

A classification of the abelian minimal closed normal subgroups of locally compact second-countable groups

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Abstract. We classify the locally compact second-countable (l.c.s.c.) groups A that are abelian and topologically characteristically simple. All such groups A occur as the monolith of some soluble l.c.s.c. group G of derived length at most 3; with known exceptions (specifically, when A is \mathbb{Q}^n or its dual for some $n \in \mathbb{N}$), we can take G to be compactly generated. This amounts to a classification of the possible isomorphism types of abelian chief factors of l.c.s.c. groups, which is of particular interest for the theory of compactly generated locally compact groups.

1 Introduction

Let G be a locally compact second-countable (l.c.s.c.) group. A *minimal closed normal subgroup* of G is a nontrivial closed normal subgroup N of G such that, whenever K is a closed normal subgroup of G contained in N , then $K = N$ or $K = \{1\}$. More specifically, we define the (*topological*) *monolith* $\text{Mon}(G)$ of G to be the intersection of all nontrivial closed normal subgroups of G .

The main goal of the article is to prove the following classification theorem.

Theorem 1.1. *The following statements hold.*

- (i) *Let $n \in \mathbb{N}$, and let p be a prime number. The following abelian groups occur as the monolith of some compactly generated soluble l.c.s.c. group of derived length at most 3:*

$$C_p^n, C_p^{\mathfrak{N}_0}, C_p^{(\mathfrak{N}_0)}, C_p^{\mathfrak{N}_0} \oplus C_p^{(\mathfrak{N}_0)}, \mathbb{R}^n, \mathbb{Q}^{(\mathfrak{N}_0)}, \widehat{\mathbb{Q}}^{\mathfrak{N}_0}, \mathbb{Q}_p^n, \mathbb{Q}_p^{(\mathfrak{N}_0)}.$$

- (ii) *The groups \mathbb{Q}^n and $\widehat{\mathbb{Q}}^n$ ($n \in \mathbb{N}$) cannot occur as minimal closed normal subgroups in any compactly generated l.c.s.c. group, but do occur as the monolith of some soluble l.c.s.c. group of derived length at most 3.*

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- (iii) *Every abelian l.c.s.c. group that is topologically characteristically simple is isomorphic to one of the groups listed in (i) or (ii).*

Here A^κ denotes the direct product of κ copies of A , $A^{(\kappa)}$ the direct sum of κ copies of A , and $\mathbb{Q}_p(\kappa)$ is the local direct product of κ copies of \mathbb{Q}_p over \mathbb{Z}_p , that is, the group of functions from a set of size κ to \mathbb{Q}_p with all but finitely many values in \mathbb{Z}_p .

The topologically characteristically simple abelian groups in Theorem 1.1 are listed in such a way that there is no redundancy, that is, no two groups listed are isomorphic unless given in the same form with the same parameters. The group $C_p^{\aleph_0} \oplus C_p^{(\aleph_0)}$ is also known as the additive group of the field $\mathbb{F}_p((t))$ of formal Laurent series over the field of p elements. The groups naturally fall into five disjoint families, namely

$$C_p^\kappa \oplus C_p^{(\kappa')}, \mathbb{R}^n, \mathbb{Q}^{(\kappa)}, \widehat{\mathbb{Q}}^\kappa, \mathbb{Q}_p(\kappa),$$

where $n \in \mathbb{N}$, p is a prime number, $\kappa \in \mathbb{N} \cup \{\aleph_0\}$ and $\kappa' \in \{0, \aleph_0\}$. For each family, we will give a characterization of that family in the class of abelian l.c.s.c. groups, in terms of basic properties of l.c.s.c. groups. These characterizations do not directly invoke the property of being topologically characteristically simple.

1.1 Context

All of the abelian groups listed in Theorem 1.1 are well known to scholars of locally compact abelian groups and are discussed in standard references on the subject, such as [1, 7]. The *homogeneous* locally compact abelian groups A , that is, such that $\text{Aut}(A)$ acts transitively on the nonzero elements of A , were classified by L. Robertson [12, Theorem 4.13]; in the second-countable case, these are

$$C_p^\kappa \oplus C_p^{(\kappa')}, \mathbb{R}^n, \mathbb{Q}^{(\kappa)}, \mathbb{Q}_p^n.$$

(The group $\mathbb{Q}_p(\aleph_0)$ is an open subgroup of the homogeneous group $\mathbb{Q} \otimes \mathbb{Z}_p^{\aleph_0}$, but the latter is not second-countable.)

There are also well-known structural descriptions of elementary abelian groups, abelian Lie groups, and divisible discrete abelian groups and their duals; these lead to characterizations of the first four families of topologically characteristically simple abelian groups. The main novelties in the present article are therefore to collect together some known constructions and obstructions of monoliths of the given isomorphism types in compactly generated and/or soluble l.c.s.c. groups (Section 3), characterize the remaining family $\mathbb{Q}_p(\kappa)$ (Proposition 3.10), and observe that this completes the classification of topologically characteristically simple abelian l.c.s.c. groups (Section 4).

Minimal closed normal subgroups, particularly those that are neither compact nor discrete, are of particular interest in the theory of compactly generated locally compact groups. (Note that if G is a compactly generated locally compact group, there are arbitrarily small compact normal subgroups N such that G/N is second-countable.) We recall two results from the recent literature.

Theorem 1.2 (Caprace–Monod; see [2]). *Let G be a compactly generated locally compact group with no nontrivial compact or discrete normal subgroups. Then every nontrivial closed normal subgroup of G contains a minimal one. Moreover, all but finitely many minimal closed normal subgroups of G are abelian.*

Using Theorem 1.1, one can sharpen Theorem 1.2 to the following.

Corollary 1.3. *Let G be a compactly generated locally compact group with no nontrivial compact or discrete normal subgroups. Then every nontrivial closed normal subgroup of G contains a minimal one. Moreover, all but finitely many minimal closed normal subgroups of G are of the form $\mathbb{F}_p((t))$, \mathbb{R}^n or $\mathbb{Q}_p(\kappa)$ for some prime p , $n \in \mathbb{N}$ and $\kappa \in \mathbb{N} \cup \{\aleph_0\}$ as applicable.*

Each of the groups $\mathbb{F}_p((t))$, \mathbb{R}^n and $\mathbb{Q}_p(\kappa)$ can occur 2^{\aleph_0} times as a minimal closed normal subgroup of a compactly generated soluble l.c.s.c. group; see Example 4.4.

Theorem 1.4 (Reid–Wesolek [11, Theorem 1.3]). *Let G be a compactly generated locally compact group. Then every finite series of closed normal subgroups of G can be refined to another such series*

$$\{1\} = G_0 < G_1 < \cdots < G_n = G$$

such that, for each $i \in \{1, \dots, n\}$, the factor G_i/G_{i-1} is compact, discrete or a chief factor of G (that is, G_i/G_{i-1} is a minimal closed normal subgroup of G/G_{i-1}).

Subsequent articles of the author and P. Wesolek [9, 10] have developed a theory of nonabelian chief factors; the present article is complementary in that it identifies the possible isomorphism types of abelian chief factors. In particular, one obtains the following structural result for abelian factors of compactly generated locally compact groups.

Corollary 1.5. *Let G be a compactly generated locally compact group, and let M and N be closed normal subgroups such that $M \leq N$ and N/M is abelian. Then there is a finite series*

$$M = G_0 < G_1 < \cdots < G_n = N$$

of closed normal subgroups of G such that, for each $0 \leq i < n$, the factor G_{i+1}/G_i is compact, discrete, or is a chief factor isomorphic to $\mathbb{F}_p((t))$, \mathbb{R}^n or $\mathbb{Q}_p(\kappa)$ for some prime p , $n \in \mathbb{N}$ and $\kappa \in \mathbb{N} \cup \{\aleph_0\}$ as applicable.

2 Preliminaries

2.1 Notation and terminology

Throughout, “l.c.s.c.” stands for “locally compact second-countable”.

We will assume all groups are equipped with a Hausdorff topology compatible with the group structure. In particular, it follows from standard results (see [7, Theorem 8.3]) that a locally compact group is first-countable if and only if it is metrizable, and second-countable if and only if it is metrizable and σ -compact.

Automorphisms or isomorphisms of topological groups are required to be homeomorphisms that preserve the group structure. In particular, a *characteristic* subgroup is one that is preserved by all automorphisms in the aforementioned sense.

