Growth rate for endomorphisms of finitely generated nilpotent groups

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Abstract. We prove that the growth rate of an endomorphism of a finitely generated nilpotent group is equal to the growth rate of the induced endomorphism on its abelianization, generalizing the corresponding result for an automorphism in [T. Koberda, Entropy of automorphisms, homology and the intrinsic polynomial structure of nilpotent groups, in: *In the Tradition of Ahlfors–Bers. VI*, Contemp. Math. 590, American Mathematical Society, Providence (2013), 87–99].

1 Introduction

In the present paper, we study purely algebraic notions of growth rate and entropy for an endomorphism of a finitely generated group.

Let π be a finitely generated group with a system $S = \{s_1, \dots, s_n\}$ of generators. Let $\phi: \pi \rightarrow \pi$ be an endomorphism. For any $\gamma \in \pi$, let $L(\gamma, S)$ be the length of the shortest word in the letters $S \cup S^{-1}$ which represents γ . Then the *growth rate* of ϕ is defined [2] to be

$$\text{GR}(\phi) := \sup \left\{ \limsup_{k \rightarrow \infty} L(\phi^k(\gamma), S)^{1/k} \mid \gamma \in \pi \right\}.$$

For each $k > 0$, we put

$$L_k(\phi, S) := \max \{ L(\phi^k(s_i), S) \mid i = 1, \dots, n \}.$$

It is known for example from [2, Proposition 1] that

$$\text{GR}(\phi) = \lim_{k \rightarrow \infty} L_k(\phi, S)^{1/k} = \inf_k \{ L_k(\phi, S)^{1/k} \},$$

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and the *algebraic entropy* of ϕ is by definition $h_{\text{alg}}(\phi) := \log \text{GR}(\phi)$. The growth rate and hence the algebraic entropy of ϕ are well-defined, i.e., independent of the choice of a set of generators [7, p. 114]. It is immediate from the definition that the growth rate and the algebraic entropy for an endomorphism of a group are invariants of conjugacy of group endomorphisms. Furthermore, for any inner automorphism τ_{γ_0} by γ_0 , we have $\text{GR}(\tau_{\gamma_0}\phi) = \text{GR}(\phi)$ and $h_{\text{alg}}(\tau_{\gamma_0}\phi) = h_{\text{alg}}(\phi)$ ([7, Proposition 3.1.10]).

Consider a continuous map f on a compact connected manifold M , and consider a homomorphism ϕ induced by f of the group of covering transformations on the universal cover of M . Then the topological entropy $h_{\text{top}}(f)$ is defined. We refer to [7] for background. Several authors, among them R. Bowen in [2] and A. Katok in [8], have proved that the topological entropy $h_{\text{top}}(f)$ of f is at least as large as the algebraic entropy $h_{\text{alg}}(\phi) = h_{\text{alg}}(f)$ of ϕ or f .

The problem of determining the growth rate of a group endomorphism, initiated by R. Bowen in [2], is now an area of active research (see detailed description in [5, 10] and references therein). For known properties of the growth of automorphisms of free groups, we refer to [1, 12, 13].

The purpose of this paper is first to study the growth rate of an endomorphism on a finitely generated nilpotent group. In [10, Theorem 1.2], it was proven that the growth rate of an automorphism of a finitely generated nilpotent group is equal to the growth rate of the induced automorphism on its abelianization. Our main result is a generalization of this result of [10] from automorphisms to endomorphisms, using completely different arguments. In Section 2, we recall some known results about the growth rate of a group endomorphism, sometimes correcting them. In Section 3, we refine the calculation in [2] of the growth rate for an endomorphism of a finitely generated torsion-free nilpotent group and prove that the growth rate is an algebraic integer.

Let π be a finitely generated torsion-free nilpotent group, and let G be its Malcev completion. Let ϕ be an endomorphism of π . Then ϕ extends uniquely to a Lie group homomorphism D of G , called the Malcev completion of ϕ . We call its differential D_* the *linearization* of ϕ . The main results are the following.

Theorem 3.3. *Let $\phi: \pi \rightarrow \pi$ be an endomorphism on a finitely generated torsion-free nilpotent group π . Let G be the Malcev completion of π . Then the linearization $D_*: \mathfrak{G} \rightarrow \mathfrak{G}$ of ϕ can be expressed as a lower triangular block matrix with diagonal blocks $\{D_j\}$ so that*

$$\text{GR}(\phi) = \max_{j \geq 1} \{\text{sp}(D_j)^{1/j}\}.$$

In particular, $\text{GR}(\phi)$ is an algebraic integer.

Theorem 3.7. *Let $\phi: \pi \rightarrow \pi$ be an endomorphism on a finitely generated torsion-free nilpotent group π with Malcev completion D . Then*

$$\text{GR}(\phi) = \text{GR}(\phi_{\text{ab}}),$$

where $\phi_{\text{ab}}: \pi/[\pi, \pi] \rightarrow \pi/[\pi, \pi]$ be the endomorphism induced by ϕ . Hence we have $\text{GR}(\phi) = \text{sp}(D_1) \leq \text{sp}(D_*)$.

2 Preliminaries

We shall assume in this article that *all groups are finitely generated* unless otherwise specified. For a given endomorphism $\phi: \pi \rightarrow \pi$, if π' is a ϕ -invariant subgroup of π , we denote by $\phi' = \phi|_{\pi'}$ the restriction of ϕ to π' . If, in addition, π' is a normal subgroup, we denote by $\hat{\phi}$ the endomorphism on π/π' induced by ϕ . Then the following are known; see for example [2, 5].

- $\text{GR}(\phi^k) = \text{GR}(\phi)^k$ for $k > 0$.
- $\text{GR}(\hat{\phi}) \leq \text{GR}(\phi)$.
- $\text{GR}(\phi) \leq \max\{\text{GR}(\phi'), \text{GR}(\hat{\phi})\}$.
- Let $\phi: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ be an endomorphism yielding an integer matrix D . Then we have $\text{GR}(\phi) = \text{sp}(D)$, the maximum of the absolute values of the eigenvalues of D .

Let S' be a finite set of generators for π' , and let $\hat{S}$ be a finite set of generators for the quotient group π/π' . Then it is possible to extend S' to a finite set S of generators for π so that S is projected onto $\hat{S}$ under the projection $\pi \rightarrow \pi/\pi'$. For any $\gamma \in \pi'$, it is true that $L(\gamma, S') \geq L(\gamma, S)$.

Consider the concentric balls $B(n) = \{\gamma \in \pi \mid L(\gamma, S) \leq n\}$ for all $n > 0$, and the *distortion function* of π' in π which is defined as

$$\Delta_{\pi'}^{\pi}(n) := \max\{L(\gamma, S') \mid \gamma \in \pi' \cap B(n)\}.$$

The notion of distortion of a subgroup was first introduced by M. Gromov in [6]. We refer to [3] for our discussion. For two functions $f, g: \mathbb{N} \rightarrow \mathbb{N}$, we say that $f \preceq g$ if there exists $c > 0$ such that $f(n) \leq cg(cn)$ for all $n > 0$. We say that two functions are equivalent, written $f \approx g$, if $f \preceq g$ and $g \preceq f$. The subgroup π' of π is *undistorted* if $\Delta_{\pi'}^{\pi}(n) \approx n$. The following facts about distortion can be found in [3].

