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On the holomorph of finite semisimple groups

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Abstract. Given a finite nonabelian semisimple group G, we describe those groups that have the same holomorph as G, that is, those regular subgroups $N \simeq G$ of $S(G)$, the group of permutations on the set G, such that $N_{S(G)}(N) = N_{S(G)}(\rho(G))$, where ρ is the right regular representation of G.

1 Introduction

Let G be any finite group, and let $S(G)$ be the symmetric group on the set of elements of G. We denote by $\rho\colon G \to S(G)$ and $\lambda\colon G \to S(G)$ respectively the right and the left regular representations of G. The normalizer

$$\mathrm{Hol}(G) = N_{S(G)}(\rho(G))$$

is the *holomorph* of G, and it is isomorphic to the natural extension of G by its automorphism group $\mathrm{Aut}(G)$. It is well known that $\mathrm{Hol}(G) = N_{S(G)}(\lambda(G))$.

In [10], the *multiple holomorph* of G has been defined as

$$\mathrm{NHol}(G) = N_{S(G)}(\mathrm{Hol}(G)),$$

and it is proved that the quotient group

$$T(G) = \mathrm{NHol}(G)/\mathrm{Hol}(G)$$

acts regularly by conjugation on the set of the regular subgroups N of $S(G)$ that are isomorphic to G and have the same holomorph as G, that is, $T(G)$ acts regularly on the set

$$\mathcal{H}(G) = \{H \le S(G) \mid H \text{ is regular},\ H \simeq G,\ N_{S(G)}(H) = \mathrm{Hol}(G)\}.$$

There has been some attention both in the distant past [11] and quite recently [3–5, 9] on the problem of determining, for G in a given class of groups, those groups that have the same holomorph as G and, in particular, the set $\mathcal{H}(G)$ and the structure of the group $T(G)$. Recently, in [5], the authors attack this problem when G is a perfect group, obtaining complete results for centerless groups [5, Theorem 7.7]. However, they leave open some interesting questions when the center is nontrivial.

The aim of this paper is to completely resolve the case when G is a finite semisimple group.

One of the main obstacles for describing the holomorph of a finite semisimple group (see [5, Remark 7.12] and also [2, ADV - 4B]) was to completely classify those finite nonabelian simple groups that admit automorphisms acting like inversion on their Schur multiplier. In Proposition 2, we produce a complete analysis, whose proof depends on the Classification of Finite Simple Groups. It turns out that there is a small list $\mathcal{L}$ (see after Proposition 2 for its definition) of related quasisimple groups having automorphisms inverting their center. Our main result can be therefore stated as follows.

Theorem. *Let G be a finite nonabelian semisimple group, and let*

$$G = A_1 A_2 \dots A_n$$

be its unique central decomposition as a product of $\operatorname{Aut}(G)$*-indecomposable factors. Assume that the number of factors A_i of G having components in $\mathcal{L}$ is exactly l for some $0 \le l \le n$. Then $T(G)$ is an elementary abelian group of order 2^h for some h with $\min\{n - l + 1, n\} \le h \le n$, acting regularly on $\mathcal{H}(G)$. Moreover, if the centers of the factors A_i are all amalgamated, then $|\mathcal{H}(G)| = 2^m$, and $T(G)$ is elementary abelian of order 2^m, where $m = \min\{n, n - l + 1\}$, and therefore $|\mathcal{H}(G)| = 2^m$.*

2 Semisimple groups

To establish the notation, note that we write permutations as exponents, and denote compositions of maps by juxtaposition. We compose maps left-to-right. We consider the right and the left regular "representations" of G, defined by

$$\begin{aligned} \rho\colon G &\to S(G) & \lambda\colon G &\to S(G), \\ g &\mapsto (x \mapsto xg), & g &\mapsto (x \mapsto gx). \end{aligned}$$

Remark 1. Since composition of maps is left-to-right, with our definition, the map λ is an *antihomomorphism*, not a homomorphism, from G to $S(G)$. We have chosen this definition over the standard one (where $\lambda(g)$ maps $x \in G$ to $g^{-1}x$) for the same reasons as in [5].

The first proposition recalls some basic facts (see [4, Proposition 2.4]). The proof is left to the reader.

Proposition 1. *Let G be any group, and let* inv *be the inversion map on G defined by* $\mathrm{inv}(g) = g^{-1}$ *for every $g \in G$. The following hold.*

(1) $C_{S(G)}(\rho(G)) = \lambda(G)$ *and* $C_{S(G)}(\lambda(G)) = \rho(G)$.

(2) $N_{S(G)}(\rho(G)) = \mathrm{Aut}(G) \ltimes \rho(G) = \mathrm{Aut}(G) \ltimes \lambda(G) = N_{S(G)}(\lambda(G))$.