Let \mathbb{N} be the set of natural numbers (not including 0), and let \mathbb{P} be the set of prime numbers.

When using a commutative topological ring R as a group, we mean its additive group with the same topology; for the multiplicative group of units of a given ring R , we write R^* and equip R^* with the subspace topology, and unless otherwise specified, R^* acts on R by multiplication. Specifically, \mathbb{Z} and \mathbb{Q} have the discrete topology, while \mathbb{R} , \mathbb{Z}_p , \mathbb{Q}_p and $\mathbb{F}_p((t))$ have their usual nondiscrete locally compact topologies.

Where it makes sense to do so, we will write abelian groups additively. Every abelian group A is naturally a \mathbb{Z} -module; given $a \in A$, we define na to be the sum of n copies of a (for $n \geq 0$) or respectively the sum of $-n$ copies of $-a$ (for $n < 0$).

Let $C_{\aleph_0} = \mathbb{Z}$ and $C_n = \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$.

The (*topological*) *monolith* of a locally compact group G is the intersection $\text{Mon}(G)$ of all nontrivial closed normal subgroups of G . If $\text{Mon}(G)$ is nontrivial, we say that G is *monolithic*.

Given a sequence $(G_i)_{i \in I}$ of topological groups, the product $\prod_{i \in I} G_i$ is also a group with the product topology. We note that $\prod_{i \in I} G_i$ is locally compact if and only if all of the groups G_i are locally compact and all but finitely many G_i are compact.

Let $(G_i, H_i)_{i \in I}$ be a sequence of pairs of topological groups, where H_i is an open subgroup of G_i . The *local direct product* $\bigoplus_{i \in I} (G_i, H_i)$ is the set of all tuples $(g_i)_{i \in I}$ such that $g_i \in G_i$ for all $i \in I$ and $g_i \in H_i$ for all but finitely many

$i \in I$. Multiplication is defined pointwise. The group $\prod_{i \in I} H_i$ is naturally embedded in $\bigoplus_{i \in I} (G_i, H_i)$ as a subgroup; we equip $\bigoplus_{i \in I} (G_i, H_i)$ with the group topology such that the natural inclusion of $\prod_{i \in I} H_i$ into $\bigoplus_{i \in I} (G_i, H_i)$ is an open embedding. Note that if H_i is locally compact for all $i \in I$ and compact for all but finitely many $i \in I$, then $\bigoplus_{i \in I} (G_i, H_i)$ is locally compact. In the case that G_i is discrete and H_i is trivial for all $i \in I$, then $\bigoplus_{i \in I} (G_i, \{1\})$ simplifies to the direct sum $\bigoplus_{i \in I} G_i$ with the discrete topology.

Given a topological group G , write G^κ for the direct product of κ copies of G (or copies indexed by κ) with the product topology; when G is discrete, write $G^{(\kappa)}$ for the direct sum of κ copies of G (or copies indexed by κ) with the discrete topology. Note that, given $n \in \mathbb{N}$ and cardinals κ, κ' , we have

$$G^{\kappa+n} \times G^{(\kappa')} \cong G^\kappa \times G^{(\kappa'+n)};$$

we will use this isomorphism without further comment when indexing local direct powers of discrete groups.

Given an abelian topological group U , a (*topological*) *direct summand* is a subgroup V of U such that U admits a direct sum decomposition $U = V \oplus W$ as a topological group, meaning that both V and W are closed subgroups and U carries the product topology.

2.2 Compact elements of abelian groups

Theorem 2.1 ([8, Theorem 25]). *Every locally compact abelian group has an open subgroup of the form $\mathbb{R}^a \oplus K$, where $a \in \mathbb{N} \cup \{0\}$ and K is compact.*

Given a locally compact abelian group G , we say $g \in G$ is *compact* if $\overline{\langle g \rangle}$ is compact. Write $P(G)$ for the set of compact elements. For a prime p , write $P_p(G)$ for the set of $g \in G$ such that $g^{p^n} \rightarrow 0$ as $n \rightarrow \infty$. Note that $P_p(G) \leq P(G)$ for all $p \in \mathbb{P}$.

Lemma 2.2. *Let G be a locally compact abelian group, and let $p \in \mathbb{P}$. Then $P(G)$ is a closed subgroup of G . If G is connected, then $P(G)$ is compact. If G is totally disconnected, then $P_p(G)$ is closed and $qP_p(G) = P_p(G)$ for all $q \in \mathbb{P} \setminus \{p\}$.*

Proof. To show that $P(G)$ is closed, it is enough to show that $P(G) \cap U$ is closed for some open subgroup U of G . By Theorem 2.1, we can take $U = \mathbb{R}^a \oplus K$ for K compact. We then see that $P(G) \cap U = K$, so $P(G)$ is closed; if G is connected, then $G = U$, so $P(G) = K$, and hence $P(G)$ is compact.

In a finite abelian group F , we see that $P_p(F)$ is the p -Sylow subgroup of F , and $qP_p(F) = P_p(F)$ for all $q \in \mathbb{P} \setminus \{p\}$. The same is then true when F is a profinite abelian group, by an inverse limit argument. In particular, $P_p(F)$ is closed.

If G is totally disconnected, then G has a profinite open subgroup U , and we have $P_p(G) \cap U = P_p(U)$, so $P_p(G)$ is closed. Given $q \in \mathbb{P} \setminus \{p\}$, for all profinite open subgroups K of $P_p(G)$, we have $qK = K$; since $P_p(G)$ is the union of such subgroups, we have $qP_p(G) = P_p(G)$. \square

If A is a totally disconnected locally compact abelian group with $P_p(A) = A$, we can naturally regard A as a \mathbb{Z}_p -module; specifically, given $\lambda_n \in \mathbb{Z}$ converging to some $\lambda \in \mathbb{Z}_p$, we define $\lambda a = \lim_{n \rightarrow \infty} \lambda_n a$. The limit exists exactly because $a \in P_p(A)$; what is more, one sees that the multiplication map $\mathbb{Z}_p \times A \rightarrow A$ is continuous. In particular, for all $a \in A$, the closed subgroup $\langle a \rangle$ is the same as the cyclic \mathbb{Z}_p -submodule $\mathbb{Z}_p a$.

2.3 Divisibility

We say $a \in A$ is *divisible* if, for all $n \in \mathbb{N}$, there exists an n -th root of a , that is, $b \in A$ such that $nb = a$, and *p -divisible* for $p \in \mathbb{P}$ if, for all $n \geq 0$, there exists $b \in A$ such that $p^n b = a$. A *divisible group* (respectively, *p -divisible group*) is one in which every element is divisible (respectively, p -divisible).

A subgroup B of an abelian group A is *essential* if B intersects nontrivially every nontrivial subgroup of A . Note that, in the case that A is torsion-free, B is essential if and only if A/B is a torsion group.

(See [1, P.31].) Every locally compact abelian group A has a largest divisible subgroup $\text{div}(A)$ and a minimal divisible extension A^{div} such that A^{div} is again a locally compact group into which A embeds as an open subgroup. The minimal divisible extension is characterized up to isomorphism by the properties that A^{div} is divisible and A is an open essential subgroup of A^{div} . In particular, if U is an open essential subgroup of A , then $A^{\text{div}} = U^{\text{div}}$.

If A is torsion-free, then roots are unique (if they exist), so we can write a/n for the n -th root of $a \in A$. There are more direct descriptions of $\text{div}(A)$ and A^{div} in this case.

Lemma 2.3. *Let A be a torsion-free abelian group.*

- (i) $\text{div}(A)$ is the set of all divisible elements of A .
- (ii) A^{div} is the tensor product $\mathbb{Q} \otimes A$ of \mathbb{Z} -modules, where A is embedded via $a \mapsto (1, a)$ and $\mathbb{Q} \otimes A$ is equipped with the topology extending that of A .
- (iii) Given a continuous homomorphism $\theta: A \rightarrow B$ between torsion-free abelian groups, then θ has a unique extension to a continuous homomorphism $\theta^{\text{div}}: A^{\text{div}} \rightarrow B^{\text{div}}$.

Proof. (i) Let D be the set of divisible elements of A . Given $a \in D$ and $m, n \in \mathbb{N}$, then by uniqueness of roots, $a/(mn)$ is the n -th root of a/m ; by letting n vary, we conclude that $a/m \in D$. It is now easy to see that D is a divisible group; it is clearly the largest such subgroup of A .