- If π' is infinite, then it is true that $n \preceq \Delta_{\pi'}^{\pi}(n)$.
- If $[\pi : \pi'] < \infty$, then π' is undistorted in π .

Assume $\Delta_{\pi'}^{\pi}(n) \leq n$. By definition, there exists $c > 0$ such that $\Delta_{\pi'}^{\pi}(n) \leq c^2 n$ for all $n > 0$. For any $\gamma \in \pi'$, let $n = L(\gamma, S)$. Then

$$L(\gamma, S') \leq \Delta_{\pi'}^{\pi}(n) \leq c^2 n = c^2 L(\gamma, S).$$

Thus $L(\gamma, S) \leq c^2 L(\gamma, S)$ for all $\gamma \in \pi'$. This inequality implies that, for all $k > 0$,

$$\begin{aligned} L_k(\phi', S') &= \max\{L(\phi'^k(\gamma_i), S') \mid \gamma_i \in S'\} \\ &\leq c^2 \max\{L(\phi'^k(\gamma_i), S) \mid \gamma_i \in S'\} \leq c^2 L_k(\phi, S), \end{aligned}$$

and so $\text{GR}(\phi') \leq \text{GR}(\phi)$. Consequently, we have the following lemma.

Lemma 2.1 ([5, Corollary 3.1]). *Let ϕ be an endomorphism of π . If π' is a ϕ -invariant undistorted subgroup in π , then $\text{GR}(\phi') \leq \text{GR}(\phi)$; hence if, in addition, π' is a normal subgroup of π , then $\text{GR}(\phi) = \max\{\text{GR}(\phi'), \text{GR}(\hat{\phi})\}$.*

Proof. Since π' is undistorted in π , we have from the definition that $\Delta_{\pi'}^{\pi}(n) \leq n$. Now the proof follows from the above observation. $\square$

Remark 2.2. Note also the following.

- If π' is of finite index in π , then π' is undistorted, and hence $\text{GR}(\phi') \leq \text{GR}(\phi)$. Example 2.4 shows that the inequality can be strict. Thus [2, Proposition 1 (3)] (see also [5, Theorem 3.1]) is not correct.
- If $\text{GR}(\phi) < \text{GR}(\phi')$, then π' is distorted, and π' is not of finite index in π .

Lemma 2.3. *Let ϕ be an endomorphism of π . If $\text{GR}(\phi) < 1$, then $\text{GR}(\phi) = 0$ and ϕ is an eventually trivial endomorphism, and vice versa.*

Proof. Let $\rho = \text{GR}(\phi)$, and let $\epsilon = 1 - \rho > 0$. Since $\lim_{m \rightarrow \infty} L_m(\phi, S)^{1/m} = \rho$, there exists $N > 0$ such that, for all $m \geq N$, we have $L_m(\phi, S)^{1/m} - \rho < \epsilon$;

$$L_m(\phi, S)^{1/m} < 1 \implies L_m(\phi, S) < 1 \implies L_m(\phi, S) = 0$$

because $L_m(\phi, S)$ is a nonnegative integer. This implies that $\rho = 0$ and the endomorphism ϕ^N is trivial or ϕ is eventually trivial. The converse is obvious. $\square$

Example 2.4. Let $\pi = \mathbb{Z} \times \mathbb{Z}_2$ with generators α and β such that $\beta^2 = 1$. Consider an endomorphism ϕ of π defined by $\phi(\alpha) = 1$ and $\phi(\beta) = \beta$. Observing that

$$\begin{aligned} L_n(\phi, S) &= \max\{L(\phi^n(\alpha), S), L(\phi^n(\beta), S)\} \\ &= \max\{L(1, S), L(\beta, S)\} = \max\{0, 1\} = 1, \end{aligned}$$

we have $\text{GR}(\phi) = 1$. Similarly, we have $\text{GR}(\phi|_{\mathbb{Z}}) = 0$ and $\text{GR}(\phi|_{\mathbb{Z}_2}) = 1$. Notice further that $\mathbb{Z}_2$ is a distorted subgroup of π because $\Delta_{\mathbb{Z}_2}^{\pi}(n) = 1$ for all n .

Lemma 2.5. *Let ϕ be an endomorphism of π .*

- (1) *If π' is a ϕ -invariant finite subgroup of π , then $\text{GR}(\phi') \leq \text{GR}(\phi)$.*
- (2) *If, in addition, π' is a normal subgroup of π , then*

$$\text{GR}(\phi) = \max\{\text{GR}(\phi'), \text{GR}(\hat{\phi})\},$$

and $\text{GR}(\phi) = \text{GR}(\hat{\phi})$ if and only if ϕ' is eventually trivial or $\hat{\phi}$ is not eventually trivial.

Proof. If the ϕ -invariant subgroup π' of π is finite, then we can show easily that $\text{GR}(\phi')$ is either 0 or 1 by taking a system of generators $S' = \pi'$ for π' . We will show that $\text{GR}(\phi') \leq \text{GR}(\phi)$. We may assume that $\text{GR}(\phi') = 1$. This implies that there is an element $x \in \pi'$ such that $\phi'^n(x) \neq 1$ for all $n > 0$. Considering any system of generators for π which contains x , we can see right away that $\text{GR}(\phi) \geq 1 = \text{GR}(\phi')$.

Assume that π' is normal in π . If $\text{GR}(\phi') = 0$, then clearly $\text{GR}(\phi) = \text{GR}(\hat{\phi})$. On the other hand, if $\text{GR}(\phi') = 1$, then $\text{GR}(\phi) = \text{GR}(\hat{\phi})$ if and only if $\text{GR}(\hat{\phi}) \geq 1$ if and only if $\hat{\phi}$ is not eventually trivial by Lemma 2.3. $\square$

Remark 2.6. However, the above lemma is not true when π' is infinite; see Example 2.7. Note further that if $\text{GR}(\phi) < \text{GR}(\phi')$, then π' is infinite.

The following is a well-known example about subgroup distortion.

Example 2.7. Let π be the Baumslag–Solitar group

$$B(1, n) := \langle a, b \mid a^{-1}ba = b^n \rangle, \quad n > 1.$$

Then $S = \{a, b\}$ is a generating set for π . Let $\pi' = \langle b \rangle$, and let $S' = \{b\}$. We observe that the subgroup π' of π is distorted. In fact, since $b^{n^k} = a^{-k}ba^k$ for all $k > 0$, we have that $L(b^{n^k}, S') = n^k$ and $L(b^{n^k}, S) = 2k + 1$. If ϕ is an endomorphism of π given by $\phi(b) = b^n$ and $\phi(a) = a$, then we can see that $\text{GR}(\phi') = n$ and $\text{GR}(\phi) = 1$.

Example 2.4 shows that [2, Proposition 1 (3)] is not correct in general, but it is almost true in the sense of Theorem 2.8. By modifying the argument of the proof of [5, Theorem 3.1], we have the following theorem.