(3) $\rho(g)^{\mathrm{inv}} = \lambda(g^{-1})$ *and* $\lambda(g)^{\mathrm{inv}} = \rho(g^{-1})$ *for every $g \in G$. In particular,* inv *conjugates $\rho(G)$ to $\lambda(G)$ (and vice versa), and it centralizes* $\mathrm{Aut}(G)$*; therefore,* inv *normalizes $N_{S(G)}(\rho(G))$, that is,* $\mathrm{inv} \in \mathrm{NHol}(G)$.

Recall that a *quasisimple group* is a perfect group X such that $X/Z(X)$ is simple, and that a *semisimple group* is a central product of quasisimple groups, that is, a group $X = X_1 X_2 \dots X_t$ with each X_i quasisimple and such that $[X_i, X_j] = 1$ for every $i \neq j$. The quasisimple normal subgroups X_i of X are called the *components* of X. Note in particular that semisimple groups are perfect.

Every finite semisimple group admits a unique decomposition as a central product of characteristic subgroups, which we call a central decomposition as in the theorem. We also remind the reader that a group H is said to be centrally indecomposable as an $\mathrm{Aut}(H)$-group if it cannot be expressed as the central product of two proper characteristic subgroups.

Lemma 1. *Let G be a finite semisimple group. Then G is a central product of perfect subgroups which are centrally indecomposable as* $\mathrm{Aut}(G)$*-subgroups,*

$$G = A_1 A_2 \dots A_n.$$

Moreover, the integer n and the subgroups A_i (for $i = 1, 2, \dots, n$) are uniquely determined (up to permutation).

Proof. Consider the Remak–Krull–Schmidt decomposition of $\mathrm{Inn}(G) \simeq G/Z(G)$ as an $\mathrm{Aut}(G)$-group, and let this be

$$\frac{G}{Z(G)} = \frac{M_1}{Z(G)} \times \frac{M_2}{Z(G)} \times \dots \times \frac{M_n}{Z(G)}. \tag{2.1}$$

Since G is perfect, each $M_i/Z(G)$ is perfect. In particular, each M_i is equal to $M_i' Z(G)$, where M_i' is perfect. Now, for every $j \neq i$, we have that

$$[M_j, M_i'] = [M_j, M_i] \leq Z(G),$$

and thus M_j induces by conjugation a central automorphism on M_i'. Since perfect groups have no nontrivial central automorphisms (see [8] or [5, Lemma 7.1]), we have $[M_i, M_j] = 1$ for every $j \neq i$. In particular, equation (2.1) and the fact that G is perfect imply the following central factorization of G:

$$G = A_1 A_2 \ldots A_n, \tag{2.2}$$

where $A_i = M_i'$ for each $i = 1, 2, \ldots, n$. Note that the A_i are perfect $\mathrm{Aut}(G)$-subgroups, which are indecomposable as $\mathrm{Aut}(G)$-subgroups. Finally, the uniqueness of the factorization (2.1) (see [12, Theorem 3.3.8]) and, again, the fact that perfect groups have no nontrivial central automorphisms imply that the central product decomposition (2.2) is also unique. □

We make use of the same notation as [5]. In particular, we define the following subsets of subgroups of $S(G)$:

$$\mathcal{H}(G) = \{H \leq S(G) \mid H \text{ is regular, } H \simeq G,\ N_{S(G)}(H) = \mathrm{Hol}(G)\},$$
$$\mathcal{I}(G) = \{H \leq S(G) \mid H \text{ is regular, } N_{S(G)}(H) = \mathrm{Hol}(G)\},$$
$$\mathcal{J}(G) = \{H \leq S(G) \mid H \text{ is regular, } N_{S(G)}(H) \geq \mathrm{Hol}(G)\}.$$

Note that $\mathcal{H}(G) \subseteq \mathcal{I}(G) \subseteq \mathcal{J}(G)$.

From now on, we assume that G is a finite semisimple group, and we write G as a central product of indecomposable $\mathrm{Aut}(G)$-subgroups, in a unique way (by Lemma 1) as $G = A_1 A_2 \ldots A_n$. Note that, by Proposition 1 (1), we have that $[\rho(A_i), \lambda(A_j)] = 1$ for every $i \neq j$.

Fix I to be the set $\{1, 2, \ldots, n\}$. For each subset J of I, we denote the central product $\prod_{j \in J} A_j$ by A_J. Then, for each subset J of I, we may define the subgroup G_J of $S(G)$ to be $G_J = \rho(A_J)\lambda(A_{J^c})$, where $J^c = I \setminus J$. Note that $G_I = \rho(G)$ and $G_\varnothing = \lambda(G)$.

Lemma 2. *Each G_J is a subgroup of $S(G)$ that lies in* $\mathrm{Hol}(G)$.