(ii) The tensor product $\mathbb{Q} \otimes A$ consists of pure tensors; given

$$q_1, q_2, r_1, r_2 \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad a_1, a_2 \in A,$$

we have

$$\left(\frac{q_1}{r_1}, a_1\right) + \left(\frac{q_2}{r_2}, a_2\right) = \left(\frac{1}{r_1 r_2}, q_1 r_2 a_1 + q_2 r_1 a_2\right).$$

From here, it is easy to see that $\mathbb{Q} \otimes A$ is divisible and torsion-free, $(\mathbb{Q} \otimes A)/A$ is torsion and that the map $a \mapsto (1, a)$ is an injective homomorphism. Thus A is essential in $\mathbb{Q} \otimes A$. We can then take the topology of $\mathbb{Q} \otimes A$ such that $a \mapsto (1, a)$ is an open embedding, with the result that $\mathbb{Q} \otimes A$ is the minimal divisible extension of A in the topological sense.

(iii) We use tensor products as in (ii). The only possible extension of θ is to define $\theta^{\text{div}}(1/n, a) = (1/n, \theta(a))$ for all $n \in \mathbb{N}$ and $a \in A$. The uniqueness of roots ensures that this is a well-defined homomorphism from $\mathbb{Q} \otimes A$ to $\mathbb{Q} \otimes B$. The continuity of θ^{div} follows from the fact that it is continuous in a neighborhood of the identity. \square

The discrete divisible abelian groups have been classified.

Lemma 2.4 ([7, Theorem A.14]). *Let A be a discrete abelian group. Then the following are equivalent:*

- (i) A is divisible;
- (ii) $A \cong \mathbb{Q}^{(\kappa)} \oplus \bigoplus_{p \in \mathbb{P}} (\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})^{(\kappa_p)}$ for some cardinals κ and κ_p .

The next two lemmas give some useful conditions under which A is *densely divisible*, meaning that $\text{div}(A)$ is a dense subgroup of A .

Lemma 2.5 (See [1, P.28 (b)]). *Let A be a compact abelian group. Then the following are equivalent:*

- (i) A is divisible;
- (ii) A is densely divisible;
- (iii) A is connected.

Lemma 2.6. *Let A be a torsion-free totally disconnected locally compact abelian group such that $P_p(A) = A$. Then $\text{div}(A)$ is dense in A if and only if pA is dense in A .*

Proof. If $\text{div}(A)$ is dense in A , then certainly pA is dense in A since we have that $pA \geq p \text{div}(A) = \text{div}(A)$. So we may suppose that pA is dense in A .

Let U be a compact open subgroup of A ; the quotient A/U is then a discrete p -group. Since pA is dense in A , we see that $p(A/U) = A/U$, that is, A/U is p -divisible. Since A/U is a p -group, we conclude that $A/U \cong (\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})^{(\kappa)}$. In particular, for all $a \in A$, there is a homomorphism $f_a: \mathbb{Z}[\frac{1}{p}] \rightarrow A/U$ such that $a + U$ is contained in the image of f_a . By [6, Theorem 1.34], f_a lifts to a homomorphism $\tilde{f}_a: \mathbb{Z}[\frac{1}{p}] \rightarrow A$ such that there exists $\tilde{a} \in a + U$ in the image of \tilde{f}_a . In particular, we see that the element \tilde{a} is p -divisible in A . Since $P_p(A) = A$, for all $n \geq 0$, the p^n -th root \tilde{a}/p^n of \tilde{a} belongs to a pro- p subgroup of A ; in particular, \tilde{a}/p^n has an m -th root for all m coprime to p . Hence, in fact, \tilde{a} is divisible in A , that is, $\tilde{a} \in \text{div}(A)$. Thus $A = \text{div}(A) + U$. Since U can be made arbitrarily small, we conclude that $\text{div}(A)$ is dense in A . \square

2.4 Pontryagin duality

Let \mathbb{T} denote the multiplicative group of complex numbers of absolute value 1. Given a locally compact abelian group A , a *character* of A is a continuous homomorphism from A to \mathbb{T} . The set \widehat{A} of characters, equipped with pointwise multiplication and the compact-open topology, is called the *character group* or *dual* of A ; it is again a locally compact abelian group. We recall some standard facts about the relationship between A and $\widehat{\widehat{A}}$; see for example [7, Chapter 6] for details.

Theorem 2.7 (Pontryagin–van Kampen; see [7, Theorem 24.8]). *There is a natural isomorphism of topological groups between a locally compact abelian group A and the double dual of A , namely $a \in A$ corresponds to the character χ_a of $\widehat{\widehat{A}}$, where $\chi_a(f) = f(a)$ for all $f \in \widehat{\widehat{A}}$.*

As such, we will simply identify A with its double dual.

Lemma 2.8. *Let A be a locally compact abelian group.*

- (i) *A is metrizable if and only if \widehat{A} is σ -compact. In particular, the class of abelian l.c.s.c. groups is closed under taking duals [8, Theorem 29].*
- (ii) *A is compact if and only if \widehat{A} is discrete [7, Theorem 23.17].*
- (iii) *A is torsion-free if and only if \widehat{A} is densely divisible (Robertson [12, Theorem 5.2]).*

(See [7, § 24].) For every continuous homomorphism $\theta: A \rightarrow B$ of locally compact abelian groups, there is a continuous homomorphism $\widehat{\theta}: \widehat{B} \rightarrow \widehat{A}$, called the *adjoint* or *dual* of θ , given by $\widehat{\theta}(\chi) = \chi \circ \theta$. The double adjoint recovers the original homomorphism, and the adjoint of an isomorphism is an isomorphism. In particular, there is a natural isomorphism between $\text{Aut}(A)$ and $\text{Aut}(\widehat{A})$ given by $\theta \mapsto \widehat{\theta}^{-1}$.

Lemma 2.9 ([7, § 23.33]). *Let $G = \bigoplus_{i \in I} (G_i, H_i)$ be a local direct product of locally compact abelian groups, where H_i is a compact open subgroup of G_i for all $i \in I$. Then*

$$\widehat{G} = \bigoplus_{i \in I} (\widehat{G}_i, H_i^\perp),$$

where $H_i^\perp := \{\chi \in \widehat{G}_i \mid \chi(H_i) = \{1\}\}$.

2.5 Torsion-free compact abelian groups

Torsion-free compact abelian groups have been classified; they are exactly the duals of divisible discrete abelian groups.

Lemma 2.10 ([7, Theorem 25.8]). *The compact torsion-free abelian groups are exactly the topological groups of the form*

$$\widehat{\mathbb{Q}}^\kappa \oplus \prod_{p \in \mathbb{P}} \mathbb{Z}_p^{\kappa_p}$$

for some cardinals κ and κ_p .

Given a torsion-free abelian pro- p group U and a subset $X = \{e_i \mid i \in I\}$ of U , we say X is a *basis* for U if U is freely generated by X as an abelian pro- p group; in other words, X converges to 0 in U (that is, every neighborhood of 0 in U contains all but finitely many elements of X), and every element of U is expressible as a sum $\sum_{i \in I} \lambda_i e_i$ with unique coefficients $\lambda_i \in \mathbb{Z}_p$ (here the infinite sum makes sense because X converges to 0). We can then write $U = \prod_{i \in I} \overline{\langle e_i \rangle}$. The *rank* of U (as a pro- p group) is the cardinality of some basis for U . A subgroup H of U is *pure* if, for all $n \in \mathbb{N}$, we have $H \cap nU = nH$; in other words, every element of H with an n -th root in U has an n -th root in H . In particular, if U is torsion-free and H is closed, then H is pure if and only if U/H is torsion-free.

Lemma 2.11. *Let U be a torsion-free abelian pro- p group.*

- (i) *U is free in the category of abelian pro- p groups; in other words, U has a basis. In particular, U is second-countable if and only if $U \cong \mathbb{Z}_p^\kappa$ for some $\kappa \leq \aleph_0$.*

- (ii) Every closed pure subgroup of U is a direct summand.
- (iii) Any two bases for U are conjugate under the action of $\text{Aut}(U)$; in particular, the rank is well-defined.
- (iv) Every \mathbb{Z}_p -submodule of U of finite rank is closed.

