The ℓ -modular representation of reductive groups over finite local rings of length two

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Abstract. Let $\mathcal{O}_2$ and $\mathcal{O}'_2$ be two distinct finite local rings of length two with residue field of characteristic p . Let $\mathbb{G}(\mathcal{O}_2)$ and $\mathbb{G}(\mathcal{O}'_2)$ be the groups of points of any reductive group scheme $\mathbb{G}$ over $\mathbb{Z}$ such that p is very good for $\mathbb{G} \times \mathbb{F}_q$ or $\mathbb{G} = \mathrm{GL}_n$. We prove that there exists an isomorphism of group algebras $K\mathbb{G}(\mathcal{O}_2) \cong K\mathbb{G}(\mathcal{O}'_2)$, where K is a sufficiently large field of characteristic different from p .

1 Introduction

Let $\mathcal{O}$ be a discrete valuation ring with a unique maximal ideal $\mathfrak{p}$ and having finite residue field $\mathbb{F}_q$, the field with q elements where q is a power of a prime p . We denote by $\mathcal{O}_r$ the reduction of $\mathcal{O}$ modulo $\mathfrak{p}^r$. Similarly, given $\mathcal{O}'$ a second discrete valuation ring with the same residue field $\mathbb{F}_q$, we can define $\mathcal{O}'_r$. The complex representation theory of general linear groups over the rings $\mathcal{O}_r$ has been heavily studied [13]. In particular, it has been conjectured by Onn [8] that there is an isomorphism of group algebras $\mathbb{C}\mathrm{GL}_n(\mathcal{O}_r) \cong \mathbb{C}\mathrm{GL}_n(\mathcal{O}'_r)$. When $r = 2$, this conjecture was proven by Singla [10]. Moreover, assuming p is odd, Singla proved a generalization of the conjecture for $r = 2$ when $\mathbb{G}$ is either SL_n with $p \nmid n$ or the classical groups Sp_n , O_n , and U_n (see [11]). Later on, Stasinski proved it for SL_n for all p (see [14]). More generally, for a possibly non-classical reductive group, Stasinski and Vera-Gajardo compared the complex representations of the two groups in question [15].

Let us briefly describe the result of Stasinski and Vera-Gajardo. First recall that, for any group scheme $\mathbb{G}$ over $\mathbb{Z}$ and any commutative ring R , we may speak of the group $\mathbb{G}(R)$ of R -points. We now let $\mathbb{G}$ be a reductive group scheme over $\mathbb{Z}$ – for details on such group schemes, see e.g. [2, 3]. Note that for any *root datum* – see e.g. [3, § II.1.13] – there is a split reductive group scheme $\mathbb{G}$ over $\mathbb{Z}$ with this root datum [2, Exp. XXV, Theorem 1.1]. The group $\mathbb{G} \times \mathbb{F}_q$ obtained by base-change

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with the finite field $\mathbb{F}_q$ of characteristic p is a reductive algebraic group over $\mathbb{F}_q$, and we want to insist that

$$\text{either } \mathbb{G} = \mathrm{GL}_n \text{ or } p \text{ is very good for } \mathbb{G} \times \mathbb{F}_q; \quad (*)$$

we observe that condition $(*)$ depends only on the root datum of $\mathbb{G}$ – see [12, Section 4] for the definition of good/very good primes.

Under assumption $(*)$, Stasinski and Vera-Gajardo proved that

$$\mathbb{C}\mathbb{G}(\mathcal{O}_2) \cong \mathbb{C}\mathbb{G}(\mathcal{O}'_2).$$

When $\mathbb{G} \times \mathbb{F}_q$ is a classical absolutely simple algebraic group not of type A , any $p > 2$ is very good for $\mathbb{G}$. When $\mathbb{G} = \mathrm{SL}_n$ or SU_n , p is very good for $\mathbb{G}$ if and only if $n \not\equiv 0 \pmod{p}$. If $\mathbb{G} \times \mathbb{F}_q$ is absolutely simple of exceptional type, any $p > 5$ is very good for $\mathbb{G}$; see [12, Section 4.3] for the precise conditions when $p \leq 5$.

We study the ℓ -modular representations of such group schemes over $\mathcal{O}_2$ a local ring of length two with finite residue field. One can show that $\mathcal{O}_2$ is isomorphic to one of the following rings: $\mathbb{F}_q[t]/t^2$ or $W_2(\mathbb{F}_q)$, the ring of Witt vectors of length two over $\mathbb{F}_q$ (see [15, Lemma 2.1]). Thus, we let $G_2 = \mathbb{G}(\mathcal{O}_2)$ and $G'_2 = \mathbb{G}(\mathcal{O}'_2)$, where $\mathbb{G}$ stands for GL_n or a reductive group scheme over $\mathbb{Z}$ such that p is very good for $\mathbb{G} \times \mathbb{F}_q$.