Proof. Since $[\rho(G), \lambda(G)] = 1$, for every $x_J, y_J \in A_J$, $x_{J^c}, y_{J^c} \in A_{J^c}$, it follows that

$$\begin{aligned}\rho(x_J)\lambda(x_{J^c})(\rho(y_J)\lambda(y_{J^c}))^{-1} &= \rho(x_J)\rho(y_J)^{-1}\lambda(x_{J^c})\lambda(y_{J^c})^{-1}\\ &= \rho(x_J y_J^{-1})\lambda(y_{J^c}^{-1} x_{J^c}),\end{aligned}$$

which lies in G_J.

Moreover, since $\rho(A_J) \leq \rho(G)$ and $\lambda(A_{J^c}) \leq \lambda(G)$, we have

$$G_J = \rho(A_J)\lambda(A_{J^c}) \leq \langle \rho(G), \lambda(G) \rangle \leq \mathrm{Hol}(G),$$

completing the proof of the lemma. □

Lemma 3. *The inversion map* inv *conjugates* G_J *to* G_{J^c} *for every* $J \subseteq I$.

Proof. The lemma is an immediate consequence of Proposition 1 (3). □

Lemma 4. *Assume that* G *is a finite semisimple group. Then, with the above notation,* $\mathcal{J}(G) = \{G_J \mid J \subseteq I\}$.

Proof. We first show that G_J acts regularly on the set G. For an arbitrary $g \in G$, write $g = x_J x_{J^c}$, with $x_J \in A_J$ and $x_{J^c} \in A_{J^c}$. Then the element $\rho(x_J)\lambda(x_{J^c}^{-1})$ of G_J sends 1 to g,

$$1^{\rho(x_J)\lambda(x_{J^c})} = x_{J^c}x_J = x_J x_{J^c} = g,$$

since $[A_J, A_{J^c}] = 1$. Moreover, the stabilizer of 1 in G_J consists of the elements $\sigma = \rho(x_J)\lambda(x_{J^c})$ such that $x_J = x_{J^c}^{-1} \in A_J \cap A_{J^c} \leq Z(G)$, and therefore σ is the identity.

To show that each G_J is normal in $\mathrm{Hol}(G)$, since $[\rho(G), \lambda(G)] = 1$, it is enough to show that $\mathrm{Aut}(G)$ normalizes every G_J. Fix $J \subseteq I$; let $\alpha \in \mathrm{Aut}(G)$, and let $\rho(x_J)\lambda(x_{J^c})$ be an arbitrary element of G_J, where $x_J \in A_J$ and $x_{J^c} \in A_{J^c}$. Then we have that, for every $g \in G$,

$$g^{\alpha^{-1}\rho(x_J)\lambda(x_{J^c})\alpha} = (x_{J^c} g^{\alpha^{-1}} x_J)^\alpha = x_{J^c}^\alpha g x_J^\alpha = g^{\rho(x_J^\alpha)\lambda(x_{J^c}^\alpha)}.$$

Therefore,

$$(\rho(x_J)\lambda(x_{J^c}))^\alpha = \rho(x_J^\alpha)\lambda(x_{J^c}^\alpha),$$

which lies in G_J, since A_J and A_{J^c} are characteristic subgroups of G.

Finally, by [5, Theorem 7.8], $|\mathcal{J}(G)| = 2^n$, and therefore, to complete the proof, it remains to show that $G_J \neq G_K$ whenever $J \neq K$. If $G_J = G_K$, then there exists an $i \in I$ for which G_J contains both $\rho(A_i)$ and $\lambda(A_i)$. But then the stabilizer of 1 in G_J would contain $\{\rho(x)\lambda(x^{-1}) \mid x \in A_i\}$, that is, all the conjugates of elements of A_i. Since A_i is not central in G, this contradicts the fact that G_J is regular. □

3 Automorphisms of finite quasisimple groups

In this section, we classify all finite quasisimple groups that admit an automorphism acting like inversion on the center. This classification, which is used in the proof of our main result, is proved in Proposition 2 using the Classification of Finite Simple Groups. This result completely answers a question posed in [5, Remark 7.12] (see also [2, ADV - 4B]), namely, whether there are finite quasisimple groups L such that $Z(L)$ is not elementary abelian, and such that $\mathrm{Aut}(L)$ does not induce inversion on $Z(L)$, or acts trivially on it.

Before stating and proving Proposition 2, we introduce some notation and terminology related to automorphisms of finite nonabelian simple groups of Lie type. We refer the interested reader to the first two chapters of [7].

Let S be any finite group of Lie type. Then, by [13, Theorem 30], any automorphism of S is a product $idfg$, where i is an inner automorphism of S, d is a diagonal automorphism of S, f is a field automorphism of S and g is a graph automorphism of S. Using the notation of [7], $\mathrm{Inndiag}(S)$, Φ_S and Γ_S denote, respectively, the group of inner-diagonal automorphisms of S, of field automorphisms of S, and of graph automorphisms of S. Further, $\mathrm{Outdiag}(S)$ is defined to be $\mathrm{Inndiag}(S)/\mathrm{Inn}(S)$ (see [7, Definition 2.5.10]).