Proof. (i) We have $U \cong \mathbb{Z}_p^\kappa$ for some cardinal κ by Lemma 2.10. We see that \mathbb{Z}_p^κ is second-countable if and only if $\kappa \leq \aleph_0$.

(ii) Given a closed pure subgroup V , then U/V is a free abelian pro- p group by (i), and hence projective in the category of abelian pro- p groups (see [13, Proposition 5.2.2]). Thus the quotient map $U \rightarrow U/V$ splits; in other words, V is a direct summand.

(iii) If $U \cong \mathbb{Z}_p^\kappa$, then $\widehat{U} \cong (\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})^{(\kappa)}$, and we can recover κ as the largest size of an independent set in $(\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})^{(\kappa)}$; thus the rank is well-defined. The fact that bases are conjugate then follows by the standard universal property argument for free objects; see for instance [13, Proposition 5.1.2].

(iv) A \mathbb{Z}_p -submodule V of U of finite rank is one of the form

$$\left\{ \sum_{i=1}^n \lambda_i v_i \mid \lambda_i \in \mathbb{Z}_p \right\}$$

for some $v_1, \dots, v_n \in V$. In other words, V is the sum in U of the compact subgroups $\mathbb{Z}_p v_i = \langle v_i \rangle$ for $1 \leq i \leq n$. Since a sum of finitely many compact subgroups is compact, it follows that V is compact and hence closed. \square

2.6 Wreath products

Given a topological group H and $\kappa \in \mathbb{N} \cup \{\aleph_0\}$, we can form the unrestricted wreath product $H \text{ Wr } C_\kappa$, which has base group H^n . If H is discrete, we can also form the restricted wreath product $H \wr \mathbb{Z}$, where the base group is $H^{(\aleph_0)}$. In each case, we think of elements of the base as functions from C_κ or \mathbb{Z} to H in the obvious manner. Here is an observation about the monolith of such a group that will be useful for examples.

Lemma 2.12. *Let H be a group with a noncentral monolith M , and $\kappa \in \mathbb{N} \cup \{\aleph_0\}$. Then $\text{Mon}(H \text{ Wr } C_\kappa) = M^{C_\kappa}$; if H is discrete, then $\text{Mon}(H \wr \mathbb{Z}) = M^{(\mathbb{Z})}$.*

Proof. Let G be $H \text{ Wr } C_\kappa$ or $H \wr \mathbb{Z}$ for $\kappa \in \mathbb{N} \cup \aleph_0$ as applicable. If $\kappa = 1$, then $G = H$, and the conclusion is trivial; let us assume $\kappa \geq 2$. Given $a \in H$, let $\delta_{i,a}$ be the function from C_κ to H such that $\delta_{i,a}(i) = a$ and $\delta_{i,a}(j) = 0$ for $j \neq i$. Note that the hypotheses on H ensure that H has trivial center.

Let N be a nontrivial closed normal subgroup of G . Then N contains some nontrivial element (f, i) , where f is in the base and $i \in C_\kappa$. If $i \neq 0$, we see that N contains elements of $H^{(C_\kappa)}$ of the form $\delta_{j,a} - \delta_{j-i,a}$, so we may assume f is nontrivial. Then $f(j) \neq 1$ for some $j \in C_\kappa$, and hence N contains the nontrivial element $\delta_{j,[a,f(j)]}$ of $H^{(C_\kappa)}$, where a is any element of $H \setminus C_H(f(j))$. Thus N contains an element of the form $\delta_{j,b}$, where $b \in H \setminus \{1\}$. By conjugating in the j -th copy of H , we deduce that N contains $\delta_{j,c}$, where c ranges over a nontrivial normal subgroup of H , and in particular for all $c \in M$. By applying the shift map, in fact, N contains $\overline{\delta_{k,c}}$ for all $k \in C_\kappa$ and $c \in M$, that is, $M^{(C_\kappa)} \leq N$. Since N is closed, we have $\overline{M^{(C_\kappa)}} \leq N$. Conversely, it is easy to see that $\overline{M^{(C_\kappa)}}$ is itself a nontrivial closed normal subgroup of G . Thus $\text{Mon}(G) = \overline{M^{(C_\kappa)}}$. \square

3 Five families of abelian groups

We consider five families of abelian l.c.s.c. groups. For each family, we give a characterization in terms of standard properties of locally compact groups; in no case do we need to include “topologically characteristically simple” as one of the properties. However, all the families indeed consist of topologically characteristically simple groups, which we demonstrate by giving examples of l.c.s.c. groups G whose monolith is the abelian group of interest. In most cases, G will be compactly generated; for the exceptions, we explain why no such G exists.

3.1 Elementary abelian groups

A group is *elementary abelian* if it is abelian and has prime exponent. The elementary abelian l.c.s.c. groups have a well-known structure.

Proposition 3.1 (see e.g. [6, Corollary 3.26]). *Let A be a nontrivial abelian l.c.s.c. group, and let $p \in \mathbb{P}$. Then the following are equivalent:*

- (i) $A \cong C_p^\kappa \oplus C_p^{(\kappa')}$ for some $\kappa \in \mathbb{N} \cup \{\aleph_0\}$ and $\kappa' \in \{0, \aleph_0\}$;
- (ii) A is isomorphic to one of $C_p^{\aleph_0}$, $C_p^{(\aleph_0)}$, $C_p^{\aleph_0} \oplus C_p^{(\aleph_0)}$ or C_p^n for some $n \in \mathbb{N}$;
- (iii) A has exponent p .

All elementary abelian l.c.s.c. p -groups occur as minimal closed normal subgroups of soluble compactly generated l.c.s.c. groups.

Proposition 3.2. *Let A be an elementary abelian l.c.s.c. group. Then A is topologically characteristically simple. Moreover, there is a compactly generated soluble l.c.s.c. group G of derived length at most 3 such that $\text{Mon}(G) \cong A$.*

Proof. We go through the possibilities in Proposition 3.1 (ii).

Case 1: A is finite. In this case, $A \cong C_p^n$ for some $n \in \mathbb{N}$, and we can regard A as the additive group of the field \mathbb{F}_{p^n} with p^n elements. The multiplicative group $\mathbb{F}_{p^n}^*$ is then an abelian group that acts transitively on the nonzero elements of \mathbb{F}_{p^n} , and so A is isomorphic to the monolith of the finite metabelian group $\mathbb{F}_{p^n} \rtimes \mathbb{F}_{p^n}^*$ (or just \mathbb{F}_2 in the case $(n, p) = (1, 2)$).

Case 2: A is isomorphic to $C_p^{\mathfrak{S}_0}$ or $C_p^{(\mathfrak{S}_0)}$. We obtain A as the monolith of a group of the form $F \text{ Wr } \mathbb{Z}$ or $F \wr \mathbb{Z}$ respectively, for some finite group F ; by Lemma 2.12, it is enough to take a finite metabelian group F with noncentral monolith isomorphic to C_p^n for some $n \geq 1$. For example, we can take $F = \mathbb{F}_{p^2} \rtimes \mathbb{F}_{p^2}^*$.

Case 3: $A \cong C_p^{\mathfrak{S}_0} \oplus C_p^{(\mathfrak{S}_0)}$. In this case, A is isomorphic to the additive group of $\mathbb{F}_p((t))$; write \cdot for multiplication in the field $\mathbb{F}_p((t))$. We consider the compactly generated metabelian l.c.s.c. group $G = \mathbb{F}_p((t)) \rtimes \langle s \rangle$, where $sgs^{-1} = t \cdot g$ for all $g \in \mathbb{F}_p((t))$. Let N be a nontrivial closed normal subgroup of G ; it is easy to see that N intersects $\mathbb{F}_p((t))$ nontrivially, so let us assume $N \leq \mathbb{F}_p((t))$. Given a nontrivial element g of N , we see that, for all $f \in \mathbb{F}_p[t, t^{-1}]$, the product $f \cdot g$ is a sum of conjugates of g in G , and hence an element of N . Since $\mathbb{F}_p[t, t^{-1}]$ is dense in $\mathbb{F}_p((t))$ and N is closed, in fact, $f \cdot g \in N$ for all $f \in \mathbb{F}_p((t))$, that is, N is a nonzero ideal of $\mathbb{F}_p((t))$. Since $\mathbb{F}_p((t))$ is a field, we conclude that $N = \mathbb{F}_p((t))$. Thus A is the monolith of G . \square

3.2 The groups \mathbb{R}^n

The characterization of the groups \mathbb{R}^n is well known.