In this paper, we generalize the previous results over $\mathbb{C}$ to results over a sufficiently large field K of characteristic $l \neq p$. More precisely, we prove that there exists an isomorphism of group algebras $KG_2 \cong KG'_2$ over a sufficiently large field K of characteristic l as long as $l \neq p$. We take K to be sufficiently large so that the representation theory of the groups over K is the same as the representation theory over an algebraically closed field of characteristic l .

In order for us to understand those two group algebras, we study their decomposition by block algebras. Given A a K -algebra, we define a block of A to be a primitive idempotent b in the center of A , which is denoted by $Z(A)$; the algebra Ab is called a block algebra of A . By an idempotent, we mean an element b such that $b^2 = b$, and it is primitive if, whenever $b = b_1 + b_2$ is an expression of b as a sum of idempotents such that $b_1 b_2 = 0$, either $b_1 = 0$ or $b_2 = 0$. Moreover, the block algebra Ab is an indecomposable two-sided ideal summand of A . Thus, for a finite group G , we can write

$$KG = B_1 \oplus B_2 \oplus \cdots \oplus B_n,$$

where the $B_i = e_i KG$ are the unique block algebras of KG up to ordering [1]. Moreover, $e_i \cdot e_j = 0$ whenever $i \neq j$.

We investigate blocks of KG which we denote as $\text{Bl}(G)$. To understand the block algebras of those two group algebras, we exploit the fact that those two groups are extensions of $\mathbb{G}(\mathbb{F}_q)$ by an abelian p -group denoted by N which is isomorphic to the Lie algebra of $\mathbb{G}(\mathbb{F}_q)$ denoted by $\mathfrak{g}$ (see [15, Lemma 2.3]). In fact, for the rest of this section, let G denote either G_2 or G'_2 . Let the map

$$\rho: G \rightarrow \mathbb{G}(\mathbb{F}_q)$$

be the surjective map obtained from the map $\mathcal{O}_2 \rightarrow \mathbb{F}_q$ with $N = \ker(\rho)$. The two groups G_2 and G'_2 act on $\mathfrak{g}$, via its quotient $\mathbb{G}(\mathbb{F}_q)$. This action is transformed by the above isomorphism into the action of G on its normal subgroup N . We explore the details of this action in Section 3. In Section 2, we use Clifford theory to relate blocks of G with blocks of N . Specifically, we have that any block $b \in \text{Bl}(G)$ is equal to $\text{Tr}_H^G(d) = \sum_{g \in G/H} g d g^{-1}$ for d in $\text{Bl}(H)$, where H is the stabilizer in G for some block e of N . In Section 4, we exploit the fact that Clifford theory, in combination with the results obtained in the characteristic zero case, gives an isomorphism between certain block algebras of the stabilizers of e in those two groups. In Section 5, we induce the previous isomorphism to an isomorphism between block algebras of G_2 and G'_2 by taking advantage of the fact that blocks of a group algebra are interior G -algebras. This isomorphism will then give rise to an isomorphism between those two group algebras.

2 Background on Clifford theory of blocks

Let G be a finite group and N a normal subgroup of G . We use Clifford theory to relate blocks of KG with blocks of KN . Given b a block of KG and e a block of KN , we say b covers e if $be \neq 0$. We denote the set of blocks of G that cover e by $\text{Bl}(G|e)$. Note that there is a natural action of G on the set of blocks of N by conjugation. Moreover, one can show that, for a fixed b , the set of blocks e of KN satisfying $be \neq 0$ is a G -conjugacy class of blocks of KN (see [5, Proposition 6.8.2]).

We have the following Clifford theorem for blocks of G .

Theorem 2.1 ([5, Theorem 6.8.3]). *Let G be a finite group and N a normal subgroup of G . Given a block $b \in \text{Bl}(G)$, we have*

- (1) *there exists a block $e \in \text{Bl}(N)$ such that $b \cdot e \neq 0$, so $b \in \text{Bl}(G|e)$.*
- (2) *If e is as in (1), then $b = \text{Tr}_H^G(d) := \sum_{g \in G/H} g d g^{-1}$ for a unique block $d \in \text{Bl}(H|e)$, where H is the stabilizer of e in G under the conjugation action.*
- (3) *The assignment $d \mapsto \text{Tr}_H^G(d) = b$ gives a bijection between $\text{Bl}(H|e)$ and $\text{Bl}(G|e)$.*

For more information about Clifford theory of blocks consult Linckelmann [5], Nagao [6], or Craven [1]. Thus, in order for us to understand the block algebras of those two groups, we start by investigating the block structure of eKH for a fixed block $e \in \text{Bl}(N)$, where as above H is the stabilizer of e in G , under the conjugation action. Let $\text{Irr}_K(N)$ be the set of irreducible characters of KN . Since l does not divide $|N|$ (because N is a p -group), any block of KN is defined by $e = e_\chi = \frac{\chi(1)}{|N|} \sum_{g \in N} \chi(g^{-1})g$ for some χ in $\text{Irr}_K(N)$ (see [7]). Specifically, KN is a semisimple algebra and eKN is a matrix algebra, also called a block of defect zero.