Theorem 2.8. *Let ϕ be an endomorphism of π , and let π' be a ϕ -invariant, finite-index subgroup of π .*

- (1) *If ϕ' is not an eventually trivial endomorphism, then $\text{GR}(\phi) = \text{GR}(\phi')$.*
 (2) *If ϕ' is an eventually trivial endomorphism of π' , then*

$$\text{GR}(\phi') = 0 \quad \text{and} \quad \text{GR}(\phi) = 0 \text{ or } 1.$$

Moreover, $\text{GR}(\phi) = 0$ if and only if ϕ is an eventually trivial endomorphism of π .

Consequently, the equality $\text{GR}(\phi) = \text{GR}(\phi')$ holds except only for the case when ϕ' is eventually trivial and ϕ is not eventually trivial. If this is the case, then $\text{GR}(\phi') = 0$ and $\text{GR}(\phi) = 1$.

Proof. Let $S' = \{\gamma_1, \dots, \gamma_t\}$ be a set of generators of π' . Let $u = [\pi : \pi']$. Then we have $\pi = \delta_1 \pi' \cup \dots \cup \delta_u \pi'$ so that $S = \{\gamma_1, \dots, \gamma_t, \delta_1, \dots, \delta_u\}$ generates π . For any $j = 1, \dots, u$, there exists a unique k_j such that $\phi(\delta_j) \in \delta_{k_j} \pi'$. We denote

$$p = \max_{1 \leq j \leq u} \{L(w_j, S') \mid \phi(\delta_j) = \delta_{k_j} w_j \in \delta_{k_j} \pi'\}.$$

Assume $p = 0$. Then $\phi(\delta_j) = \delta_{k_j}$ for all $j = 1, \dots, u$. For each $j = 1, \dots, u$, we write $\phi^m(\delta_j) = \delta_{j_m}$. Hence it follows that $L(\phi^m(\delta_j), S) = 0$ or 1 according to whether $\delta_{j_m} = 1$ or $\delta_{j_m} \neq 1$.

Suppose that there is $N > 0$ such that $\phi^N(\delta_j) = 1$ for all $j = 1, \dots, u$ and hence $L(\phi^m(\delta_j), S) = 0$ for all $m \geq N$. Since π' is undistorted in π , there exists some $c > 0$ such that

$$L(\gamma, S') \leq c^2 \cdot L(\gamma, S) \quad \text{for all } \gamma \in \pi'.$$

It is clear that

$$L(\gamma, S) \leq L(\gamma, S') \quad \text{for all } \gamma \in \pi'.$$

Thus

$$\begin{aligned} L_m(\phi', S') &\leq c^2 \cdot L_m(\phi, S), \\ L_m(\phi, S) &= \max\{L(\phi^m(\gamma_i), S)\} \leq L_m(\phi', S'). \end{aligned}$$

This implies that $\text{GR}(\phi') = \text{GR}(\phi)$.

Suppose on the contrary, for any $m > 0$, there is some j such that $\phi^m(\delta_j) \neq 1$. Then $\max\{L(\phi^m(\delta_j), S)\} = 1$. Hence

$$\begin{aligned} L_m(\phi, S) &= \max\{L(\phi^m(\gamma_i), S), L(\phi^m(\delta_j), S)\} \\ &= \max\{L(\phi^m(\gamma_i), S), 1\} \leq \max\{L_m(\phi', S'), 1\}. \end{aligned}$$

This implies that $\text{GR}(\phi') \leq \text{GR}(\phi) \leq \max\{\text{GR}(\phi'), 1\}$. Since ϕ' is not eventually trivial, Lemma 2.3 implies that $\text{GR}(\phi') \geq 1$, and hence $\text{GR}(\phi) = \text{GR}(\phi')$.

Next we assume that $p \geq 1$. For each $j = 1, \dots, u$, we write $\phi(\delta_j) = \delta_{j_1} w_1$ for some j_1 and $w_1 \in \pi'$. Then

$$\phi^m(\delta_j) = \delta_{j_m} w_m \phi(w_{m-1}) \cdots \phi^{m-1}(w_1),$$

and thus

$$L(\phi^m(\delta_j), S) \leq 1 + p + pL_1(\phi', S') + pL_2(\phi', S') + \cdots + pL_{m-1}(\phi', S').$$

Let $L = \text{GR}(\phi')$. By the assumption of our proposition, $L \geq 1$. Let $\epsilon > 0$ be given. Since

$$\lim_{m \rightarrow \infty} L_m(\phi', S')^{1/m} = L,$$

there is some $N > 0$ such that if $m > N$, then $L_m(\phi', S') < (L + \epsilon)^m$. Choose $q_1, \dots, q_N > 0$ such that $L_i(\phi', S') < q_i(L + \epsilon)^i$ for $i = 1, \dots, N$. Put

$$q = \max\{q_1, \dots, q_N, 1\} \geq 1.$$

Then $L_m(\phi', S') < q(L + \epsilon)^m$ for all $m \geq 1$. Hence we have

$$\begin{aligned} L(\phi^m(\delta_j), S) &\leq 1 + p + pq(L + \epsilon) + pq(L + \epsilon)^2 + \cdots + pq(L + \epsilon)^{m-1} \\ &\leq 1 + pq \frac{(L + \epsilon)^m - 1}{(L + \epsilon) - 1}. \end{aligned}$$

Since $pq \neq 0$, this implies that

$$\lim_{m \rightarrow \infty} \sqrt[m]{\max_j \{L(\phi^m(\delta_j), S)\}} \leq L + \epsilon.$$

Since π' is undistorted in π , there exists some $c > 0$ such that

$$\begin{aligned} L_m(\phi', S') &= \max_i \{L(\phi'^m(\gamma_i), S')\} \\ &\leq c^2 \cdot \max_{i,j} \{L(\phi'^m(\gamma_i), S), L(\phi^m(\delta_j), S)\} = c^2 L_m(\phi, S), \end{aligned}$$

and hence we obtain

$$L = \text{GR}(\phi') \leq \text{GR}(\phi) = \lim_{m \rightarrow \infty} \sqrt[m]{L_m(\phi, S)} \leq L + \epsilon$$

for all $\epsilon > 0$. Consequently, $\text{GR}(\phi) = \text{GR}(\phi')$.

Suppose that ϕ' is an eventually trivial endomorphism of π' . Then it is clear that $\text{GR}(\phi') = 0$. Consider a set $S = \{\gamma_1, \dots, \gamma_t, \delta_1, \dots, \delta_u\}$ of generators for π

as above. For any $m > 0$, we observe that $\phi^m(\gamma_i) = 1$ and $\phi^m(\delta_j) = \delta_{jm} w_m$ for some $j_m \in \{1, \dots, u\}$ and w_m in a finite subset of π' . This implies that the sequence $\{L_m(\phi, S)\}$ is bounded. Because $L_m(\phi, S) = 0$ or ≥ 1 , it follows that $\text{GR}(\phi) = 0$ or 1 respectively.

When $\text{GR}(\phi) = 0$, Lemma 2.3 says that ϕ is an eventually trivial endomorphism. Next we consider the case when $\text{GR}(\phi) = 1$. From the definition, we can choose $N > 0$ so that, for $m \geq N$, we have $1/2^m < L_m(\phi, S)$, which implies that $L_m(\phi, S) \geq 1$ because $L_m(\phi, S)$ is an integer. Therefore, for each $m \geq N$, we can choose $\gamma \in S$ such that $\phi^m(\gamma) \neq 1$. This shows that ϕ is not eventually trivial even though ϕ' is eventually trivial. $\square$

Before leaving this section, we observe the following elementary fact.