Proposition 3.3. *Let A be a nontrivial abelian locally compact group. Then the following are equivalent:*

- (i) $A \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$ (in particular, A is second-countable);
- (ii) A is connected and $P(A) = \{0\}$.

Proof. Clearly, \mathbb{R}^n is connected, abelian and $P(\mathbb{R}^n) = \{0\}$. Conversely, suppose A is a connected abelian l.c.s.c. group such that $P(A) = \{0\}$. Then A has no proper open subgroup and no nontrivial compact subgroup, so $A \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$ by Theorem 2.1. \square

We note also that $\mathbb{R}^m \cong \mathbb{R}^n$ if and only if $m = n$ (see [7, Theorem 9.12]).

Example 3.4. Fix $n \in \mathbb{N}$. By Lemma 2.12, we see that \mathbb{R}^n is the monolith of the virtually connected soluble Lie group $(\mathbb{R} \rtimes \mathbb{R}_{>0}^*) \wr C_n$.

3.3 The groups $\mathbb{Q}^{(\kappa)}$

The structure of divisible countable torsion-free abelian groups is well known, but the circumstances in which such groups can occur as minimal closed normal subgroups are more interesting.

Proposition 3.5. *Let A be a nontrivial abelian l.c.s.c. group. Then the following are equivalent:*

- (i) $A \cong \mathbb{Q}^{(\kappa)}$ for some $\kappa \in \mathbb{N} \cup \{\aleph_0\}$;
- (ii) A is discrete, torsion-free and divisible;
- (iii) A is totally disconnected, pA is dense in A for all $p \in \mathbb{P}$, and $P(A) = \{0\}$.

Proof. (i) and (ii) are equivalent by Lemma 2.4; note that we can have at most countably many copies of \mathbb{Q} since A is second-countable. Clearly, (ii) implies (iii).

Suppose that A is totally disconnected, pA is dense in A for all $p \in \mathbb{P}$, and $P(A) = \{0\}$. We see that A has no nontrivial compact subgroups; by Theorem 2.1, it follows that A is discrete, and clearly, also A is torsion-free. We also have $pA = A$ for all $p \in \mathbb{P}$. It is then clear (by induction on the prime factorization of n) that $nA = A$ for all $n \in \mathbb{N}$; hence A is divisible. Thus (iii) implies (ii), completing the proof that (i)–(iii) are equivalent. \square

If $A \cong \mathbb{Q}^{(\kappa)}$ for $\kappa \in \mathbb{N} \cup \{\aleph_0\}$, then κ is the supremum of the number of generators of any finitely generated subgroup of A . In particular, κ is an invariant of the group A .

For whether or not $\mathbb{Q}^{(\kappa)}$ appears as a minimal normal subgroup of a compactly generated (soluble) group, we appeal to an article [5] by P. Hall.

Theorem 3.6 (Hall [5, Theorem 2]). *There is a finitely generated soluble group G of derived length 3 such that $\text{Mon}(G) \cong \mathbb{Q}^{(\aleph_0)}$.*

More specifically, writing $\mathbb{Q}^{(\aleph_0)}$ as a \mathbb{Q} -vector space

$$\mathbb{Q}^{(\mathbb{Z})} \quad \text{with basis} \quad \{v_m \mid m \in \mathbb{Z}\},$$

Hall's construction is

$$G = \mathbb{Q}^{(\mathbb{Z})} \rtimes \langle \eta, \xi \rangle,$$

where ξ and η act on $\mathbb{Q}^{(\mathbb{Z})}$ via $\xi.v_m = v_{m+1}$ and $\eta.v_m = \rho(m)v_m$, where ρ is some bijection from \mathbb{Z} to \mathbb{P} . One sees that the conjugates of η in $\langle \eta, \xi \rangle$ commute with one another, so $\langle \eta, \xi \rangle \cong \mathbb{Z} \wr \mathbb{Z}$.

By contrast, Hall shows that, for $n \in \mathbb{N}$, the group \mathbb{Q}^n cannot occur as a minimal normal subgroup of a finitely generated group. We adapt the argument to

show that \mathbb{Q}^n cannot occur as a minimal closed normal subgroup of a compactly generated group, which leads to another characterization of the group $\mathbb{Q}^{(\aleph_0)}$.

Proposition 3.7. *The group $\mathbb{Q}^{(\aleph_0)}$ is the only discrete, abelian and torsion-free group that can occur as a minimal closed normal subgroup of a compactly generated l.c.s.c. group.*

Proof. The group $\mathbb{Q}^{(\aleph_0)}$ is discrete, abelian and torsion-free, and we have seen that it can occur as a minimal closed normal subgroup of a compactly generated l.c.s.c. group.

Now suppose that G is a compactly generated l.c.s.c. group and that M is a minimal closed normal subgroup of G that is discrete, abelian and torsion-free. In particular, M is characteristically simple. Since M is torsion-free, we must have $nM = M$ for all $n \in \mathbb{N}$, that is, M is divisible.

We may assume for a contradiction that M is not isomorphic to $\mathbb{Q}^{(\aleph_0)}$, so by Proposition 3.5, we have $M \cong \mathbb{Q}^n$ for some $n \in \mathbb{N}$. Let $\{v_1, \dots, v_n\}$ be a basis for M as a \mathbb{Q} -vector space. Since M is countable, the group $C_G(v_i)$ is a closed subgroup of G of countable index, and hence $C_G(v_i)$ is open for $1 \leq i \leq n$. Moreover, we observe that, since elements of M are uniquely divisible, $C_G(v_i)$ centralizes all rational multiples of v_i . Consequently, $C_G(M)$ is the following open subgroup of G :

$$C_G(M) = \bigcap_{i=1}^n C_G(v_i).$$

Clearly, also $C_G(M)$ is normal; the quotient $G/C_G(M)$ is then a finitely generated group, so we can write $G = \langle x_1, \dots, x_m \rangle C_G(M)$.

The rest of the argument is as in [5]. Writing M additively for the moment, and letting x_1, \dots, x_m act on M by left conjugation, we have equations of the form

$$x_\alpha \cdot v_i = \sum_{j=1}^n \xi_{ij\alpha} v_j, \quad x_\alpha^{-1} \cdot v_i = \sum_{j=1}^n \eta_{ij\alpha} v_j,$$

where i runs from 1 to n and α from 1 to m . The coefficients $\xi_{ij\alpha}$ and $\eta_{ij\alpha}$ are rational numbers, which between them involve only a finite set of primes π in their denominators. Let M_1 be the group generated by all G -conjugates of v_1, \dots, v_n . Then every element of M_1 is of the form $\sum_{j=1}^n \lambda_j v_j$, where each of the coefficients λ_j belongs to the proper subring $\mathbb{Z}[\frac{1}{p} : p \in \pi]$ of \mathbb{Q} . In particular, M_1 is a proper nontrivial subgroup of M that is normal in G , contradicting the minimality of M . \square

On the other hand, \mathbb{Q}^n occurs as the monolith of a countable soluble group G . For example, similar to Example 3.4, we can take $G = (\mathbb{Q} \times \mathbb{Q}_{>0}^*) \wr C_n$. The difference is that $\mathbb{R}_{>0}^*$ is compactly generated, whereas $\mathbb{Q}_{>0}^*$ is not.

3.4 The groups $\widehat{\mathbb{Q}}^\kappa$

The character group $\widehat{\mathbb{Q}}$ of \mathbb{Q} is perhaps less widely known than \mathbb{Q} itself, but still a basic object in the theory of locally compact abelian groups. It can be constructed as follows (see for instance [4] for details).

Given a p -adic number a , write $\{a\}_p$ for the p -adic fractional part of a ; in other words, $\{a\}_p$ is the unique element $r \in \mathbb{Z}[\frac{1}{p}]$ such that $0 \leq r < 1$ and $a \in r + \mathbb{Z}_p$.