Proposition 2. *Let K be a finite quasisimple group. Then there exists an automorphism of K that inverts $Z(K)$ if and only if K is not isomorphic to one of the following groups:*

(1) *a covering of* $\mathrm{L}_3(4)$*, with center containing* $Z_2 \times Z_2 \times Z_3$*,*

(2) *a covering of* $\mathrm{U}_4(3)$*, with center containing* $Z_3 \times Z_4$*,*

(3) $\widetilde{\mathrm{U}_6(2)}$*, the universal covering of* $\mathrm{U}_6(2)$*,*

(4) $\widetilde{{}^2\mathrm{E}_6(2)}$*, the universal covering of* ${}^2\mathrm{E}_6(2)$*.*

Proof. As is well known (see for example [1]), any finite quasisimple group K is isomorphic to a quotient of the universal covering group of its simple quotient $K/Z(K)$. Also, $\mathrm{Aut}(K) \simeq \mathrm{Aut}(K/Z(K))$ (see for instance [1, Section 33]). Therefore, for our purposes, it is enough to consider the action of $\mathrm{Aut}(S)$ on the Schur multiplier $M(S)$ when S varies among all finite nonabelian simple groups. In this situation, the outer automorphism group $\mathrm{Out}(S)$ acts on $M(S)$, which, by [6, Section 5, 6-1], is isomorphic to the direct product of two factors of relatively prime orders, $M_c(S)$ and $M_e(S)$. The actions of $\mathrm{Out}(S)$ on both factors are completely described in [7, Theorem 6.3.1 and Theorem 2.5.12] for every finite nonabelian simple group S. In particular, $\mathrm{Outdiag}(S)$ centralizes $M_c(S)$, and there is an isomorphism of $\mathrm{Outdiag}(S)$ on $M_c(S)$ preserving the action of $\mathrm{Out}(S)$. Note also that if one of the factors $M_c(S)$ or $M_e(S)$ has order at most 2, since they have coprime orders, to prove our statement, it is enough to see if inversion is induced by $\mathrm{Out}(S)$ on the other factor. We may therefore consider the two cases:

(1) $|M_e(S)| \leq 2$,

(2) $|M_e(S)| > 2$.

Case (1) $|M_e(S)| \leq 2$. We prove that, in this case, there is always an automorphism of S inverting $M(S)$. As noted above, it is enough to consider the action of

$\mathrm{Out}(S)$ on $\mathrm{Outdiag}(S)$, which is $\mathrm{Aut}(S)$-isomorphic to $M_c(S)$. But $\mathrm{Outdiag}(S)$ is always inverted by $\mathrm{Out}(S)$ since, by [7, Theorem 2.5.12], we have that,

(i) if $S \in \{A_m(q), D_{2m+1}(q), \mathrm{E}_6(q)\}$, then $\mathrm{Outdiag}(S)$ is inverted by a graph automorphism (by [7, Theorem 2.5.12 (i)]),

(ii) if $S \in \{{}^2A_m(q), {}^2D_{2m+1}(q), {}^2\mathrm{E}_6(q)\}$, $\mathrm{Outdiag}(S)$ is inverted by a field automorphism (by [7, Theorem 2.5.12 (g)]),

(iii) in all other cases, $\mathrm{Outdiag}(S)$ is either trivial or an elementary abelian 2-group, and therefore it is inverted by the trivial automorphism.

Case (2) $|M_e(S)| > 2$. From [7, Table 6.3.1] and the knowledge of the corresponding factor $M_c(S)$ ([7, Theorem 2.5.12]), we can see that if S is not isomorphic to one of the simple groups

$$\mathrm{L}_3(4), \mathrm{U}_6(2), {}^2\mathrm{E}_6(2), \mathrm{U}_4(3),$$

then there exists an automorphism of S that inverts $M(S)$ and, therefore, any quasisimple group K such that $K/Z(K) \simeq S$ admits an automorphism inverting its center. We now consider separately the four special cases listed above.