Proposition 2.9. *Let ϕ be an endomorphism of π with a finite set S of generators. Let*

$$\text{GR}_i(\phi) = \lim_{k \rightarrow \infty} L(\phi^k(s_i), S)^{1/k}$$

for each $s_i \in S$. Then $\text{GR}(\phi) = \max\{\text{GR}_i(\phi) \mid s_i \in S\}$.

Proof. Since $L(\phi^k(s_i), S) \leq L_k(\phi, S)$, it follows that $\text{GR}_i(\phi) \leq \text{GR}(\phi)$. Assume $\text{GR}_i(\phi) < \text{GR}(\phi)$ for all $s_i \in S$. Thus there exists $K > 0$ such that if $k \geq K$ and $s_i \in S$, then $L(\phi^k(s_i), S)^{1/k} < \text{GR}(\phi)$. Because S is finite, it follows that $L_k(\phi, S)^{1/k} < \text{GR}(\phi)$ for all $k \geq K$. However, since

$$\lim_{k \rightarrow \infty} L_k(\phi, S)^{1/k} = \lim_{k \geq K} L_k(\phi, S)^{1/k} = \inf_{k \geq K} L_k(\phi, S)^{1/k},$$

we obtain a contradiction: $\text{GR}(\phi) = \inf_{k \geq K} L_k(\phi, S)^{1/k} < \text{GR}(\phi)$. $\square$

3 Finitely generated nilpotent groups

Consider the lower central series of a finitely generated group π ,

$$\pi = \pi_1 \supset \pi_2 \supset \cdots,$$

where $\pi_j = [\pi, \pi_{j-1}]$ is the j -fold commutator subgroup $\gamma_j(\pi)$ of π . The endomorphism $\phi: \pi \rightarrow \pi$ induces endomorphisms

$$\phi_j: \pi_j \rightarrow \pi_j, \quad \hat{\phi}_j: \pi/\pi_j \rightarrow \pi/\pi_j, \quad \bar{\phi}_j: \pi_j/\pi_{j+1} \rightarrow \pi_j/\pi_{j+1}.$$

Then it is known from [2] that $\text{GR}(\phi) \geq \text{GR}(\bar{\phi}_j)^{1/j}$ for all $j \geq 1$. The group π is called *nilpotent* if $\pi_j = 1$ for some j . When $\pi_c \neq 1$ but $\pi_{c+1} = 1$, we say that it is *c-step*.

Lemma 3.1 ([2, Proposition 2]). *If π is c -step nilpotent, then*

$$\mathrm{GR}(\phi) = \max\{\mathrm{GR}(\hat{\phi}_c), \mathrm{GR}(\phi_c)^{1/c}\}.$$

If π is nilpotent, then

$$\mathrm{GR}(\phi) = \max_{j \geq 1} \{\mathrm{GR}(\bar{\phi}_j)^{1/j}\}.$$

Recall, for example from [10, Proposition 3.1], that a finitely generated nilpotent group π is virtually torsion-free. Thus there exists a finite-index, torsion-free, normal subgroup Γ of π . Following the proof of [11, Lemma 3.1], we can see that there exists a fully invariant subgroup $\Lambda \subset \Gamma$ of π which is of finite index. Therefore, any endomorphism $\phi: \pi \rightarrow \pi$ restricts to an endomorphism $\phi': \Lambda \rightarrow \Lambda$. By Theorem 2.8, we may consider only the case when ϕ' is not eventually trivial, and hence we may assume that $\mathrm{GR}(\phi) = \mathrm{GR}(\phi')$. Consequently, for the computation of $\mathrm{GR}(\phi)$, we may assume that π is a finitely generated torsion-free nilpotent group.

Consider the lower central series of a finitely generated torsion-free c -step nilpotent group π ,

$$\pi = \pi_1, \quad \pi_{j+1} = [\pi, \pi_j], \quad \pi_c \neq 1 \quad \text{and} \quad \pi_{c+1} = 1.$$

For each $j = 1, \dots, c$, we consider the isolator of π_j in π ,

$$\sqrt{\pi_j} = \sqrt[\pi]{\pi_j} := \{x \in \pi \mid x^k \in \pi_j \text{ for some } k \geq 1\}.$$

Then it is known that $\sqrt{\pi_j}$ is a characteristic subgroup of π with $[\sqrt{\pi_j} : \pi_j]$ finite. Furthermore, $\sqrt{\pi_j}/\pi_j$ is precisely the set of all torsion elements in the nilpotent group π/π_j , and $\sqrt{\pi_j}/\sqrt{\pi_{j+1}} \cong \mathbb{Z}^{k_j}$ for some integer $k_j > 0$. Hence we obtain the *adapted central series* [4, p. 3]

$$\pi = \sqrt{\pi_1} \supset \sqrt{\pi_2} \supset \dots \supset \sqrt{\pi_c} \supset \sqrt{\pi_{c+1}} = 1.$$

The following lemma plays a crucial role in our study of growth rates for endomorphisms of finitely generated nilpotent groups.

Lemma 3.2 ([15, Lemma 3.7]). *Let π be a finitely generated c -step nilpotent group with lower central series $\pi = \pi_1 \supset \pi_2 \supset \dots \supset \pi_c \supset \pi_{c+1} = 1$. Then there are finite sets $T_j = \{\tau_{j1}, \dots, \tau_{jk_j}\} \subset \pi_j$ such that*

- (1) *if $p_j: \pi_j \rightarrow \pi_j/\pi_{j+1}$ denotes the projection, then $p_j(T_j)$ is an independent set of generators for the finitely generated abelian group π_j/π_{j+1} ,*
- (2) *if $j > 1$, then every τ_{jr} is of the form $[\tau_{1i}, \tau_{j-1, \ell}]$,*
- (3) *T_1 generates π .*

Let G be the Malcev completion of a finitely generated torsion-free nilpotent group π , and let ϕ be an endomorphism of π . Then ϕ extends uniquely to a Lie group homomorphism D of G , called the Malcev completion of ϕ . We call its differential D_* the *linearization* of ϕ .

Theorem 3.3. *Let $\phi: \pi \rightarrow \pi$ be an endomorphism on a finitely generated torsion-free nilpotent group π . Let G be the Malcev completion of π . Then the linearization $D_*: \mathfrak{G} \rightarrow \mathfrak{G}$ of ϕ can be expressed as a lower triangular block matrix with diagonal blocks $\{D_j\}$ so that*

$$\text{GR}(\phi) = \max_{j \geq 1} \{\text{sp}(D_j)^{1/j}\}.$$

In particular, $\text{GR}(\phi)$ is an algebraic integer.

