A note on involution centralizers in black box groups

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Abstract. Here we note a minor variation on the method in [J.N. Bray, An improved method for generating the centralizer of an involution, *Arch. Math. (Basel)* **74** (2000), no. 4, 241–245] which enables calculations of $C_H(t)$ for H a subgroup of a black box group G and t an involution of G . We then give some applications of this, the first to investigate the action of the Lyons sporadic group via conjugation on its involutions, a number of calculations in $E_6(2)$ and $E_8(2)$, then finally to carry out some calculations in $\text{GL}_{670}(2)$.

1 Introduction

In [7], Bray revealed a method for calculating centralizers of involutions in black box groups with an order oracle. This method extended one introduced earlier by R. Parker (see [18]). In recent times, the Bray method has had many ramifications in computational group theory (for a fraction of these, consult [2, 6, 8, 11–14, 19, 20, 25, 26]). The purpose of this short note is to observe a further twist to this story. Although this perturbation to [7] is very minor, it results in some substantial gains. In Section 2, we make use of it to determine the suborbits of the Lyons sporadic simple group in its action by conjugation on its involutions. A further demonstration of the efficacy of this variation, in Section 2.4, are some calculations in $\text{GL}_{670}(2)$. Also in Section 2, we include statistics of some sample calculations in $E_6(2)$ and $E_8(2)$.

Suppose G is a black box group with an order oracle. Assume t is an involution of G . In [7], the elements $\mathcal{K}(t, g)$ of G are key. For $g \in G$ and letting n be the order of $[t, g]$, we define

$$\mathcal{K}(t, g) = \begin{cases} [t, g]^m & \text{if } n = 2m, \\ g[t, g]^m & \text{if } n = 2m + 1. \end{cases}$$

These elements $\mathcal{K}(t, g)$ (and also $\mathcal{K}(t, g^{-1})$ if n is even) supply elements in $C_G(t)$. Those $\mathcal{K}(t, g)$ obtained when n is odd have the property observed by R. Parker that they are uniformly distributed throughout $C_G(t)$.

For $H \leq G$, set $\mathcal{O}_H = \{h \mid h \in H \text{ and } [t, h] \text{ is of odd order}\}$.

Lemma 1.1. *Suppose t is an involution in G , $H \leq G$, and let $c \in C_H(t)$. Then $|\{h \in \mathcal{O}_H \mid \mathcal{K}(t, h) = c\}|$ is independent of c .*

Proof. Since $c \in C_H(t)$, $[t, h] = [t, ch]$ for all $h \in H$, so we only need prove the lemma for each right coset $C_H(t)h$ ($h \in H$) for which $[t, h]$ has odd order. For $eh \in C_H(t)h$, we have

$$\mathcal{K}(t, eh) = eh[t, eh]^m = eh[t, h]^m = e\mathcal{K}(t, h),$$

where $[t, h]$ has order $2m + 1$. Hence each such coset contributes 1 to

$$|\{h \in \mathcal{O}_H \mid \mathcal{K}(t, h) = c\}|,$$

so giving the lemma. □

Theorem 3.1 of [7] is the case $H = G$, and its proof is virtually identical to that for Lemma 1.1. The point is that t does not need to be in H . So to compute $C_H(t)$, we may proceed as follows.

- (1) Initialize $S := \{\}$.
- (2) Choose a random element $h \in H$.
- (3) Compute $\mathcal{K}(t, h)$.
- (4) Check whether $\mathcal{K}(t, h) \in H$; if yes, then add $\mathcal{K}(t, h)$ to S .
- (5) Go to step (2).

Then $\langle S \rangle$ will be a subgroup of $C_H(t)$. All the analysis and caveats discussed in [7] will apply here. Lemma 1.1 shows that the set of elements passing test (4) will be uniformly distributed in $C_H(t)$. Also, the membership problem raises its head in step (4), and the exact nature of H may help in resolving this. For example, we may have $H = C_G(X)$ ($X \subseteq G$) in which case step (4) can be settled by checking whether $\mathcal{K}(t, h)$ commutes with the set X . The other conundrum facing us is what is the probability that the elements $\mathcal{K}(t, h)$ are actually in H . Without further specific information about H , there does not seem much that can be said in general. Indeed, we could have that $C_H(t) = 1$. In this connection, in Section 2, we gather statistics of some actual calculations.