Let $\mathbb{Q}_\infty = \mathbb{Z}_\infty = \mathbb{R}$. Let \mathbb{A} be the ring of adèles over \mathbb{Q} ; specifically,

$$\mathbb{A} = \bigoplus_{p \in \mathbb{P} \cup \{\infty\}} (\mathbb{Q}_p, \mathbb{Z}_p).$$

We identify \mathbb{A} with its additive group, which is a locally compact abelian group. For each $a \in \mathbb{A}$, there is a character ξ_a of \mathbb{Q} defined by

$$\xi_a(r) = e^{-2\pi i r a_\infty} \prod_{p \in \mathbb{P}} e^{2\pi i \{ra_p\}_p}.$$

The map $a \mapsto \xi_a$ defines a quotient map $\theta: \mathbb{A} \rightarrow \widehat{\mathbb{Q}}$. The kernel consists of those $a \in \mathbb{A}$ such that a_p is a constant rational number over all $p \in \mathbb{P} \cup \{\infty\}$; in particular, $\ker \theta \cong \mathbb{Q}$ as a discrete subgroup of \mathbb{A} . Thus $\widehat{\mathbb{Q}}$ can be identified with the quotient \mathbb{A}/\mathbb{Q} .

We can characterize the groups $\widehat{\mathbb{Q}}^\kappa$ in a similar way to their dual groups $\mathbb{Q}^{(\kappa)}$.

Proposition 3.8. *Let A be a nontrivial abelian l.c.s.c. group. Then the following are equivalent:*

- (i) $A \cong \widehat{\mathbb{Q}}^\kappa$ for some $\kappa \in \mathbb{N} \cup \{\aleph_0\}$;
- (ii) A is compact, torsion-free and densely divisible;
- (iii) A is torsion-free and connected, and $P(A) = A$.

Proof. Suppose $A \cong \widehat{\mathbb{Q}}^\kappa$ for some $\kappa \in \mathbb{N} \cup \{\aleph_0\}$. Then by Lemma 2.9, A is the character group of $\mathbb{Q}^{(\kappa)}$. Since $\mathbb{Q}^{(\kappa)}$ is discrete and torsion-free, it follows by Lemma 2.8 that A is compact and densely divisible. Thus (i) implies (ii).

Suppose (ii) holds. Clearly, $P(A) = A$, and by Lemma 2.5, A is connected. Thus (ii) implies (iii).

Suppose (iii) holds. By Lemma 2.2, A is compact. We deduce that (i) holds from Lemma 2.10 and the fact that A is second-countable. This completes the proof that (i)–(iii) are equivalent. \square

If $A \cong \widehat{\mathbb{Q}}^\kappa$, then the cardinal κ is uniquely determined by \widehat{A} , and hence by A . Using the natural isomorphism between $\text{Aut}(\mathbb{Q}^{(\kappa)})$ and $\text{Aut}(\widehat{\mathbb{Q}}^\kappa)$, Hall's construction from the last subsection immediately gives rise to a compactly generated soluble l.c.s.c. group G of the form $G = \widehat{\mathbb{Q}}^{\aleph_0} \rtimes (\mathbb{Z} \wr \mathbb{Z})$ with $\text{Mon}(G) \cong \widehat{\mathbb{Q}}^{\aleph_0}$; similarly, for $n \in \mathbb{N}$, we obtain $\widehat{\mathbb{Q}}^n$ as the monolith of a soluble l.c.s.c. group of the form $(\widehat{\mathbb{Q}} \times \mathbb{Q}_{>0}^*) \wr C_n$.

On the other hand, the same obstacle as for \mathbb{Q}^n prevents $\widehat{\mathbb{Q}}^n$ from occurring as a minimal closed normal subgroup of a compactly generated l.c.s.c. group and gives a characterization of $\widehat{\mathbb{Q}}^{\aleph_0}$.

Proposition 3.9. *The group $\widehat{\mathbb{Q}}^{\aleph_0}$ is the only compact, abelian and torsion-free group that can occur as a minimal closed normal subgroup of a compactly generated l.c.s.c. group.*

Proof. The group $\widehat{\mathbb{Q}}^{\aleph_0}$ is compact, abelian and torsion-free, and we have seen that it can occur as a minimal closed normal subgroup of a compactly generated l.c.s.c. group.

Now suppose that G is a compactly generated l.c.s.c. group and that M is a minimal closed normal subgroup of G that is compact, abelian and torsion-free. In particular, M is characteristically simple, so as a compact group, M must either be connected or pro- p for some p . The latter case can be ruled out; if M were to be a torsion-free pro- p group, then pM would be a proper nontrivial closed characteristic subgroup of M , which would contradict the minimality of M . So M is connected.

We may assume for a contradiction that M is not isomorphic to $\widehat{\mathbb{Q}}^{\aleph_0}$, so by Proposition 3.8, we have $M \cong \widehat{\mathbb{Q}}^n$ for some $n \in \mathbb{N}$, and hence $\widehat{M} \cong \mathbb{Q}^n$. Let $\{\chi_1, \dots, \chi_n\}$ be a basis for \widehat{M} as a \mathbb{Q} -vector space. We have a natural action of G on \widehat{M} given by

$$(g \cdot \chi)(m) = \chi(g^{-1}mg) \quad (g \in G, \chi \in \widehat{M}, m \in M).$$

Since \widehat{M} is countable, the stabilizer of χ_i ,

$$G_i := \{g \in G \mid \text{for all } m \in M, \chi_i(g^{-1}mg) = \chi_i(m)\},$$

has countable index in G . Moreover, G_i is closed; specifically, we have

$$G_i / \ker \chi_i = C_H(M / \ker \chi_i),$$

where $H = N_G(\ker \chi_i) / \ker \chi_i$, and we see that H is a locally compact group since both $\ker \chi_i$ and $N_G(\ker \chi_i)$ are closed. Thus G_i is open. As in the proof

of Proposition 3.7, we see that $\bigcap_{i=1}^n G_i$ acts trivially on \widehat{M} . Since the characters of M distinguish points, we have

$$C_G(M) = \bigcap_{i=1}^n G_i,$$

so $C_G(M)$ is open and $G = \langle x_1, \dots, x_m \rangle C_G(M)$.

As in the proof of Proposition 3.7, we obtain a proper nontrivial subgroup V_1 of \widehat{M} that is preserved by G . The annihilator of V_1 in M ,

$$V_1^\perp = \bigcap_{\chi \in V_1} \ker \chi,$$

is then a proper nontrivial closed subgroup of M that is normal in G , contradicting the minimality of M . \square

3.5 The groups $\mathbb{Q}_p(\kappa)$

The group $\mathbb{Q}_p(\kappa)$ is the local direct product $\bigoplus_X (\mathbb{Q}_p, \mathbb{Z}_p)$, where the indexing set X has size κ . The group $\mathbb{Q}_p(\aleph_0)$ has been proposed (for example [3]) as a p -adic version of the infinite-dimensional separable Hilbert space, over which one can define operator algebras and representation theory analogous to the classical theory over \mathbb{R} and \mathbb{C} . However, in contrast to the other families, the author is not aware of a standard characterization of the groups $\mathbb{Q}_p(\kappa)$ as abelian l.c.s.c. groups. We provide a new characterization here.

Proposition 3.10. *Let A be an abelian l.c.s.c. group, and let $p \in \mathbb{P}$. Then the following are equivalent:*

- (i) $A \cong \mathbb{Q}_p(\kappa)$ for some $\kappa \in \mathbb{N} \cup \{\aleph_0\}$;
- (ii) A is totally disconnected and torsion-free, and both $P_p(A)$ and pA are dense in A ;
- (iii) A is totally disconnected and torsion-free, $P_p(A) = A$, and $\text{div}(A)$ is dense in A .

Proof. It is clear that (i) implies (ii), and we see that (ii) implies (iii) by Lemmas 2.2 and 2.6. It now remains to show that (iii) implies (i). So from now on, we assume that A is totally disconnected and torsion-free, and that both $P_p(A)$ and $\text{div}(A)$ are dense in A . In fact, $P_p(A) = A$ by Lemma 2.2, so we can regard A as a \mathbb{Z}_p -module; in particular, since A is torsion-free, it follows that A is not discrete.

Fix a compact open subgroup U of A . Since U is an infinite torsion-free second-countable pro- p group, by Lemma 2.11, we have $U \cong \mathbb{Z}_p^\kappa$ for some $\kappa \in \mathbb{N} \cup \{\aleph_0\}$.