Let $S = \mathrm{L}_3(4)$. Then $M_e(S) \simeq Z_4 \times Z_4$ and $M_c(S) \simeq Z_3$. Here

$$\mathrm{Out}(S) = \Sigma \times \langle u \rangle,$$

with $\Sigma = \mathrm{Outdiag}(S)\Gamma_S \simeq S_3$ and u the image in $\mathrm{Out}(S)$ of a graph-field automorphism of order 2. By [7, Theorem 6.3.1], u is the only element of $\mathrm{Out}(S)$ that induces inversion of $M_e(S)$. Now, u is $\mathrm{Aut}(S)$-conjugate to an element of the form ϕi, with ϕ a field automorphism and i a graph automorphism, where ϕ and i are commuting involutions (note that $\Phi_S\Gamma_S \simeq Z_2 \times Z_2$). The action of $\Phi_S\Gamma_S$ on $M_c(S)$ is the same as on $\mathrm{Outdiag}(S)$. Thus, by [7, Theorem 2.5.12 (g) and (i)], both ϕ and i invert $M_c(S)$, and therefore u acts trivially on it. This argument shows that, when K is the universal covering group of S, no inversion on $Z(K)$ is induced by an automorphism of K. Assume now that K is a covering of S different from the universal one. If $3 \nmid |Z(K)|$, then u inverts $Z(K)$. Assume therefore that $3 \mid |Z(K)|$. If $Z(K)$ is cyclic of order 3 or 6, then ϕ inverts $Z(K)$. Otherwise, we may argue as follows. Since $M_c(S)$ and $\mathrm{Outdiag}(S)$ are $\mathrm{Aut}(S)$-isomorphic, the elements of $\mathrm{Out}(S)$ that induce inversion on $M_c(S)$ are the six non-central involutions, that is, the elements of the set

$$T = \mathrm{Outdiag}(S)\phi \cup \mathrm{Outdiag}(S)i.$$

Note that, from [7, Proposition 6.2.2 and the proof of Theorem 6.3.1], $\mathrm{Outdiag}(S)$ acts faithfully on the quotient group $M_e(S)/\Phi(M_e(S))$, and hence on $M_e(S)$.

Next, let t be an element of T. Since t inverts $\mathrm{Outdiag}(S)$,

$$C_{M_e(S)/\Phi(M_e(S))}(t) \simeq C_{\Phi(M_e(S))}(t) \neq \Phi(M_e(S)),$$

so $C_{M_e(S)}(t) = \langle b \rangle \simeq Z_4$, and t inverts a unique cyclic subgroup of order 4. This in particular shows that, when K is a covering extension with

$$Z(K) \simeq Z_2 \times Z_2 \times Z_3 \quad \text{or} \quad Z(K) \simeq Z_2 \times Z_4 \times Z_3,$$

no inversion is induced by automorphisms of K on $Z(K)$, while inversion is induced if $Z(K) \simeq Z_4 \times Z_3$.

Let $S = \mathrm{U}_4(3)$. Then $M_e(S) \simeq Z_3 \times Z_3$ and $M_c(S) \simeq Z_4$. Here

$$\mathrm{Out}(S) = \mathrm{Outdiag}(S)\Phi_S$$

is isomorphic to a dihedral group of order 8 acting faithfully on $M_e(S)$. In particular, the nontrivial central element of $\mathrm{Out}(S)$ is the unique element inducing inversion on $M_e(S)$. Note that this element belongs to $\mathrm{Outdiag}(S)$, and therefore it centralizes $M_c(S)$. This implies that no inversion can be induced on $M(S)$ by automorphisms of S. Therefore, the universal covering group $\widetilde{S}$ has no automorphisms inverting its center. This argument can be extended to show that the same situation occurs in any covering having center of order 12. However, for all other coverings $\widehat{S}$ of S, it can be easily checked that inversion on the center is induced either by the nontrivial central element of $\mathrm{Outdiag}(S)\Phi_S$ when 3 divides $|Z(\widehat{S})|$, or by a field automorphism of order two when $3 \nmid |Z(\widehat{S})|$.

Let $S = \mathrm{U}_6(2)$, or $S = {}^2\mathrm{E}_6(2)$. In both cases, we have that

$$M_e(S) \simeq Z_2 \times Z_2, \quad M_c(S) \simeq Z_3,$$

and $\mathrm{Out}(S) \simeq S_3$ acts faithfully on $M_e(S)$ (see [7, Proposition 6.2.2]). In particular, the trivial outer automorphism is the only one that inverts $M_e(S)$. Since it does not invert $M_c(S)$, the universal covering groups $\widetilde{S}$ have no automorphisms inverting their centers. On the contrary, every covering $\widehat{S}$ different from the universal one possesses such automorphisms, which are either trivial if $3 \nmid |Z(\widehat{S})|$, or are field automorphisms. □

For convenience, we write $\mathcal{L}$ for the set of quasisimple groups that appear as exceptions in Proposition 2; thus

$$\mathcal{L} = \Big\{\widehat{\mathrm{L}_3(4)} \text{ (with } Z(\widehat{\mathrm{L}_3(4)}) \geq Z_2 \times Z_2 \times Z_3),$$
$$\widehat{\mathrm{U}_4(3)} \text{ (with } Z(\widehat{\mathrm{U}_4(3)}) \geq Z_3 \times Z_4),\ \widetilde{\mathrm{U}_6(2)},\ \widetilde{{}^2\mathrm{E}_6(2)}\Big\}.$$

Remark 2. According to [7, Theorem 6.3.2], when $S \simeq \mathrm{L}_3(4)$ (or $S \simeq \mathrm{U}_4(3)$), there are precisely two non-isomorphic covering groups $\widehat{S}$ such that $\widehat{S}/Z(\widehat{S}) \simeq S$ and $Z(\widehat{S}) \simeq Z_2 \times Z_4 \times Z_3$ (respectively, $Z(\widehat{S}) \simeq Z_3 \times Z_4$). In all other cases in the list $\mathcal{L}$, the covering group of the associated finite simple group is unique. Therefore, up to isomorphism, $|\mathcal{L}| = 9$.