Suppose $H = C_G(s)$, where s is an involution of G (in fact, the situation that sparked this note); experimentally, the following works well. In place of (2)–(5), do the following.

- (2') Compute $\mathcal{K}(s, g)$, where g is a random element of G (so applying the Bray method for $C_G(s)$).
- (3') Compute $\mathcal{K}(t, \mathcal{K}(s, g))$.
- (4') Check whether s and $\mathcal{K}(t, \mathcal{K}(s, g))$ commute; if yes, then add $\mathcal{K}(t, \mathcal{K}(s, g))$ to S .
- (5') Go to step (2').

Observe that Lemma 1.1 applied twice shows that these will be uniformly distributed in $C_G(t) \cap C_G(s)$. Of course, we could determine generating sets for $C_G(t)$ and $C_G(s)$ using the Bray method, and then we could attempt to compute $C_G(t) \cap C_G(s)$. In computationally hard groups, the latter step may prove impossible. To illustrate this with an example, take $G = E_6(2)$, as given in the electronic ATLAS [24] in its 27-dimension GF(2) representation. There $G = \langle a, b \rangle$, where a has order 2, b has order 3 with ab of order 62. Taking $t = a$ and $s = (ab)^{31}$, using the method discussed here (that is, steps (1), (2'), (3'), (4'), (5')) with 10,000 random elements $g \in G$, we get $\langle S \rangle = C_G(s) \cap C_G(t)$ with $|\langle S \rangle| = 2^{12} \cdot 3 \cdot 7$. This is done in the blink of an eye, whereas first calculating $C_G(s)$ and $C_G(t)$ (which is quick) and then $C_G(s) \cap C_G(t)$ takes forever (1552 seconds on a 16×1248 MHz machine running MAGMA [9] version 222-10). This disparity will be even greater for larger groups.

Finally, we observe that the above process for the centralizer of two involutions may be iterated so as to find $C_G(H)$, where H is generated by involutions – see Section 2.4 for an example of this.

2 Some applications

2.1 Involutions in the Lyons group

The sporadic simple group Ly, first pin-pointed by Lyons in [15], has order $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$. Its smallest non-trivial permutation degree is 8,835,156 – indeed, this representation was built in the 1970s by Sims [22] in the course of showing the existence of Ly. Despite considerable advances in computer algebra, such a permutation representation is not viable for probing the structure of Ly. More tractable is the 111-dimensional representation over GF(5), first displayed in [17], which we employ here. From [24], we obtain this representation where Ly is generated by the standard generators a, b with $a \in 2A$, $b \in 5A$ and $ab \in 14A$ (using ATLAS [10] names for the conjugacy classes of Ly).

For many of the sporadic simple groups G , the action (by conjugation) of G on a G -conjugacy class of involutions has been analyzed. See [4, 5, 21, 23], where the

suborbits are determined. This type of information is valuable when studying such topics as commuting involution graphs [3], C -strings and polytopes [16].

Here we investigate the action of $G \cong \text{Ly}$ upon $X = 2A$. Before stating our main conclusion, we introduce two pieces of notation. Let $t \in X$. For C a G -conjugacy class, $X_C = \{x \in X \mid tx \in C\}$, and for $y \in G$,

$$C_G^*(y) = \{g \in G \mid y^g = y \text{ or } y^g = y^{-1}\}.$$

Observe that X_C will either be empty or the union of $C_G(t)$ -orbits (under conjugation) and, of course, $X = \bigcup X_C$, as C runs through all the G -conjugacy classes. As is well known, $|X_C|$ equals a particular structure constant which may be calculated using the complex irreducible character table of G . Also, we have $[C_G^*(y) : C_G(y)] = 1$ or 2 .

The following easy lemma will be used in the proof of Theorem 2.2.