Proof. Let π be a finitely generated torsion-free c -step nilpotent group with the adapted central series

$$\pi = \sqrt{\pi_1} \supset \sqrt{\pi_2} \supset \cdots \supset \sqrt{\pi_c} \supset \sqrt{\pi_{c+1}} = 1.$$

Let $q_j: \sqrt{\pi_j} \rightarrow \sqrt{\pi_j} / \sqrt{\pi_{j+1}}$ denote the projection. We choose $\{T_1, \dots, T_c\}$ as in Lemma 3.2. Since π_2 is a fully invariant, finite-index subgroup of $\sqrt{\pi_2}$, it induces a short exact sequence

$$1 \rightarrow \sqrt{\pi_2}/\pi_2 \rightarrow \pi_1/\pi_2 \rightarrow \pi_1/\sqrt{\pi_2} = \sqrt{\pi_1}/\sqrt{\pi_2} \rightarrow 1.$$

Since $\sqrt{\pi_2}/\pi_2$ is finite, it follows that $\sqrt{\pi_1}/\sqrt{\pi_2} \cong \mathbb{Z}^{k_1}$ can be regarded as the free part of the finitely generated abelian group π_1/π_2 . Hence we can choose $S_1 \subset T_1$ such that $p_1(S_1)$ is an independent set of free generators of $\sqrt{\pi_1}/\sqrt{\pi_2}$ and $p_1(T_1 - S_1)$ is an independent set of torsion generators of π_1/π_2 .

Next we consider the short exact sequence

$$1 \rightarrow \sqrt{\pi_3}/\pi_3 \rightarrow \sqrt{\pi_2}/\pi_3 \rightarrow \sqrt{\pi_2}/\sqrt{\pi_3} \rightarrow 1.$$

Since $\pi_2/\pi_3 \subset \sqrt{\pi_2}/\pi_3$, we obtain the following commutative diagram between exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \sqrt{\pi_3}/\pi_3 & \longrightarrow & \sqrt{\pi_2}/\pi_3 & \longrightarrow & \sqrt{\pi_2}/\sqrt{\pi_3} \cong \mathbb{Z}^{k_2} \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & (\pi_2 \cap \sqrt{\pi_3})/\pi_3 & \longrightarrow & \pi_2/\pi_3 & \longrightarrow & \pi_2/(\pi_2 \cap \sqrt{\pi_3}) \longrightarrow 1. \\ & & & & & & = \pi_2 \cdot \sqrt{\pi_3}/\sqrt{\pi_3} \end{array}$$

where all vertical maps are inclusions of finite index. So we can choose $S_2 \subset T_2$ such that $p_2(S_2)$ is an independent set of free generators of the free abelian group $(\pi_2 \cdot \sqrt{\pi_3})/\sqrt{\pi_3}$ and $p_2(T_2 - S_2)$ is an independent set of torsion generators of π_2/π_3 . Note that $S_2 \subset \pi_2 \subset \sqrt{\pi_2}$. Because the right-most vertical inclusion is of finite index, we can choose $S_2 \subset \sqrt{\pi_2}$ such that $q_2(S_2)$ is an independent set of free generators of $\sqrt{\pi_2}/\sqrt{\pi_3}$, and for each $\sigma_2 \in S_2$, there are unique $\ell_2 \geq 1$ and unique $\tau_{2*} \in S_2$ such that $\sigma_2^{\ell_2} = \tau_{2*}$ modulo $\sqrt{\pi_3}$. We remark also that $\#S_2 = \#S_2$.

Continuing in this way, we obtain $\{S_1, \dots, S_c\} \subset \{T_1, \dots, T_c\}$ such that

- $S_j \subset T_j$, $\#S_j = \#S_j$,
- $p_j(S_j)$ is an independent set of free generators of π_j/π_{j+1} ,
- $p_j(T_j - S_j)$ is an independent set of torsion generators of π_j/π_{j+1} ,
- $q_j(S_j)$ is an independent set of free generators of $\sqrt{\pi_j}/\sqrt{\pi_{j+1}}$,
- for each $\sigma_j \in S_j \subset \sqrt{\pi_j}$, there exist unique $\ell_j \geq 1$ and $\tau_{j*} \in S_j$ such that

$$\sigma_j^{\ell_j} = \tau_{j*} \text{ mod } \sqrt{\pi_{j+1}}.$$

The adapted central series of π allows us to choose a *preferred basis* $\mathbf{a}$ of π ; we can choose $\mathbf{a}$ to be $\{S_1, \dots, S_c\}$ so that it generates π and π can be embedded as a lattice of a connected, simply connected nilpotent Lie group G , the Malcev completion of π . Its Lie algebra $\mathcal{G}$ has a linear basis $\log \mathbf{a} = \{\log S_1, \dots, \log S_c\}$. From $\sigma_j^{\ell_j} = \tau_{j*} \text{ mod } \sqrt{\pi_{j+1}}$, we have

$$\ell_j \log(\sigma_j) = \log(\sigma_j^{\ell_j}) = \log(\tau_{j*}) \text{ mod } \gamma_{j+1}(\mathcal{G}). \quad (\text{B})$$

This implies that $\{\log S_1, \dots, \log S_c\}$ is also a linear basis of $\mathcal{G}$.

Let $\phi: \pi \rightarrow \pi$ be an endomorphism. Then ϕ induces endomorphisms

$$\phi_j: \pi_j \rightarrow \pi_j, \quad \hat{\phi}_j: \pi/\pi_j \rightarrow \pi/\pi_j, \quad \bar{\phi}_j: \pi_j/\pi_{j+1} \rightarrow \pi_j/\pi_{j+1}$$

and

$$\begin{aligned} \varphi_j: \sqrt{\pi_j} &\rightarrow \sqrt{\pi_j}, & \hat{\varphi}_j: \pi/\sqrt{\pi_j} &\rightarrow \pi/\sqrt{\pi_j}, \\ \bar{\varphi}_j: \sqrt{\pi_j}/\sqrt{\pi_{j+1}} &\rightarrow \sqrt{\pi_j}/\sqrt{\pi_{j+1}}. \end{aligned}$$

Moreover, any endomorphism ϕ on π extends uniquely to a Lie group endomorphism D on G , the Malcev completion of ϕ . With respect to the preferred basis $\log \mathbf{a}$ of the Lie algebra $\mathcal{G}$ of G , we can express the linearization D_* of ϕ as a lower triangular block matrix; each diagonal block D_j is an integer matrix representing the endomorphism $\bar{\varphi}_j: \sqrt{\pi_j}/\sqrt{\pi_{j+1}} \cong \mathbb{Z}^{k_j} \rightarrow \sqrt{\pi_j}/\sqrt{\pi_{j+1}} \cong \mathbb{Z}^{k_j}$. For

details, we refer to [9] for example. When the new basis $\{\log S_1, \dots, \log S_c\}$ is used instead of $\log \mathbf{a}$, the integer entries of block matrices D_j will be changed to rational entries because of identities (B), but the eigenvalues of D_j will be unchanged. This means that, whenever the eigenvalues of D_* are concerned, we may assume that π_j/π_{j+1} is torsion-free, or $\pi_j = \sqrt{\pi_j}$. Consequently, we may assume that $\text{GR}(\bar{\phi}_j) = \text{GR}(\bar{\varphi}_j)$. Since $\sqrt{\pi_j}/\sqrt{\pi_{j+1}} \cong \mathbb{Z}^{k_j}$, by taking the tensor product with $\mathbb{R}$, it is known that $\text{GR}(\bar{\varphi}_j) = \text{sp}(D_j)$. Thus $\text{GR}(\bar{\phi}_j) = \text{sp}(D_j)$. Now the theorem follows from Lemma 3.1. $\square$

Remark 3.4. From Theorem 3.3, it follows that the growth rate of any endomorphism on a finitely generated torsion-free nilpotent group is an algebraic integer. The question of determining groups for which the growth rate of a group endomorphism is an algebraic number was raised by R. Bowen in [2, p. 27].