As an application of Proposition 2, we prove the following.

Corollary 1. *Assume that X is a finite semisimple group with all components not in $\mathcal{L}$. Then there exists an automorphism of X that inverts the center $Z(X)$.*

Proof. Let X be a finite semisimple group. Then X is isomorphic to D/N, with $D = K_1 \times K_2 \times \cdots \times K_t$ the direct product of nonabelian quasisimple groups $K_i \notin \mathcal{L}$ and $N \leq D$ such that $N \cap K_i = 1$ for each $i \in \{1, \dots, t\}$. Note that

$$N \leq Z(D) = Z(K_1) \times Z(K_2) \times \cdots \times Z(K_t).$$

By Proposition 2, for each $i \in \{1, \dots, t\}$ there exists an automorphism α_i of K_i that acts like the inversion on $Z(K_i)$. We may therefore define $\alpha \in \mathrm{Aut}(D)$ by $x^\alpha = (x_1, x_2, \dots, x_t)^\alpha = (x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_t^{\alpha_t})$ for $x \in D$ (where each $x_i \in K_i$). Note that every element of $Z(D)$ is inverted by α. In particular, $N^\alpha = N$, and α induces an automorphism on D/N that inverts its center. □

4 The holomorph of a semisimple group

We first consider the case in which G has no components belonging to $\mathcal{L}$.

Proposition 3. *Assume that G is a finite nonabelian semisimple group, and let $G = A_1 A_2 \dots A_n$ be its unique central decomposition as a product of* $\mathrm{Aut}(G)$*-indecomposable factors. Suppose that the components of G do not belong to $\mathcal{L}$. Then $\mathcal{H}(G) = \{G_J \mid J \subseteq I\}$. Moreover, the group $T(G) = \mathrm{NHol}(G)/\mathrm{Hol}(G)$ is elementary abelian of order 2^n.*

Proof. By Lemma 4, we have $\mathcal{J}(G) = \{G_J \mid J \subseteq I\}$, so $\mathcal{H}(G) \subseteq \{G_J \mid J \subseteq I\}$. We fix a subset J of I and, using Corollary 1, choose an automorphism α_{J^c} of A_{J^c} that inverts $Z(A_{J^c})$. Define the map $\varphi_J \colon G \to G$ by $\varphi_J(x_J x_{J^c}) = x_J (x_{J^c})^{-\alpha_{J^c}}$. Since α_{J^c} inverts $A_J \cap A_{J^c} \leq Z(G)$, the map φ_J is a well-defined bijection of G, that is, an element of $S(G)$. We claim that the following hold:

(1) φ_J conjugates G_I to G_J;

(2) $\varphi_J \in \mathrm{NHol}(G)$ and, if $J \neq I$, then $\varphi_J \notin \mathrm{Hol}(G)$;

(3) $(\varphi_J)^2 \in \mathrm{Hol}(G)$.

Note that, by the arbitrary choice of $J \subseteq I$, once proved, (1) will imply that $\mathcal{H}(G) = \mathcal{J}(G) = \{G_J \mid J \subseteq I\}$, while (2), (3) and the fact that $T(G)$ acts regularly on $\mathcal{H}(G)$ will imply that $T(G)$ is an elementary abelian 2-group of rank n.

(1) $(G_I)^{\varphi_J} = G_J$. We claim that, for $x_J \in A_J$ and $x_{J^c} \in A_{J^c}$,

$$(\rho(x_J x_{J^c}))^{\varphi_J} = \rho(x_J)\lambda(x_{J^c}^{-\alpha_{J^c}})$$

or, equivalently, that

$$\rho(x_J x_{J^c}) \cdot \varphi_J = \varphi_J \cdot \rho(x_J)\lambda(x_{J^c}^{-\alpha_{J^c}}).$$