Lemma 2.1. *For $t, x \in X$, put $z = tx$. Then*

$$C_{C_G(t)}(x) = C_G(t) \cap C_G(x) = C_{C_G(z)}(t) = C_{C_G(z)}(x).$$

So, for $z = tx \in C$, where $x \in X$ if we know the structure of $C_G(z)$, we may be able to calculate $C_G(t) \cap C_G(x)$. In particular, note that $[C_G^*(z) : C_G(z)] = 2$ and, as $G = \text{Ly}$ has just one conjugacy class of involutions, $C_{C_G(z)}(y)$ as y ranges through all involutions in $C_G^*(z) \setminus C_G(z)$ gives all the possible structures for

$$C_G(t) \cap C_G(y), \quad y \in X_C,$$

that must occur (but not the possible multiplicities).

As we shall see, a number of the X_C may be easily broken into $C_G(t)$ -orbits. However, for three G -conjugacy classes, it is not so straightforward, and we make use of the routine (1), (2'), (3'), (4'), (5') given in Section 1.

Theorem 2.2. *Suppose that $G = \text{Ly}$, $X = 2A$ and $t \in X$. Then the permutation rank of G on X is 78 and the suborbit (that is, $C_G(t)$ -orbit) sizes are as follows.*

C	$ X_C $	$C_G(t)$ -orbit sizes
1A	1	1
2A	34,650	34,650
3A	5,040	5,040
3B	677,600	554,400 + 123,200

C	$ X_C $	$C_G(t)$ -orbit sizes
4A	1,663,200	831,600 + 831,600
5A	199,584	299,584
5B	5,322,240	1,330,560 + 3,991,680
6A	831,600	831,600
6B	6,098,400	1,108,800 + 4,989,600
6C	13,305,600	3,326,400 + 3,326,400 + 3,326,400 + 3,326,400
7A	19,958,400	19,958,400
8A	979,200	4,989,600 + 4,989,600
8B	19,958,400	9,979,200 + 9,979,200
9A	26,611,200	19,958,400 + 6,652,800
10A	4,989,600	4,989,600
10B	39,916,800	19,958,400 + 19,958,400
12A	14,968,800	4,989,600 + 9,979,200
12B	26,611,200	6,652,800 + 19,958,400
14A	19,958,400	19,958,400
15A	7,983,360	1,995,840 + 1,995,840 + 1,995,840 + 1,995,840
15B	39,916,800	19,958,400 + 19,958,400
15C	39,916,800	39,916,800
18A	39,916,800	19,958,400 + 19,958,400
20A	39,916,800	19,958,400 + 19,958,400
21A	39,916,800	19,958,400 + 19,958,400
21B * *	39,916,800	19,958,400 + 19,958,400
24A	39,916,800	19,958,400 + 19,958,400
24B	39,916,800	19,958,400 + 19,958,400
24C *	39,916,800	19,958,400 + 19,958,400
25A	39,916,800	39,916,800
28A	39,916,800	19,958,400 + 19,958,400
30A	39,916,800	19,958,400 + 19,958,400
30B	39,916,800	19,958,400 + 19,958,400
31A	39,916,800	39,916,800
31B * 2	39,916,800	39,916,800

C	$ X_C $	$C_G(t)$ -orbit sizes
$31C * 4$	39,916,800	39,916,800
$31D * 8$	39,916,800	39,916,800
$31E * 16$	39,916,800	39,916,800
$37A$	39,916,800	39,916,800
$37B*$	39,916,800	39,916,800
$40A$	39,916,800	19,958,400 + 19,958,400
$40B*$	39,916,800	19,958,400 + 19,958,400
$42A$	39,916,800	19,958,400 + 19,958,400
$42B*$	39,916,800	19,958,400 + 19,958,400
$67A$	39,916,800	39,916,800
$67B * 2$	39,916,800	39,916,800
$67C * 4$	39,916,800	39,916,800

Proof. Let $x \in X$, and set $z = tx \in C$, C a G -conjugacy class. We recall that $C_G(t) \cong 2 \cdot \text{Alt}(11)$, so $|C_G(t)| = 39,916,800 = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$.

