b_{i'_1} \dots b_{i'_{k-1}} b_{i'_{k+1}}}$$

for some $1 \le i'_1, \dots, i'_{k-1}, i'_{k+1} \le r$ so that

$$x^y = x^{b_{i'_1} \dots b_{i'_{k-1}} b_{i'_{k+1}} b_{i_k} b_{i_{k+2}} \dots b_{i_j}}.$$

This contradicts the choice of the product $b_{i_1} \dots b_{i_j}$ because

$$b_{i_1} \dots b_{i_{k-1}} b_{i_k} b_{i_{k+1}} b_{i_{k+2}} \dots b_{i_j} > b_{i'_1} \dots b_{i'_{k-1}} b_{i'_{k+1}} b_{i_k} b_{i_{k+2}} \dots b_{i_j}.$$

Thus $x^y = x^{b_{i_1} \ldots b_{i_j}}$ with $i_1 \geq i_2 \geq \ldots \geq i_j$ or, equivalently,

$$x^y = x^{b_r{}^{e_r} \ldots b_1{}^{e_1}}$$

for some non-negative integers $e_r, \ldots, e_1$.

By Lemma 2.4, there exists a positive integer e such that $b_i{}^e \in Z(\langle x, B \rangle)$ for all i. Hence we may assume that $e_i < e$ for all i, and so $|x^{v(G)}| \leq e^r$. Thus $x^{v(G)}$ is finite for all $x \in G$. We conclude therefore that $v(G) = [w(G), G]$ is FC-embedded in G, as required. □

3 Proof of Theorem B

The proof of Theorem B is very similar to that of Theorem A; the only important difference is the presence of bounds. In what follows, the term "$\{a, b, c, \ldots\}$-bounded" means "bounded from above by some function depending only on the parameters $a, b, c, \ldots$".

Before proving Theorem B, we recall briefly the ultraproduct construction of groups (see, for instance, [4] for more details).

For a non-empty set I, a filter over I is a set $\mathcal{F}$ with the following properties:

(i) $\emptyset \notin \mathcal{F}$, $I \in \mathcal{F}$;

(ii) if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$;

(iii) if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{F}$.

The filter $\mathcal{F}$ is principal if there exists a non-empty set $Y \subseteq I$ such that

$$\mathcal{F} = \{X \subseteq I \mid Y \subseteq X\},$$

and non-principal otherwise. An example of a non-principal filter over an (infinite) set I is the so-called cofinite filter

$$\mathcal{F} = \{X \subseteq I \mid I \setminus X \text{ is finite}\}.$$

A filter $\mathcal{U}$ over I is called an ultrafilter if, for every $X \subseteq I$, either $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$. This is equivalent to saying that $\mathcal{U}$ is a maximal filter over I. Also, $\mathcal{U}$ is a non-principal ultrafilter if and only if it contains the cofinite filter (see [4, Proposition 1.4]).

Given an ultrafilter $\mathcal{U}$ over I and a family $\{G_i\}_{i \in I}$ of groups, the ultraproduct modulo $\mathcal{U}$ is the quotient set of the Cartesian product $\prod_{i \in I} G_i$ with respect to the equivalence relation defined as follows: the tuples $(g_i)_{i \in I}$ and $(h_i)_{i \in I}$ of the Cartesian product are equivalent modulo $\mathcal{U}$ if and only if

$$\{i \in I \mid g_i = h_i\} \in \mathcal{U}.$$

Thus the ultraproduct modulo $\mathcal{U}$ can be seen as the quotient of the unrestricted direct product of groups G_i by the subgroup consisting of all tuples $(g_i)_{i\in I}$ such that

$$\{i \in I \mid g_i = 1\} \in \mathcal{U}.$$

The following is a consequence of Łoś's theorem (see [4, Corollary 3.2]), in the case of a sentence in the first-order language of groups.