Let $g \in G$, and write g as $g = y_J y_{J^c}$ (with $y_J \in A_J$ and $y_{J^c} \in A_{J^c}$). Then

$$\begin{aligned} g^{\rho(x_J x_{J^c})\cdot\varphi_J} &= (y_J y_{J^c} x_J x_{J^c})^{\varphi_J} = (y_J x_J y_{J^c} x_{J^c})^{\varphi_J} \\ &= y_J x_J (y_{J^c} x_{J^c})^{-\alpha_{J^c}} = y_J x_J x_{J^c}^{-\alpha_{J^c}} y_{J^c}^{-\alpha_{J^c}}, \\ g^{\varphi_J\cdot\rho(x_J)\lambda(x_{J^c}^{-\alpha_{J^c}})} &= (y_J y_{J^c}^{-\alpha_{J^c}})^{\rho(x_J)\lambda(x_{J^c}^{-\alpha_{J^c}})} = x_{J^c}^{-\alpha_{J^c}} y_J y_{J^c}^{-\alpha_{J^c}} x_J \\ &= y_J x_J x_{J^c}^{-\alpha_{J^c}} y_{J^c}^{-\alpha_{J^c}}. \end{aligned}$$

Therefore, (1) is proved. Note that, together with Lemma 4, we have proved that $\mathcal{J}(G) \subseteq \mathcal{H}(G)$ and therefore $\mathcal{H}(G) = \mathcal{I}(G) = \mathcal{J}(G)$.

(2) $\varphi_J \in \mathrm{NHol}(G)$. By (1), we have that

$$(N_{S(G)}(G_I))^{\varphi_J} = N_{S(G)}(G_I^{\varphi_J}) = N_{S(G)}(G_J) = N_{S(G)}(G_I)$$

since each G_J lies in $\mathcal{H}(G) = \mathcal{I}(G)$, and therefore $N_{S(G)}(G_J) = N_{S(G)}(G_I)$; thus $\varphi_J \in \mathrm{NHol}(G)$.

Furthermore, $G_J \neq G_I$ for $J \neq I$; thus we trivially have that $\varphi_J \notin \mathrm{Hol}(G)$ for $J \neq I$.

(3) $(\varphi_J)^2 \in \mathrm{Hol}(G)$. We claim that, for every $x_J x_{J^c} \in G$,

$$(\rho(x_J x_{J^c}))^{\varphi_J^2} = \rho(x_J x_{J^c}^{\alpha_{J^c}^2})$$

or, equivalently, that

$$\rho(x_J x_{J^c}) \cdot \varphi_J^2 = \varphi_J^2 \cdot \rho(x_J x_{J^c}^{\alpha_{J^c}^2}).$$

Let $g \in G$, and write $g = y_J y_{J^c}$, where $y_J \in A_J$, $y_{J^c} \in A_{J^c}$. Then

$$\begin{aligned} g^{\rho(x_J x_{J^c})\cdot\varphi_J^2} &= (y_J y_{J^c} x_J x_{J^c})^{\varphi_J^2} = (y_J x_J y_{J^c} x_{J^c})^{\varphi_J^2} \\ &= y_J x_J (y_{J^c} x_{J^c})^{\alpha_{J^c}^2} = y_J y_{J^c}^{\alpha_{J^c}^2} x_J x_{J^c}^{\alpha_{J^c}^2}, \end{aligned}$$

while

$$g^{\varphi_J^2 \cdot \rho(x_J x_{J^c}^{\alpha_{J^c}^2})} = (y_J y_{J^c}^{\alpha_{J^c}^2})^{\rho(x_J x_{J^c}^{\alpha_{J^c}^2})} = y_J y_{J^c}^{\alpha_{J^c}^2} x_J x_{J^c}^{\alpha_{J^c}^2}.$$

This completes the proof of Proposition 3. □

Remark 3. For each fixed subset J of I, we may define an operation $\circ_J$ on the set of elements of G as follows:

$$g \circ_J h = g_J g_{J^c} \circ_J h_J h_{J^c} = g_J h_J h_{J^c} g_{J^c}$$

for each $g = g_J g_{J^c}, h = h_J h_{J^c} \in G$, where $g_J, h_J \in A_J$ and $g_{J^c}, h_{J^c} \in A_{J^c}$. Then $(G, \circ_J)$ is a group. Note that $\circ_I$ coincides with the group operation of G, while $\circ_\varnothing$ with the opposite multiplication in G, that is, $g_1 \circ_\varnothing g_2 = g_2 g_1$ for every $g_1, g_2 \in G$. With this notation, it is straightforward to prove that

(1) $(G, \circ_J)$ is a group isomorphic to G_J for each $J \subseteq I$,

(2) $\mathrm{Aut}(G) = \mathrm{Aut}(G, \circ_J)$ for each $J \subseteq I$ (see [5, Theorem 5.2 (d)],

(3) if G satisfies the assumptions of Proposition 3, each map φ_J is an isomorphism between $(G, \circ_I)$ and $(G, \circ_J)$.