Suppose that $C_G(z) = \langle z \rangle$. Then if z has odd order, $C_G(t) \cap C_G(y) = 1$ for all $y \in X_C$, and if z has even order, $|C_G(t) \cap C_G(y)| = 2$ for all $y \in X_C$. This deals with X_C for

$$C \in \left\{ 18A, 24A, 24B, 24C*, 25C, 28A, 31A, 31B * 2, 31C * 4, 31D * 8, \right. \\ \left. 31E * 16, 37A, 37B*, 40A, 40B*, 42A, 42B*, 67A, 67B * 2, 67C * 4 \right\}.$$

Also if z has odd order and $[C_G(z) : \langle z \rangle] = 2$, then $|C_G(t) \cap C_G(y)| = 2$ for all $y \in X_C$. This observation yields that X_{21A} and X_{21B*} both break into two $C_G(t)$ -orbits.

If z has even order, then, as G has only one involution conjugacy class, we may regard $C_G(t) \cap C_G(x)$ as a subgroup of $2 \cdot \text{Alt}(11)$. So we may calculate in $2 \cdot \text{Alt}(11)$, using the permutation representation of degree 5,040 in [24]. This approach allows us to break X_C into $C_G(t)$ -orbits for

$$C \in \{2A, 4A, 6A, 6B, 6C, 8A, 8B, 10A, 10B, 12A, 12B, 14A, 20A\}.$$

Note that, for $z \in 7A$, $C_G^*(z) = C_G^*(w)$, where $w^2 = z$ and $w \in 14A$, so X_{7A} is a $C_G(t)$ -orbit just as X_{14A} is.

At this point, it remains to examine X_C for

$$C \in \{3A, 3B, 5A, 5B, 9A, 15A, 15B, 15C\}.$$

For $z \in C = 3A$, $C_G^*(z) \sim 3 \cdot \text{McL} : 2$ and $C_{C_G(z)}(t) \cong M_{11}$ (see [10]), and therefore X_{3A} is a $C_G(t)$ -orbit. Suppose z has order 15. For $z \in 15B$, we have $C_G^*(z) = C_G(w)$ for $w \in 30B$ and $w^2 = z$, so X_{15B} breaks into two $C_G(t)$ -orbits just as X_{30B} does. If $z \in 15A \cup 15C$, then $z^5 \in 3A$, and we may calculate in $\text{McL} : 2$, using the permutation representation of $\text{McL} : 2$ of degree 275. Suppose $z \in 9A$. Then $[C_G(z) : \langle z \rangle] = 6$. Since G has no elements of order 18, $C_G(z)/\langle z \rangle$ cannot be cyclic, and so $C_G(z)/\langle z \rangle \cong \text{Sym}(3)$. If y is an involution inverting z , then y must induce inner automorphisms on $C_G(z)/\langle z \rangle$. Since $C_{C_G(z)}(y) \cap \langle z \rangle = 1$, we conclude that $C_{C_G(z)}(y) = 2$ or 6 ; whence it follows that X_{9A} had two $C_G(t)$ -orbits of sizes 19,958,400 and 6,652,800.

We are now left with the cases $C \in \{3B, 5A, 5B\}$. The structure of $C_G^*(z)$ in these cases is more intricate, so we use the procedure discussed in Section 1.