Next, we note that $\text{div}(A) \cap U$ is a dense \mathbb{Z}_p -submodule of U ; given $\lambda \in \mathbb{Z}_p$ and $a \in \text{div}(A)$, then λa has n -th root $\lambda(a/n)$ in A for all $n \in \mathbb{N}$, so we have $\lambda a \in \text{div}(A) \cap U$. In the case that $\kappa \in \mathbb{N}$, all \mathbb{Z}_p -submodules of U are closed by Lemma 2.11, so $\text{div}(A) \geq U$; in this case, A is divisible, so $A = U^{\text{div}} \cong \mathbb{Q}_p^\kappa$. Thus we may assume that $\kappa = \aleph_0$. More explicitly, let us write $\mathbb{Q}_p(\aleph_0)$ as

$$\mathbb{Q}_p(\aleph_0) := \bigoplus_{\mathbb{N}} (\mathbb{Q}_p, \mathbb{Z}_p),$$

and let δ_i be the element of $\bigoplus_{\mathbb{N}} (\mathbb{Q}_p, \mathbb{Z}_p)$ with i -th entry 1 and all other entries 0. Note that $\{\delta_i \mid i \in \mathbb{N}\}$ is a basis for $\mathbb{Z}_p^{\aleph_0} := \prod_{\mathbb{N}} \mathbb{Z}_p$.

Note that, since $\mathbb{Q}_p(\aleph_0)/\mathbb{Z}_p^{\aleph_0}$ is torsion, we have $\mathbb{Q}_p(\aleph_0) \leq (\mathbb{Z}_p^{\aleph_0})^{\text{div}}$. Similarly, A/U is torsion since U is open and the p^n -th powers of every element of A converge to the identity as $n \rightarrow \infty$; thus $A \leq U^{\text{div}}$. We aim to find an isomorphism $\theta: U \rightarrow \mathbb{Z}_p^{\aleph_0}$ of topological groups whose divisible extension θ^{div} satisfies $\theta^{\text{div}}(A) = \mathbb{Q}_p(\aleph_0)$.

Notice that we can write A as a subgroup $M^{\text{div}} + U$ of U^{div} , where M is a \mathbb{Z}_p -submodule of U . Indeed, since $A = \text{div}(A) + U$ and U is essential, the submodule $M = \text{div}(A) \cap U$ will suffice. In fact, it suffices to take a submodule M that contains some multiple of an element of $a + U$ for each $a \in A$. Since A/U is countable, we can take M to be countably generated as a \mathbb{Z}_p -module. That is, we can take $M = \bigcup_{i \in \mathbb{N}} M_i$, where $(M_i)_{i \in \mathbb{N}}$ is an increasing sequence of finite-rank \mathbb{Z}_p -submodules of U .

Given one such \mathbb{Z}_p -submodule M_i , let M'_i be the set of roots of M_i in U . Since M_i has finite rank, it is closed by Lemma 2.11; hence M_i is isomorphic as a topological group to \mathbb{Z}_p^r for some $r \in \mathbb{N}$. Ignoring the topology, we see that M'_i must correspond to a subgroup of $M_i^{\text{div}} \cong \mathbb{Q}_p^r$. Since $M'_i \in U$, there are no nontrivial p -divisible elements of M'_i , from which we deduce that $|M'_i : M_i|$ is finite. We also see that $M'_i \leq \text{div}(A)$ and that M'_i is pure in U , so M'_i is a direct summand of U of the same rank as M_i . Thus, by replacing M_i with M'_i , we may assume $M = \bigcup_{i \in \mathbb{N}} M_i$, where $(M_i)_{i \in \mathbb{N}}$ is an increasing sequence of direct summands of U .

By refining the sequence, we may also assume that M_i has rank i ; in other words, we have $M_1 = \langle a_1 \rangle$, and thereafter, $M_{i+1} = M_i \oplus \langle a_{i+1} \rangle$ for some sequence $(a_i)_{i \in \mathbb{N}}$ in $\text{div}(A) \cap U$.

Choose an ordered basis $(e_i)_{i \in \mathbb{N}}$ for U .

Claim. *Subject to a suitable reordering of $(e_i)_{i \in \mathbb{N}}$, there are $\lambda_{ij} \in \mathbb{Z}_p$ with $\lambda_{ij} = 0$ for $i > j$ and $\lambda_{ij} = 1$ for $i = j$, and $a_i := \sum_{j \in \mathbb{N}} \lambda_{ij} e_j$, such that $\{a_1, \dots, a_n\}$ forms a basis for M_n for all $n \in \mathbb{N}$.*

We proceed by induction starting from $n = 0$, where $M_0 = \{0\}$; there is nothing to prove in the base case. Let $n \geq 1$. Given $k \in \mathbb{N}$, let π_k be the projection of $U = \prod_{i \in \mathbb{N}} \overline{\langle e_i \rangle}$ onto the direct summand $\prod_{i \leq k} \overline{\langle e_i \rangle}$ with respect to the basis $(e_i)_{i \in \mathbb{N}}$. By the inductive hypothesis, we have a suitable basis $\{a_1, \dots, a_{n-1}\}$ for M_{n-1} , and we see that

$$\pi_{n-1}(M_{n-1}) = \prod_{i \leq n-1} \overline{\langle e_i \rangle} = \pi_{n-1}(M_n),$$

so $M_n = M_{n-1} + \ker \pi_{n-1}$. Considering ranks, it follows that, in fact, we have $M_n = M_{n-1} \oplus \overline{\langle a_n \rangle}$ for some $a_n \in \ker \pi_{n-1}$. Since M_n is a direct summand of U , we see that $a_n \notin pU$, and hence if we write a_n as $\prod_{j \geq n} \lambda_j e_j$, there is some $k \geq n$ such that $\lambda_k \in \mathbb{Z}_p^*$. By replacing a_n with $\lambda_k^{-1} a_n$, we may assume $\lambda_k = 1$, and then, by swapping e_k and e_n , we may assume that $k = n$. (Note that this last step does not change e_1, \dots, e_{n-1} , so as we run the argument, we will eventually arrive at a fixed ordered basis for U with the required properties.) We now see that $\{a_1, \dots, a_n\}$ has the required form, completing the proof of the claim.

We now consider the set $X = \{a_i \mid i \in \mathbb{N}\}$ provided to us by the claim. Note that the restriction on the coefficients of a_i with respect to the basis $\{e_i \mid i \in \mathbb{N}\}$ ensures that $a_i \rightarrow 0$ as $i \rightarrow \infty$. The fact that M is dense in U ensures that X is a topological generating set for U . Moreover, X is topologically linearly independent over \mathbb{Z}_p ; given an equation of the form $\sum_{i \in \mathbb{N}} \mu_i a_i = 0$ with $\mu_i \in \mathbb{Z}_p$, we see by applying the maps π_n that $\mu_n = 0$ for all $n \in \mathbb{N}$. Thus X is a basis for U .

We carry out the same argument for writing $\mathbb{Q}_p(\mathbb{N})$ as a subgroup $N^{\text{div}} + \mathbb{Z}_p^{\mathbb{N}}$ of $(\mathbb{Z}_p^{\mathbb{N}})^{\text{div}}$. In this case, we see that it suffices to take $N = \bigcup_{i \in \mathbb{N}} N_i$, where $N_i = \langle \delta_1, \dots, \delta_i \rangle$.

Since $(a_i)_{i \in \mathbb{N}}$ and $(\delta_i)_{i \in \mathbb{N}}$ are bases of U and $\mathbb{Z}_p^{\mathbb{N}}$ respectively, there is a unique isomorphism $\theta: U \rightarrow \mathbb{Z}_p^{\mathbb{N}}$ of topological groups such that $\theta(a_i) = \delta_i$; this extends to an isomorphism $\theta^{\text{div}}: U^{\text{div}} \rightarrow (\mathbb{Z}_p^{\mathbb{N}})^{\text{div}}$. It is then clear that $\theta^{\text{div}}(M) = N$, and hence $\theta^{\text{div}}(A) = \mathbb{Q}_p(\mathbb{N})$, so θ^{div} restricts to an isomorphism from A to $\mathbb{Q}_p(\mathbb{N})$. This completes the proof that (iii) implies (i) and hence that all three statements are equivalent. \square

If $A \cong \mathbb{Q}_p(\kappa)$, then the cardinal κ is the rank in the sense of Lemma 2.11 of any compact open subgroup of A . Thus κ is uniquely determined by A . Clearly, also p is uniquely determined.