We consider now the general situation in which exactly l components of G do belong to $\mathcal{L}$. If $K \in \mathcal{L}$, we call *$\mathcal{L}$-critical* any subgroup U of $Z(K)$ such that, respectively,

- $U \simeq Z_2 \times Z_2 \times Z_3$ if $K \simeq \widehat{\mathrm{L}_3(4)}$,
- $U \simeq Z_3 \times Z_4$ if $K \simeq \widehat{\mathrm{U}_4(3)}$,
- $U = Z(K)$ if $K \simeq \widetilde{\mathrm{U}_6(2)}$ or if $K = \widetilde{{}^2\mathrm{E}_6(2)}$.

Theorem 1. *Let G be a finite nonabelian semisimple group, and let*

$$G = A_1 A_2 \dots A_n$$

be its unique central decomposition as a product of $\mathrm{Aut}(G)$*-indecomposable factors. Assume that the number of factors A_i of G having components in $\mathcal{L}$ is exactly l for some $0 \le l \le n$. Then $T(G)$ is an elementary abelian group of order 2^h for some h with $\min\{n - l + 1, n\} \le h \le n$, acting regularly on $\mathcal{H}(G)$. Moreover, if the centers of the factors A_i are all amalgamated, then $|\mathcal{H}(G)| = 2^m$, and $T(G)$ is elementary abelian of order 2^m, where $m = \min\{n, n - l + 1\}$, and therefore $|\mathcal{H}(G)| = 2^m$.*

Proof. By Proposition 3, the result is clear for $l = 0$, so assume $l > 0$.

Without loss of generality, we may assume that $A_1, A_2, \dots, A_l$ are the central $\mathrm{Aut}(G)$-indecomposable factors having components in $\mathcal{L}$. We set $L = \{1, 2, \dots, l\}$ and claim that $\mathcal{H}(G)$ contains the set $\mathcal{K} = \{G_J, G_{J^c} \mid J \cap L = \varnothing\}$, whose cardinality is $2 \cdot |\mathcal{P}(L^c)| = 2^m$. By the definition of $\mathcal{H}(G)$ and Lemmas 3 and 2, it is enough to show that $G_{J^c} \simeq G_I$ for each subset J of L^c. By Corollary 1, there exists an automorphism α_J of A_J that inverts $Z(A_J)$. Therefore, the map φ_{J^c} defined as in Proposition 3 by $\varphi_{J^c}(x_J x_{J^c}) = (x_J)^{-\alpha_J} x_{J^c}$ for each $x_J \in A_J$, $x_{J^c} \in A_{J^c}$ is a well-defined bijection of G that conjugates G_I to G_{J^c}. Note that $T(G)$ contains the elementary abelian 2-subgroup $\{\varphi_{J^c} \mid J \cap L = \varnothing\}$.

Assume now that $\mathcal{K} \subset \mathcal{H}(G)$. Note that $G_R \in \mathcal{H}(G) \setminus \mathcal{K}$ for some $R \subset I$ if and only if $G_{R^c} \in \mathcal{H}(G) \setminus \mathcal{K}$. This shows in particular that $|\mathcal{H}(G)|$ is even.

Moreover, by Remark 3, $G_R \in \mathcal{H}(G) \setminus \mathcal{K}$ if and only if G_R is isomorphic to the group $(G, \circ_R)$. Now, by Remark 3, any possible isomorphism α from G to $(G, \circ_R)$ maps each A_i to itself. In particular, if we take $r \in R \cap L$, $s \in R^c \cap L$, the isomorphism α induces an automorphism on A_r and an antihomomorphism on A_s, as we have

$$(a_r b_r)^\alpha = a_r^\alpha b_r^\alpha = a_r^\alpha \circ_J b_r^\alpha \quad \text{for each } a_r, b_r \in A_r, \tag{4.1}$$

$$(a_s b_s)^\alpha = a_s^\alpha b_s^\alpha = b_s^\alpha \circ_J a_s^\alpha \quad \text{for each } a_s, b_s \in A_s. \tag{4.2}$$

Condition (4.1) implies that the restriction of α to $W = A_r \cap A_s$ is a homomorphism, while condition (4.2) implies that the restriction of $\alpha \circ \mathrm{inv}$ to W is a homomorphism. We deduce that the inversion map on W is induced by an automorphism of A_r, which is in contradiction with Proposition 2 if W contains an $\mathcal{L}$-critical subgroup for some component of A_r. Thus we have proved that $G_R \in \mathcal{H}(G) \setminus \mathcal{K}$ if and only if, for every $r \in R \cap L$ and every $s \in R^c \cap L$, the subgroup $A_r \cap A_s$ does not contain $\mathcal{L}$-critical subgroups of components. Note that this is equivalent to saying that the involutory map φ_R is an element of $T(G)$. In particular, $T(G)$ is an elementary abelian 2-group of order $|\mathcal{H}(G)|$, which is therefore a power of 2.

When $Z(A_i) = Z(G)$ for each $i = 1, 2, \dots, n$, the result is clear since each $A_r \cap A_s = Z(G)$. □

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