(2.2.1) $C = 3B$. For different choices of $x_1, x_2 \in X_{3B}$, we obtain

$$L_i \leq C_G(t) \cap C_G(x_i),$$

where $|L_1| = 2^2 \cdot 3^4$, $|L_2| = 2^3 \cdot 3^2$. In fact, $L_i = C_G(t) \cap C_G(x_i)$ for $i = 1, 2$, and we explain how this was established for $i = 1$, $i = 2$ being similar. Using the `LMGRadicalQuotient` command in `MAGMA` on $C_G(t) \cong 2 \cdot \text{Alt}(11)$, we obtain a map from $C_G(t)$ to $C_G(t)/\langle t \rangle \cong \text{Alt}(11)$, with the latter given as a permutation group of degree 11. Let $\overline{L_1}$ be the projection of L_1 into $\text{Alt}(11)$ (and note $L_1 \cap \langle t \rangle = 1$). Because $|\overline{L_1}| = 3^4 \cdot 2^2$, the only possible maximal subgroups of $\text{Alt}(11)$ containing $\overline{L_1}$ are isomorphic to either $\text{Sym}(9)$ or $\text{Alt}(10)$. Within $\text{Alt}(10)$, the only proper overgroups might be $\text{Alt}(9)$, and then within $\text{Alt}(9)$ only $3^3 : \text{Sym}(4)$. Within $\text{Sym}(9)$, the only proper subgroup that could contain $\overline{L_1}$ has shape $3^3 : (\text{Sym}(4) \times 2)$. So if $C_G(t) \cap C_G(x_1) > L_1$, we must have 2^3 dividing $|C_G(t) \cap C_G(x_1)|$. Selecting a Sylow 2-subgroup of $\overline{L_1}$, $\overline{S_1}$, we calculate $N_{C_G(t)/\langle t \rangle}(\overline{S_1})$ and then take its full inverse image in $C_G(t)$, say K . Since we have $|K| = 2^6 \cdot 3$, we may quickly enumerate $C_K(x_1)$ finding that $|C_K(x_1)| = 2^2$. So 2^3 does not divide $|C_G(t) \cap C_G(x_1)|$, and we conclude that

$$C_G(t) \cap C_G(x_1) = L_1.$$

This gives a $2^6 \cdot 5^2 \cdot 7 \cdot 11 = 123,200$ size $C_G(t)$ -orbit of X_{3B} .

(2.2.2) $C = 5A$. Selecting a random $x \in X_{5A}$, we use the procedure in Section 1 and obtain $L \leq C_G(t) \cap C_G(x)$ with $L \sim 5^2 : 2$. When taking the image of L (as for X_{3B}), say $\overline{L}$, in $C_G(t)/\langle t \rangle \cong \text{Alt}(11)$, we find

$$\overline{L} \leq \overline{K} \cong (\text{Alt}(5) \times \text{Alt}(5)).4.$$

Taking the full inverse image of $\overline{K}$, say K , in $C_G(t)$ (which has order $2^8 \cdot 3^2 \cdot 5^2$), we calculate that $|K \cap C_G(x)| = 2^3 \cdot 5^2$. Projecting $K \cap C_G(x)$ back into $\text{Alt}(11)$, we see that $K \cap C_G(x) = C_G(t) \cap C_G(x)$. Consequently, X_{5A} consists of a single $C_G(t)$ -orbit.

(2.2.3) $C = 5B$. Again selecting a random $x \in X_{5B}$ and employing the procedure in Section 1, we locate $x_1, x_2 \in X_{5B}$ with

$$L_1 \leq C_G(t) \cap C_G(x_1), \quad L_2 \leq C_G(t) \cap C_G(x_2),$$

$L_1 \cong \text{Dih}(10)$ and $L_2 \cong \text{Dih}(30)$. Let $\overline{L_1}$ be the image of L_1 in $C_G(t)/\langle t \rangle$, and let $\overline{N_2}$ and $\overline{N_5}$ be the normalizers in $\text{Alt}(11)$ of respectively a Sylow 2- and 5-subgroup of $\overline{L_1}$. If N_2 and N_5 are respective full inverse images of $\overline{N_2}$ and $\overline{N_5}$, we have $|N_2| = 2 \cdot 1,152$ and $|N_5| = 2 \cdot 100$; whence we quickly check that

$$|N_2 \cap C_G(x)| = 2 \quad \text{and} \quad |N_5 \cap C_G(x)| = 10.$$

Hence we conclude that $C_G(t) \cap C_G(x_1) = L_1 \cong \text{Dih}(10)$. Very similar arguments show that $C_G(t) \cap C_G(x_2) = L_2 \cong \text{Dih}(30)$, and therefore X_{5B} breaks into two $C_G(t)$ -orbits of sizes $2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$ and $2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$.

Since we have now dealt with all the non-empty subsets X_C , the proof of Theorem 2.2 is complete. $\square$

2.2 Calculations in $E_6(2)$

We now look further at the calculation in $E_6(2)$, mentioned in Section 1. Repeating the calculation for the given t and s 100 times, in each case using 10,000 random elements, gave the following five outcomes with the following frequencies.