Example 3.5. Let Nil be the 3-dimensional Heisenberg group. That is,

$$\text{Nil} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Consider the subgroups Γ_k , $k \in \mathbb{N}$, of Nil,

$$\Gamma_k = \left\{ \begin{bmatrix} 1 & n & \frac{\ell}{k} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \mid m, n, \ell \in \mathbb{Z} \right\}.$$

These are lattices of Nil, and every lattice of Nil is isomorphic to some Γ_k . Let

$$a_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 1 & 0 & \frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $S = \{a_1, a_2, a_3\}$ is a generating set of Γ_k satisfying

$$[a_1, a_2] = a_3^{-k}, [a_1, a_3] = [a_2, a_3] = 1,$$

and in fact,

$$\begin{bmatrix} 1 & n & \frac{\ell}{k} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} = a_1^m a_2^n a_3^\ell.$$

Let

$$\pi = \Gamma_k = \langle a_1, a_2, a_3 \mid [a_1, a_2] = a_3^{-k}, [a_1, a_3] = [a_2, a_3] = 1 \rangle.$$

Let $\pi' = \langle a_3 \rangle$ and $S' = \{a_3\}$. Since $(a_3^{-k})^{n^2} = [a_1^n, a_2^n]$, we have

$$L((a_3^{-k})^{n^2}, S') = kn^2 \quad \text{and} \quad L((a_3^{-k})^{n^2}, S) = 4n.$$

Hence

$$L((a_3^{-k})^{n^2}, S') > L((a_3^{-k})^{n^2}, S)$$

for all n with $n > 4/k$. It follows that π' is distorted.

Consider any endomorphism $\phi: \pi \rightarrow \pi$. Then ϕ must be of the form

$$\phi(a_1) = a_1^{m_{11}} a_2^{m_{21}} a_3^p, \quad \phi(a_2) = a_1^{m_{12}} a_2^{m_{22}} a_3^q, \quad \phi(a_3) = a_3^{m_{11}m_{22} - m_{12}m_{21}}.$$

We will compute $\text{GR}(\phi)$. The lower central series of π is $\pi = \pi_1 \supset \pi_2 = \langle a_3^k \rangle$, and its adapted central series is $\pi = \pi_1 \supset \sqrt{\pi_2} = \langle a_3 \rangle$. We observe that

$$T_1 = \{a_1, a_2, a_3\} \quad \text{and} \quad T_2 = \{a_3^k\}$$

are sets satisfying the conditions of Lemma 3.2. Then we can see that

$$S_1 = \{a_1, a_2\} \subset T_1, \quad S_2 = \{a_3^k\} \subset T_2 \quad \text{and} \quad S_1 = \{a_1, a_2\}, \quad S_2 = \{a_3\}.$$

Furthermore, $\{S_1, S_2\} = \{a_1, a_2, a_3\}$ is a preferred basis for π . The linearization of ϕ with respect to this preferred basis has two integer blocks D_1 and D_2 , where

$$D_1 = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad D_2 = [m_{11}m_{22} - m_{12}m_{21}] = [\det(D_1)].$$

By Theorem 3.3, we have $\text{GR}(\phi) = \max\{\text{sp}(D_1), \text{sp}(D_2)^{1/2}\}$. Let μ, ν be the eigenvalues of D_1 . Then

$$\text{GR}(\phi) = \max\{|\mu|, |\nu|, \sqrt{|\mu\nu|}\} = \max\{|\mu|, |\nu|\} = \text{sp}(D_1).$$

In fact, we show in Theorem 3.7 that it is always the case that $\text{GR}(\phi) = \text{sp}(D_1)$.

We consider another example in which we obtain much information about linearizations of endomorphisms, and then we obtain an idea of proving the next result, Theorem 3.7.

Example 3.6. Consider a 2-step torsion-free nilpotent group π generated by

$$\tau_1, \tau_2, \tau_3, \sigma_{12}, \sigma_{13}$$

satisfying the relations

$$\begin{aligned} [\tau_1, \tau_2] &= \sigma_{12}, & [\tau_1, \tau_3] &= \sigma_{13}, & [\tau_2, \tau_3] &= \sigma_{12}^m \sigma_{13}^n, \\ [\tau_i, \sigma_{jk}] &= [\sigma_{12}, \sigma_{13}] = 1. \end{aligned}$$

Since $\pi_2 = \langle \sigma_{12}, \sigma_{13} \rangle \cong \mathbb{Z}^2$ and $\pi/\pi_2 = \langle \bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3 \rangle \cong \mathbb{Z}^3$, it follows that the set $\{T_1, T_2\} = \{\tau_1, \tau_2, \tau_3, \sigma_{12}, \sigma_{13}\}$ satisfies the conditions of Lemma 3.2 and forms a preferred basis of our group π . Let ϕ be an endomorphism of π . A direct computation shows that if

$$\phi(\tau_i) = \tau_1^{d_{1i}} \tau_2^{d_{2i}} \tau_3^{d_{3i}} \mod \pi_2,$$

i.e., if the first block of the linearization of ϕ is

$$D_1 = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix},$$

then, with $\sigma_{23} = [\tau_2, \tau_3]$, we have

$$\begin{aligned} \phi(\sigma_{12}) &= \sigma_{12}^{M_{33}} \sigma_{13}^{M_{23}} \sigma_{23}^{M_{13}}, \\ \phi(\sigma_{13}) &= \sigma_{12}^{M_{32}} \sigma_{13}^{M_{22}} \sigma_{23}^{M_{12}}, \\ \phi(\sigma_{23}) &= \sigma_{12}^{M_{31}} \sigma_{13}^{M_{21}} \sigma_{23}^{M_{11}}, \end{aligned}$$

where M_{ij} denote the (i, j) -minor of D . These yield a matrix

$$K = \begin{bmatrix} M_{33} & M_{32} & M_{31} \\ M_{23} & M_{22} & M_{21} \\ M_{13} & M_{12} & M_{11} \end{bmatrix} = \bigwedge^2(D_1),$$

the second exterior power of D_1 . On the other hand, since $\sigma_{23} = \sigma_{12}^m \sigma_{13}^n$, we have

$$\phi(\sigma_{12}) = \sigma_{12}^{M_{33}} \sigma_{13}^{M_{23}} \sigma_{23}^{M_{13}} = \sigma_{12}^{M_{33}+mM_{13}} \sigma_{13}^{M_{23}+nM_{13}}, \quad (3.1)$$

$$\phi(\sigma_{13}) = \sigma_{12}^{M_{32}} \sigma_{13}^{M_{22}} \sigma_{23}^{M_{12}} = \sigma_{12}^{M_{32}+mM_{12}} \sigma_{13}^{M_{22}+nM_{12}}, \quad (3.2)$$

$$\begin{aligned} \phi(\sigma_{23}) &= \phi(\sigma_{12})^m \phi(\sigma_{13})^n \\ &= \sigma_{12}^{M_{31}} \sigma_{13}^{M_{21}} \sigma_{23}^{M_{11}} = \sigma_{12}^{M_{31}+mM_{11}} \sigma_{13}^{M_{21}+nM_{11}}. \end{aligned} \quad (3.3)$$