Lemma 3.1. *Let $\{G_i\}_{i\in I}$ be a family of groups, and let $\mathcal{U}$ be an ultrafilter over I. Then a sentence in the first-order language of groups holds in the ultraproduct modulo $\mathcal{U}$ if and only if the set of all $i \in I$ for which the sentence holds in G_i is a member of $\mathcal{U}$.*

Recall that the width of a group-word w in a group G is the supremum, as g ranges over the verbal subgroup $w(G)$, of the minimum length of all decompositions of g as a product of elements of G_w^*. Clearly, for a given positive integer k, the word w has finite width at most k in G if and only if the product of any $k+1$ elements of G_w^* can be expressed as a product of at most k elements of G_w^*. Our next result relies on Lemma 3.1 and the following two facts, which have been stated in [5, proof of Theorem A.1]:

(a) for a given integer m, the property that a given word takes at most m values in a group can be expressed as a sentence in the first-order language of groups;

(b) for a given positive integer k, the property that a given word has finite width at most k in a group can be expressed as a sentence in the first-order language of groups.

Proposition 3.2. *Let $m \geq 1$. Suppose that w is a semiconcise word and G is a group in which w takes precisely m values. Then the order of $[w(G), G]$ is m-bounded.*

Proof. Assuming that w involves n variables, write $w = w(x_1, \ldots, x_n)$, and set $v(x_1, \ldots, x_n, x_{n+1}) = [w(x_1, \ldots, x_n), x_{n+1}]$. Then $[w(G), G] = v(G)$.

By way of contradiction, suppose there exists a family of groups $\mathcal{G} = \{G_i\}_{i\in\mathbb{N}}$ with the property that $|(G_i)_w| \leq m$ for all $i \in \mathbb{N}$ but

$$\lim_{i\to\infty} |v(G_i)| = \lim_{i\to\infty} |[w(G_i), G_i]| = \infty.$$

Consider a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$, and let Q be the ultraproduct modulo $\mathcal{U}$ of $\mathcal{G}$. Then, by (a) and Lemma 3.1, we have $|Q_w| \leq m$. As w is semiconcise, it follows that $v(Q) = [w(Q), Q]$ is finite. In particular, v has finite width, say k, in Q. Now (b) and Lemma 3.1 yield that there exists $X \in \mathcal{U}$ such that v has finite width at most k in G_i for all $i \in X$. Hence every element of $v(G_i) = [w(G_i), G_i]$ can be written as a product of at most k elements of $(G_i)^*_v$. Moreover, as

$$v(x_1, \ldots, x_n, x_{n+1}) = w(x_1, \ldots, x_n)^{-1} w(x_1, \ldots, x_n)^{x_{n+1}},$$

from $|(G_i)_w| \leq m$, we get $|(G_i)_v| \leq m^2$ for all $i \in \mathbb{N}$. Therefore, $|v(G_i)| \leq m^{2k}$ for all $i \in X$. As noted above, $\mathcal{U}$ contains the cofinite filter over $\mathbb{N}$, so $X \cap Y \in \mathcal{U}$ for every cofinite subset Y of $\mathbb{N}$. In particular, $X \cap Y$ is non-empty. Therefore, every cofinite subset of $\mathbb{N}$ contains some element i for which $|v(G_i)| \leq m^{2k}$. This is incompatible with the assumption that $|v(G_i)|$ goes to infinity. □

Lemma 3.3. *Let $w = w(x_1, \ldots, x_n)$ be a group-word. If G is a* BFC(w)*-group such that $|x^{G_w}| \leq m$ for all $x \in G$, then the conjugacy class $x^{G^*_w}$ has $\{m, n\}$-bounded order for all $x \in G$.*

Proof. By the second statement in [2, Proposition 2.9], a group G is a BFC(w)-group if and only if it is a BFC(w^{-1})-group. More precisely, if G is a BFC(w)-group such that $|x^{G_w}| \leq m$ for all $x \in G$, then $x^{G_{w^{-1}}}$ has $\{m, n\}$-bounded order. Since $G^*_w = G_w \cup G_{w^{-1}}$, the result follows. □

Lemma 3.4. *Let $w = w(x_1, \ldots, x_n)$ be a group-word, and set*

$$v = [w(x_1, \ldots, x_n), x_{n+1}].$$

If G is a BFC(w)*-group such that $|x^{G_w}| \leq m$ for all $x \in G$, then G is a* BFC(v)*-group and x^{G_v} has $\{m, n\}$-bounded order for all $x \in G$.*

Proof. This is similar to the proof of Lemma 2.3 taking into account that x^y can only take $\{m, n\}$-boundedly many values by Lemma 3.3. □