$ \langle S \rangle $ obtained	$2^{12} \cdot 3 \cdot 7$	$12^2 \cdot 3$	2^{11}	$2^{10} \cdot 3$	2^{10}
Number of times	69	20	2	4	5

On the other hand, using 12,000 random elements from $E_6(2)$ in each of 100 experiments always gave $|\langle S \rangle| = 2^{12} \cdot 3 \cdot 7$.

2.3 Calculations in $E_8(2)$

We again go to [24] for our copy of $G = E_8(2)$, using the 2-generator version there. So $G = \langle a, b \rangle$ with a and b being 248×248 matrices over $\text{GF}(2)$. Let V

denote the 248-dimensional $\text{GF}(2)G$ -module. The order of a is 30 and the order of b is 10. We shall employ the naming system in [1] (which follows ATLAS conventions) for the conjugacy classes of G . So $a^{15} \in 2D$ and $b^5 \in 2B$ (because $\dim C_V(a^{15}) = 128$ and $\dim C_V(b^5) = 156$). In each of the following cases, we did 10 experiments each using 40,000 random elements from $E_8(2)$.

- (i) $t = a^{15}$ and $s = t^g$, where $g = b^3ab^{-1}aba^{11}$. Then ts has order 17 with $\dim C_V(ts) = 32$, so, using [1], $ts \in 17AB$ with $C_G(ts) \cong 17 \times \Omega_8^-(2)$.

$ S $	8	5	7	7	4	6	7	3	9	10
$ \langle S \rangle $	$2^8 \cdot 5$	$2^6 \cdot 3$	$2^9 \cdot 3$	$2^9 \cdot 3$	$2^6 \cdot 3$	$2^8 \cdot 3$	$2^9 \cdot 3$	$2^4 \cdot 3$	$2^8 \cdot 3$	$2^9 \cdot 3$

- (ii) $t = b^5$ and $s = t^g$, where $g = ab^3a^7b^4a^{19}$. This time ts has order 8.

$ S $	78	78	79	68	92	53	75	63	74	91
$ \langle S \rangle $	2^{18}	2^{18}	2^{21}	2^{23}	2^{30}	2^{22}	2^{23}	2^{21}	2^{19}	2^{26}

In case we are getting too rosy a picture, we have the following.

- (iii) $t = b^5$ and $s = t^g$, $g = ab^3a^7$. Here ts has order 5, $\dim C_V(ts) = 68$. So $ts \in 5A$ with $C_G(ts) \cong 5 \times \Omega_{12}^-(2)$. Each of the 10 experiments yielded no elements of S at all. Moreover, note that, by Lemma 2.1, $C_G(t) \cap C_G(s)$ must be non-trivial.

2.4 Centralizers in a 670×670 matrix group

Using the $\text{GF}(2)$ representation of $O_{10}^+(2)$ of degree 670 given in [24], we may regard $G = O_{10}^+(2)$ as a subgroup of $\text{GL}_{670}(2)$. In the notation of [24], $G = \langle a, b \rangle$, where a has order 2 and b has order 20. Set $H = \langle a, a^b, a^{(bab^3)} \rangle$. We have that $H \cong L_3(2)$, and we seek to calculate $C_G(H)$. First employing the procedure in Section 1, we calculate $C_G(\langle a, a^b \rangle)$ (which has order $2^7 \cdot 3^3 \cdot 5 \cdot 7$). Then using random elements selected from $C_G(\langle a, a^b \rangle)$, we repeat the procedure to calculate elements in $C_{C_G(\langle a, a^b \rangle)}(a^{(bab^3)})$ (and note that $a^{(bab^3)} \notin C_G(\langle a, a^b \rangle)$), thus delivering a subgroup of $C_G(H)$ of order $2^2 \cdot 3^2$. There is of course the perennial problem of knowing whether we have obtained all of $C_G(H)$ – though there are instances where just having some elements of $C_G(H)$ may be helpful. As it happens, $|C_G(H)| = 2^2 \cdot 3^2$ (which may be checked by working in the 496 degree permutation representation for $O_{10}^+(2)$ available in [24]). Finally, we remark that the standard MAGMA functions could not calculate $C_G(H)$.

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