From (3.1) and (3.2), the second block of the linearization of ϕ is

$$D_2 = \begin{bmatrix} M_{33} + mM_{13} & M_{32} + mM_{12} \\ M_{23} + nM_{13} & M_{22} + nM_{12} \end{bmatrix}.$$

Plugging (3.1) and (3.2) into (3.3), we have

$$\begin{cases} \begin{bmatrix} M_{31} \\ M_{21} \\ M_{11} \end{bmatrix} = m \begin{bmatrix} M_{33} \\ M_{23} \\ M_{13} \end{bmatrix} + n \begin{bmatrix} M_{32} \\ M_{22} \\ M_{12} \end{bmatrix} & \text{when } (m, n) \neq (0, 0), \\ \begin{bmatrix} M_{31} \\ M_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{when } m = n = 0. \end{cases} \quad (3.4)$$

When $(m, n) \neq (0, 0)$, because of (3.4), K is column equivalent to the matrix K' with the zero third column, and then, by doing some row operations on K' , we can see that K' is row equivalent to the matrix K'' , where

$$K \rightsquigarrow K' = \begin{bmatrix} M_{33} & M_{32} & 0 \\ M_{23} & M_{22} & 0 \\ M_{13} & M_{12} & 0 \end{bmatrix} \rightsquigarrow K'' = \left[\begin{array}{cc|c} M_{33} + mM_{13} & M_{32} + mM_{12} & 0 \\ M_{23} + nM_{13} & M_{22} + nM_{12} & 0 \\ \hline M_{13} & M_{12} & 0 \end{array} \right].$$

Thus the second block D_2 of the linearization D_* is a block submatrix of K'' . This is obtained by removing the row and column of K'' that are determined by (3.3) or by the relation $[\tau_2, \tau_3] = \sigma_{12}^m \sigma_{13}^n$. Note also that K , K' and K'' have the same eigenvalues which are 0 and the eigenvalues of D_2 . When $(m, n) = (0, 0)$, because of (3.4), we have

$$K = \left[\begin{array}{cc|c} M_{33} & M_{32} & 0 \\ M_{23} & M_{22} & 0 \\ \hline M_{13} & M_{12} & M_{11} \end{array} \right].$$

Thus D_2 of D_* is a block submatrix of K , and K has M_{11} and the eigenvalues of D_2 as its eigenvalues.

On the other hand, if μ_1, μ_2, μ_3 are the eigenvalues of D_1 , as $K = \bigwedge^2(D_1)$, the eigenvalues of K are $\mu_i \mu_j$ ($i < j$). Consequently, we have

$$\text{sp}(D_1) = \max_{i=1,2,3} \{|\mu_i|\} \geq \max_{i \neq j} \{\sqrt{|\mu_i \mu_j|}\} = \text{sp}(K) \geq \text{sp}(D_2)^{1/2}.$$

This proves that $\text{GR}(\phi) = \text{sp}(D_1)$.

The following result was proved in [10] when ϕ is an automorphism using the intrinsic polynomial structure of nilpotent groups. We will now improve [10, Theorem 1.2] from automorphisms to endomorphisms by using completely different arguments.

Theorem 3.7. *Let $\phi: \pi \rightarrow \pi$ be an endomorphism on a finitely generated torsion-free nilpotent group π with Malcev completion D . Then*

$$\mathrm{GR}(\phi) = \mathrm{GR}(\phi_{\mathrm{ab}}),$$

where $\phi_{\mathrm{ab}}: \pi/[\pi, \pi] \rightarrow \pi/[\pi, \pi]$ be the endomorphism induced by ϕ . Hence we have $\mathrm{GR}(\phi) = \mathrm{sp}(D_1) \leq \mathrm{sp}(D_*)$.

Proof. Let π be c -step and choose a family of finite sets $\{T_1, \dots, T_c\}$ satisfying the conditions of Lemma 3.2. As was observed in the proof of Theorem 3.3, we can choose $\{S_1, \dots, S_c\}$ such that each $S_j \subset T_j \subset \pi_j$ projects onto free generators of π_j/π_{j+1} and a preferred basis $\{S_1, \dots, S_c\}$ of π so that each block matrix D_j of the linearization D_* of ϕ which is determined by $\log S_j$ may be assumed to be determined by $\log S_j$.

Indeed, for each j with $1 \leq j \leq c$, we write $S_j = \{\tau_{j1}, \dots, \tau_{jk_j}\} \subset T_j$; then, if $j > 1$, every τ_{jr} is of the form $[\tau_{1i}, \tau_{j-1, \ell}]$. For $1 \leq j \leq c$, if

$$\phi(\tau_{j\ell}) = \tau_{j1}^{d_{1\ell}^j} \cdots \tau_{jk_j}^{d_{k_j\ell}^j} \text{ modulo } \pi_{j+1},$$

then the j th block of the linearization D_* of ϕ is

$$D_j = \begin{bmatrix} d_{11}^j & \cdots & d_{1k_j}^j \\ \vdots & & \vdots \\ d_{k_11}^j & \cdots & d_{k_jk_j}^j \end{bmatrix}.$$

In order to compare first the eigenvalues of D_1 with those of D_2 , we use the following new notation: $D_1 = [d_{ij}^1] = [d_{ij}]$, $\sigma_{ij} = [\tau_{1i}, \tau_{1j}]$ for all $1 \leq i < j \leq k_1$. Then $\sigma_{ij} = \tau_{2, \ell}^{\pm 1} \in S_2$ for some ℓ , or σ_{ij} is a word of elements in $S_2^{\pm 1}$ modulo π_3 (see the presentation of π in Example 3.6). Let $S = \{\sigma_{ij} \mid 1 \leq i < j \leq k_1\}$; then we may assume that $S_2 \subset S$. Further, S_2 differs from S except possibly by σ_{ij} 's, words of elements in $S_2^{\pm 1}$ modulo π_3 (note in Example 3.6 that $S_2 = \{\sigma_{12}, \sigma_{13}\}$ and $S = \{\sigma_{12}, \sigma_{13}, \sigma_{12}^m \sigma_{13}^n\}$).