Lemma 3.5. *Let $w = w(x_1, \ldots, x_n)$ be a group-word, and set*

$$v = [w(x_1, \ldots, x_n), x_{n+1}].$$

Let G be a BFC(w)*-group such that $|x^{G_w}| \leq m$ for all $x \in G$, and let B be a finite subset of G^*_v. Then, for any $x \in G$, there exists an $\{m, n, |B|\}$-bounded positive integer e such that $b^e \in Z(\langle x, B\rangle)$ for all $b \in B$.*

Proof. It follows as in the proof of Lemma 2.4: by [2, Lemma 2.7 (ii)], the set $(J/Z(J))_w$ is finite of $\{m, n, |B|\}$-bounded order and, by Proposition 3.2, the number e is $\{m, n, |B|\}$-bounded. □

Proof of Theorem B. Set $v = [w(x_1, \dots, x_n), x_{n+1}]$. Then Lemma 3.4 tells us that G is a BFC(v)-group and x^{G_v} has $\{m, n\}$-bounded order for all $x \in G$. Let x be an arbitrary element of G, and choose

$$b_1, \dots, b_r \in G_v^* \quad \text{such that} \quad x^{G_v^*} = \{x^{b_1}, \dots, x^{b_r}\}.$$

Write $B = \{b_1, \dots, b_r\}$. Define the order $<$ on the set of all (formal) products of the form $b_{i_1} \dots b_{i_j}$, with $1 \le i_k \le r$ and $j \ge 1$, as in (2.1) in the proof of Theorem A.

Let y be an arbitrary element of $v(G)$. Arguing as in the proof of Theorem A, write

$$x^y = x^{{b_r}^{e_r} \dots {b_1}^{e_1}}$$

for some non-negative integers $e_r, \dots, e_1$. Since the number r is $\{m, n\}$-bounded by Lemma 3.4, it follows from Lemma 3.5 that there exists an $\{m, n\}$-bounded positive integer e such that ${b_i}^e \in Z(\langle x, B\rangle)$ for all i. Hence we may assume that $e_i < e$ for all i, and so $|x^{v(G)}| \le e^r$. Thus $x^{v(G)}$ is finite of $\{m, n\}$-bounded order for all $x \in G$, and so $v(G)$ is BFC-embedded in G. □

4 Examples

In this section, we first provide some new examples of semiconcise words, and then we give an example of a word which is not semiconcise.

The following lemma is well known (see, for instance, [10, Lemma 4.28]).

Lemma 4.1. *Let $G = \langle g_1, \dots, g_m\rangle$ be a group, and suppose that, for a group-word w, the set $\{[x, g_i] \mid x \in G_w, i = 1, \dots, m\}$ is finite. Then G_w is contained in finitely many right cosets of $w(G) \cap Z(G)$.*

Proof. Suppose that there exists an infinite sequence $x_1, x_2, \dots$ of elements of G_w belonging to distinct right cosets of $C = w(G) \cap Z(G)$. For any $i = 1, \dots, m$, put $S_i = \{[x, g_i] \mid x \in G_w\}$. Denote by P the Cartesian product $S_1 \times \dots \times S_m$, and let π be the map sending $x \in G_w$ to $([x, g_1], \dots, [x, g_m]) \in P$. Now we have that $x_i\pi = x_j\pi$ implies $x_i x_j^{-1} \in C$, or $Cx_i = Cx_j$ which is impossible. It follows that P must be infinite, a contradiction. □

Proposition 4.2. *Let $w = w(x_1, \dots, x_n)$ be a group-word, and set*

$$v = [w(x_1, \dots, x_n), x_{n+1}].$$

If w is semiconcise, then v is semiconcise.