Example 3.11. Fix $p \in \mathbb{P}$ and $\kappa \in \mathbb{N} \cup \{\aleph_0\}$. We have a compactly generated metabelian group $G = \mathbb{Q}_p \rtimes \langle s \rangle$, where s acts on \mathbb{Q}_p by multiplication by p . Note that the proper nontrivial closed subgroups of \mathbb{Q}_p are exactly its compact open subgroups, which are all conjugate in G ; from here, it is easy to see that $\text{Mon}(G) = \mathbb{Q}_p$.

Generalizing this example, we realize $\mathbb{Q}_p(\kappa)$ as the monolith of

$$\bigoplus_{C_\kappa} (\mathbb{Q}_p \rtimes \langle s \rangle, \mathbb{Z}_p) \rtimes C_\kappa,$$

where C_κ acts by shifting the indexing set of the local direct product.

4 The classification of topologically characteristically simple abelian l.c.s.c. groups

We now show that the five families described in the previous section comprise the whole of the class \mathcal{A} of topologically characteristically simple abelian l.c.s.c. groups. With the characterizations of the previous section in hand, the proof is straightforward; we show that any $A \in \mathcal{A}$ must have one of the sets of properties characterizing one of the families.

Lemma 4.1. *Let $A \in \mathcal{A}$. Then the following holds:*

- (1) *either A has prime exponent, or else A is torsion-free;*
- (2) *if A is torsion-free, then pA is dense for all $p \in \mathbb{P}$;*
- (3) *A is either connected or totally disconnected;*
- (4) *$P(A)$ is trivial or equal to A ;*
- (5) *if A is totally disconnected and $P(A) = A$, then $P_p(A) = A$ for some $p \in \mathbb{P}$.*

Proof. (1) We note that, for all $p \in \mathbb{P}$, the subgroup $\Omega_p(A) := \{a \in A \mid pa = 0\}$ is closed, and hence belongs to $\{\{0\}, A\}$. If $\Omega_p(A) = A$ for some $p \in \mathbb{P}$, then A has exponent p . Otherwise, $\Omega_p(A) = \{0\}$ for all $p \in \mathbb{P}$, from which it follows for all $n \in \mathbb{N}$ (by induction on the number of prime factors of n) that the only solution in A to the equation $na = 0$ is $a = 0$, that is, A is torsion-free.

(2) Suppose A is torsion-free, and let $p \in \mathbb{P}$. Then pA is an injective image of A , hence nontrivial; thus pA is dense in A .

(3) The connected component A° of the identity in A is a closed characteristic subgroup, so either $A^\circ = A$ or $A^\circ = \{0\}$.

(4) This is clear, as $P(A)$ is a closed characteristic subgroup.

(5) If A has exponent $p \in \mathbb{P}$, then $P_p(A) = A$, so by (1), we may assume A is torsion-free. In particular, $P(A)$ is nontrivial and torsion-free, so A is not discrete. It follows that A has an infinite profinite open subgroup U . Since U is nontrivial, it must have a nontrivial p -Sylow subgroup S for some $p \in \mathbb{P}$. We then see that $P_p(A) \cap U = S$; in particular, $P_p(A)$ is nontrivial and closed. Since $P_p(A)$ is clearly also characteristic, we deduce that $A = P_p(A)$. \square

Theorem 4.2. *Let A be a nontrivial abelian l.c.s.c. group. Then the following are equivalent.*

- (i) A is isomorphic to one of the following, where $n \in \mathbb{N}$, $\kappa \in \mathbb{N} \cup \{\aleph_0\}$ and $p \in \mathbb{P}$ as applicable:

$$C_p^n, C_p^{\aleph_0}, C_p^{(\aleph_0)}; C_p^{\aleph_0} \oplus C_p^{(\aleph_0)}, \mathbb{R}^n, \mathbb{Q}^{(\kappa)}, \widehat{\mathbb{Q}}^\kappa, \mathbb{Q}_p^{(\kappa)}.$$

- (ii) A is isomorphic to the monolith of some soluble l.c.s.c. group of derived length at most 3.
- (iii) A is topologically characteristically simple.
- (iv) A satisfies statements (1)–(5) of Lemma 4.1.

Proof. The examples of the previous section show that (i) implies (ii). Clearly, (ii) implies (iii), and (iii) implies (iv) by Lemma 4.1. It remains to show that (iv) implies (i).

If A has prime exponent, then $A \cong C_p^\kappa \oplus C_p^{(\kappa')}$ for some $\kappa \in \mathbb{N} \cup \{\aleph_0\}$ and $\kappa' \in \{0, \aleph_0\}$ by Proposition 3.1, so we may assume that A does not have prime exponent; thus A is torsion-free by (1), and hence, by (2), pA is dense in A for all $p \in \mathbb{P}$.

By (3), A is either connected or totally disconnected, and by (4), $P(A)$ is either trivial or equal to A . Consider the four cases in turn.

A is connected and $P(A) = \{0\}$. By Proposition 3.3, we have $A \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$.

A is connected and $P(A) = A$. Since A is also torsion-free, we conclude by Proposition 3.8 that $A \cong \widehat{\mathbb{Q}}^\kappa$ for some $\kappa \in \mathbb{N} \cup \{\aleph_0\}$.

A is totally disconnected and $P(A) = \{0\}$. Since pA is dense in A for all $p \in \mathbb{P}$, we conclude by Proposition 3.5 that $A \cong \mathbb{Q}^{(\kappa)}$ for some $\kappa \in \mathbb{N} \cup \{\aleph_0\}$.

A is totally disconnected and $P(A) = A$. In this case, $A = P_p(A)$ for some $p \in \mathbb{P}$ by (5). Since also A is torsion-free and pA is dense, we conclude by Proposition 3.10 that $A \cong \mathbb{Q}_p^{(\kappa)}$ for some $\kappa \in \mathbb{N} \cup \{\aleph_0\}$.

In all cases, (i) holds, completing the cycle of implications. \square

We note the following consequence of Theorem 4.2 together with the constructions of the previous section.

Corollary 4.3. *With the exception of the groups \mathbb{Q}^n and $\widehat{\mathbb{Q}}^n$ for $n \in \mathbb{N}$, every abelian topologically characteristically simple l.c.s.c. group occurs as the monolith of a compactly generated soluble l.c.s.c. group of derived length at most 3.*

Theorem 1.1 is now clear.

Example 4.4. Let $p \in \mathbb{P}$, and let \mathbb{K} be the additive group of one of the fields $\mathbb{F}_p((t))$, \mathbb{R} or \mathbb{Q}_p . If \mathbb{K} is $\mathbb{F}_p((t))$ or \mathbb{Q}_p , let $A_{\mathbb{K}} = \langle s \rangle$, where s acts on $\mathbb{F}_p((t))$ or \mathbb{Q}_p as multiplication by t or p respectively, and if $\mathbb{K} = \mathbb{R}$, we set $A_{\mathbb{K}} = \mathbb{R}_{>0}^*$. Thus $\mathbb{K} \rtimes A_{\mathbb{K}}$ is a compactly generated metabelian l.c.s.c. group with \mathbb{K} as its monolith. We then form a group $B_{\mathbb{K}} = (\mathbb{K} \times \mathbb{K}) \rtimes A_{\mathbb{K}}$, where $A_{\mathbb{K}}$ has the same action on each copy of \mathbb{K} . This time, we see that $B_{\mathbb{K}}$ has 2^{\aleph_0} minimal closed normal subgroups isomorphic to \mathbb{K} ; namely, for each $\lambda \in \mathbb{K}$, there is a minimal closed normal subgroup

$$L_{\mathbb{K},\lambda} := \{(a, \lambda a) \mid a \in \mathbb{K}\}.$$

In particular, this shows that $\mathbb{F}_p((t))$, \mathbb{R} and \mathbb{Q}_p can occur 2^{\aleph_0} times as a minimal closed normal subgroup of a compactly generated l.c.s.c. group. Generalizing this example, we see for $n \in \mathbb{N}$ that \mathbb{K}^n occurs 2^{\aleph_0} times as a minimal closed normal subgroup of $B_{\mathbb{K}} \wr C_n$ by taking a subgroup of the base group of the form $(L_{\mathbb{K},\lambda})^n$. In the case $\mathbb{K} = \mathbb{Q}_p$, we can form the group

$$\bigoplus_{\mathbb{Z}} (B_{\mathbb{Q}_p}, \mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z},$$

where \mathbb{Z} acts by shifting the indexing set; this time, $\mathbb{Q}_p(\aleph_0)$ occurs 2^{\aleph_0} times as a minimal closed normal subgroup.

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