Now we can express $\phi(\sigma_{ij})$ as follows:

$$\phi(\sigma_{ij}) = \sigma_{12}^{M_{1,2}^{i,j}} \sigma_{13}^{M_{1,3}^{i,j}} \cdots \sigma_{1k_1}^{M_{1,k_1}^{i,j}} \cdots \sigma_{k_1-1,k_1}^{M_{k_1-1,k_1}^{i,j}} \text{ modulo } \pi_3 \quad (\text{P})$$

for some integers $M_{p,q}^{i,j}$. We denote by K the $\binom{k_1}{2} \times \binom{k_1}{2}$ matrix $[M_{p,q}^{i,j}]$,

$$K = \begin{bmatrix} M_{1,2}^{1,2} & M_{1,2}^{1,3} & \cdots & M_{1,2}^{k_1-1,k_1} \\ M_{1,3}^{1,2} & M_{1,3}^{1,3} & \cdots & M_{1,3}^{k_1-1,k_1} \\ \vdots & \vdots & & \vdots \\ M_{k_1-1,k_1}^{1,2} & M_{k_1-1,k_1}^{1,3} & \cdots & M_{k_1-1,k_1}^{k_1-1,k_1} \end{bmatrix}.$$

We will refer to the column vector $(M_{1,2}^{i,j}, M_{1,3}^{i,j}, \dots, M_{1,k_1}^{i,j}, \dots, M_{k_1-1,k_1}^{i,j})^t$ of K as the (i, j) -column of K . Note the following.

- (i) For any $\sigma_{ij} \in S$, $M_{p,q}^{i,j}$ is unique for which $\sigma_{pq} \in S_2$.
- (ii) If $\sigma_{ij} \in S - S_2$, then σ_{ij} is a word w of elements in $S_2^{\pm 1}$ modulo π_3 . If $w \neq 1$ modulo π_3 , the (i, j) -column of K is an integer combination of (p, q) -columns of K corresponding to the elements σ_{pq} appearing in the word w . If $w \equiv 1$, then $M_{p,q}^{i,j} = 0$ for which $\sigma_{pq} \in S_2$.
- (iii) The right-hand side of the expression (P) can be rewritten in terms of only the elements of S_2 using the words $\sigma_{ij} \equiv w(\sigma_{pq})$. This yields the second block D_2 .

Since $\sigma_{ij} = [\tau_{1i}, \tau_{1j}]$, taking ϕ on both sides, we have (see [5, Lemma 4.1] or [14, p. 93, Lemma 4.1])

$$\begin{aligned} & \sigma_{12}^{M_{1,2}^{i,j}} \sigma_{13}^{M_{1,3}^{i,j}} \cdots \sigma_{1k_1}^{M_{1,k_1}^{i,j}} \cdots \sigma_{k_1-1,k_1}^{M_{k_1-1,k_1}^{i,j}} \\ &= [\tau_{11}^{d_{1,i}} \cdots \tau_{1k_1,i}^{d_{k_1,i}}, \tau_{11}^{d_{1,j}} \cdots \tau_{1k_1,j}^{d_{k_1,j}}] \\ &= \prod_p \prod_q [\tau_{1p}^{d_{p,i}}, \tau_{1q}^{d_{q,j}}] = \prod_p \prod_q [\tau_{1p}, \tau_{1q}]^{d_{p,i} d_{q,j}} \\ &= \prod_{1 \leq p < q \leq k_1} \sigma_{pq}^{d_{p,i} d_{q,j} - d_{p,j} d_{q,i}} \text{ modulo } \pi_3. \end{aligned}$$

This shows that K is the second exterior power of D_1 , i.e., $K = \bigwedge^2(D_1)$. Hence, if μ_i ($1 \leq i \leq k_1$) are the eigenvalues of the matrix D_1 , then $\mu_i \mu_j$ ($i < j$) are the eigenvalues of K .

From part (ii) of the above remarks, we see that K is column equivalent to the matrix K' with zero (i, j) -column for which $\sigma_{ij} = w(\sigma_{pq}) \neq 1$ modulo π_3 . We rearrange the elements of S so that $S = S_2 \cup (S - S_2) = S_2 \cup S_2^1 \cup S_2^2$, where $S_2^1 = \{\sigma_{ij} \in S - S_2 \mid \sigma_{ij} = 1\}$ and $S_2^2 = \{\sigma_{ij} \in S - S_2 \mid \sigma_{ij} \neq 1\}$. Rearranging S to $S_2 \cup (S - S_2)$, we have

$$K \sim_C K' = \left[\begin{array}{c|c} K' & 0 \\ \hline * & * \end{array} \right].$$

The effect of part (iii) on K and hence on K' is doing some row operations using the (i, j) -rows in the last block of K' for which $\sigma_{ij} = w(\sigma_{pq}) \neq 1$ modulo π_3 . By rearranging S further to $S_2 \cup S_2^1 \cup S_2^2$, we have

$$K' \sim_R K'' = \left[\begin{array}{c|c|c} D_2 & 0 & 0 \\ \hline * & * & 0 \\ \hline * & * & 0 \end{array} \right].$$

The middle block column is determined by the fact that if $\sigma_{ij} \equiv w \equiv 1$, then $M_{p,q}^{i,j} = 0$ for which $\sigma_{pq} \in S_2$.

Consequently, the second block D_2 of D_* is a block submatrix of K'' which is obtained by removing the rows and columns associated to $S - S_2$. Note also that K, K' and K'' have the same eigenvalues which contain the eigenvalues of D_2 . This observation shows that

$$\text{sp}(D_1) = \max\{|\mu_i|\} \geq \max\{\sqrt{|\mu_i \mu_j|}\} = \text{sp}(K)^{1/2} \geq \text{sp}(D_2)^{1/2}.$$

For the next inductive step, we recall that every element of $S_3(\subset T_3)$ is of the form $[\tau_{1\ell}, \sigma_{ij}]$, where $i < j$. Taking ϕ , we have

$$\begin{aligned} \phi([\tau_{1\ell}, \sigma_{ij}]) &= \left[\prod_r \tau_{1r}^{d_{r,\ell}}, \prod_{1 \leq p < q \leq k_1} \sigma_{pq}^{d_{p,i}d_{q,j} - d_{p,j}d_{q,i}} \right] \\ &= \prod_r \prod_{1 \leq p < q \leq k_1} [\tau_{1r}, \sigma_{pq}]^{d_{r,\ell}(d_{p,i}d_{q,j} - d_{p,j}d_{q,i})} \text{ modulo } \pi_4. \end{aligned}$$

This expression is unique except possibly for the exponents of the elements

$$[\tau_{1r}, \sigma_{pq}] = 1 \text{ modulo } \pi_4.$$

This produces the matrix $K = D_1 \otimes \bigwedge^2 D_1$. First if $[\tau_{1r}, \sigma_{pq}] = w(S_3) \neq 1$ modulo π_4 , by doing some column operations and then by doing some row operations, we obtain a matrix K'' , which can be regarded as a lower triangular block matrix. Finally, we remove the columns and rows from K'' which are associated with the elements $[\tau_{1r}, \sigma_{pq}] = w(S_3)$ modulo π_4 . This gives rise to the third block D_3 of D_* . Hence $\text{sp}(D_3) \leq \text{sp}(D_1)^3$. Continuing in this way, we may assume that the j th block D_j of D_* is obtained from $(\bigotimes_{j-2} D_1) \otimes \bigwedge^2 D_1$ so that

$$\text{sp}(D_1) \geq \text{sp}(D_j)^{1/j}.$$

Consequently, $\text{GR}(\phi) = \max\{\text{sp}(D_j)^{1/j}\} = \text{sp}(D_1) = \text{GR}(\phi_{\text{ab}}) \leq \text{sp}(D_*)$. $\square$

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