Proof. Let G be a group, and assume that G_v is finite. Since $v(G)'$ is finite (see, for instance, [6, Proposition 1]), we may assume that $v(G)$ is abelian. It follows that every subgroup of $v(G)$ is finitely generated. Let $K = \langle g_1, \ldots, g_m \rangle$ be a finitely generated subgroup of G such that $v(G) = v(K)$. For an arbitrary $g = g_0 \in G$, put $H = \langle g_0, K \rangle$. Of course, $v(G) = v(H)$. Also, $|H : C_H(v(H))|$ is finite because every $h \in H_v$ has only finitely many conjugates in H. We claim that $|v(H) : v(H) \cap Z(H)|$ is also finite, from which it follows that $[v(H), H]$ is finite by a result of Baer (see [10, Corollary, p. 103]). In fact, the set

$$\{[x, g_i] \mid x \in H_w, i = 0, 1, \ldots, m\}$$

is finite, and therefore, by Lemma 4.1, H_w is contained in finitely many right cosets of $w(H) \cap Z(H)$. Hence $(H/Z(H))_w$ is finite. Since w is semiconcise, we obtain that

$$[w(H), H]Z(H)/Z(H) \simeq v(H)/v(H) \cap Z(H)$$

is finite.

Then $[v(H), H] = [v(G), H]$ is finite. In particular, $[v(G), \langle g \rangle]$ is finite for any $g \in G$. Thus $[v(G), G]$ is a finitely generated periodic abelian group, and so $[v(G), G]$ is finite. This proves that v is semiconcise. □

As an immediate consequence of Proposition 4.2, we get new examples of semiconcise words starting from Engel words, which are semiconcise by [6, Proposition 4].

Corollary 4.3. *Let $w = [x,_n y]$ be the n-Engel word, and set*

$$v = [x,_n y, z_1, \ldots, z_m],$$

where the variables $x, y, z_1, \ldots, z_m$ are all different. Then v is semiconcise.

According to [8], for any odd integer $n > 10^{10}$ and any prime number $p > 5000$, the word

$$v(x, y) = [[x^{pn}, y^{pn}]^n, y^{pn}]^n$$

is not concise. Indeed, Ivanov constructed a 2-generator torsion-free group A, whose center is cyclic and $A/Z(A)$ is infinite of exponent $p^2 n$, such that v takes only two values in A and the nontrivial value is a generator of $Z(A)$.

In [1, Section 4], the authors considered a modification of Ivanov's example, namely the wreath product

$$G = A \text{ wr } B, \tag{4.1}$$

where A is as above and $B = \langle b \rangle$ is a cyclic group of order 2. Furthermore, taking

$$w(x, y) = v(x^2, y^2), \tag{4.2}$$

they showed that $|G_w| \leq 4$ and $b^{w(G)}$ is infinite. In a similar way, we now prove that $b^{[w(G),G]}$ is also infinite. This implies that w is not semiconcise.

Proposition 4.4. *There exist a group-word w (which is not semiconcise) and a* BFC(w)*-group G such that* $[w(G), G]$ *is not* FC*-embedded in G.*

Proof. Let G and w be as in (4.1) and (4.2), respectively. Then, by [1, Proposition 4.1], G is a BFC(w)-group.

Denote by $K = A \times A^b$ the base group of G. For any odd integer $m \geq 1$, let $N = \langle v_0^m, (v_0^b)^m \rangle$, where $v_0 \in A$ is the nontrivial value of $v(x, y)$ in A. Notice that N is central in K and closed under conjugation by b so that N is a normal subgroup of G. Also, since K/N has odd exponent p^2mn and $|G/K| = 2$, we have

$$K/N = \{g^2 N \mid g \in G\}$$

and consequently

$$(K/N)_v = (G/N)_w.$$

Hence $v_0 N \in (G/N)_w$, and therefore $v_0^k N \in w(G/N)$ for any integer k. It follows that

$$b^{[b,v_0^k]} N \in (bN)^{[w(G/N),G/N]}.$$

Now

$$b^{[b,v_0^k]} = b[b, v_0^k, b]^{-1} = b[(v_0^b)^{-k} v_0^k, b]^{-1} = b(v_0^2 (v_0^b)^{-2})^k,$$

where $v_0^2 (v_0^b)^{-2}$ has order m in G/N. Thus

$$|\{b^{[b,v_0^k]} N \mid k \in \mathbb{Z}\}| = m,$$

and so

$$|(bN)^{[w(G/N),G/N]}| \geq m.$$

In particular, $|b^{[w(G),G]}| \geq m$. Since m is an arbitrary odd positive integer, we conclude that $b^{[w(G),G)]}$ is infinite. This proves that $[w(G), G)]$ is not FC(w)-embedded in G. □

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