For any α in $Z^2(G; K^*)$, we denote by $K^\alpha G$ the twisted group algebra of G by α . It has basis as a K -algebra the elements of G and, given $x, y \in G$, we define $x \cdot y = \alpha(x, y)xy$, where xy is the product of x and y in G . Furthermore, given a different β in $Z^2(G; K^*)$, there is a G -graded algebra isomorphism $K^\alpha G \cong K^\beta G$ if and only if the classes of α and β are equal in $H^2(G; K^*)$ (see [4]). Thus, for the purpose of this paper, α is defined up to an element in the second cohomology group $H^2(G; K^*)$ with G acting trivially on K^* .

Proposition 2.2 ([5, Theorem 6.8.13]). *Let H be a finite group, N a normal subgroup of H , and e an H -stable block of defect zero of KN . Set $S = KNe$, and suppose that K is a splitting field for S . Set $L = H/N$. For any $x \in H$, there is $s_x \in S^*$ such that $xtx^{-1} = s_x t (s_x)^{-1}$ for all $t \in S$ and such that $s_x s_y = s_{xy}$ if at least one of x, y is in N . Then the 2-cocycle $\alpha \in Z^2(H; K^*)$ defined by $s_x s_y = \alpha(x, y) s_{xy}$ for $x, y \in N$ depends only on the images of x, y in L and induces a 2-cocycle, still denoted α , in $Z^2(L; K^*)$, and we have an isomorphism of K -algebras*

$$eKH \xrightarrow{\sim} S \otimes_K K^{\alpha^{-1}} L$$

sending xe to $s_x \otimes \bar{x}$, where $x \in H$ and $\bar{x}$ is the image of x in L .

Thus, we can take advantage of the previous proposition to show that

$$e_\chi KH \cong e_\chi KN \otimes_K K^{\alpha^{-1}} L.$$

3 The action of G_2 and G'_2 on the kernel N

In this section, we will let G denote either G_2 or G'_2 . We study the action of G on the kernel N so we can understand the stabilizer $H \bmod N$ and thus we can take advantage of Proposition 2.2. The two groups G_2 and G'_2 act on $\mathfrak{g}$, via its quotient $\mathbb{G}(\mathbb{F}_q)$. There is a natural adjoint action of $\mathbb{G}(\mathbb{F}_q)$ on $\mathfrak{g}$,

$$\text{Ad}: \mathbb{G}(\mathbb{F}_q) \rightarrow \text{GL}(\mathfrak{g}).$$

Also, there is an automorphism of $\sigma: \mathbb{G}(\mathbb{F}_q) \rightarrow \mathbb{G}(\mathbb{F}_q)$ given by raising each matrix entry to the p -th power. Note that σ composed with Ad gives rise to a second action of $\mathbb{G}(\mathbb{F}_q)$ on $\mathfrak{g}$. Thus, given $X \in \mathfrak{g}$, we note that the action of $G_2 = \mathbb{G}(\mathbb{F}_q[t]/t^2)$ and $G'_2 = \mathbb{G}(W_2(\mathbb{F}_q))$ are as follows:

$$\begin{aligned} g \cdot_1 X &= \text{Ad}(\bar{g})X & \text{for } g \in G_2, \\ g \cdot_2 X &= \text{Ad}(\sigma(\bar{g}))X & \text{for } g \in G'_2, \end{aligned}$$

where $\bar{g} = \rho(g)$; see [15, Section 2.3] for more details. Moreover, we also have actions of both G_2 and G'_2 on $\mathfrak{g}^* = \text{Hom}_{\mathbb{F}_q}(\mathfrak{g}, \mathbb{F}_q)$ and $\text{Irr}_K(\mathfrak{g})$. Given F an element of $\mathfrak{g}^*$ or $\text{Irr}_K(\mathfrak{g})$ and $X \in \mathfrak{g}$, we define

$$\begin{aligned} (g \cdot_1 F)(X) &= F(g^{-1} \cdot_1 X) & \text{for } g \in G_2, \\ (g \cdot_2 F)(X) &= F(g^{-1} \cdot_2 X) & \text{for } g \in G'_2. \end{aligned}$$

Given a non-trivial irreducible character $\phi: \mathbb{F}_q \rightarrow K^*$ and for each $\beta \in \mathfrak{g}^*$, we define the character $\chi_\beta \in \text{Irr}_K(\mathfrak{g})$ by

$$\chi_\beta(X) = \phi(\beta(X)).$$

Lemma 3.1 ([15, Lemma 4.1]). *The map $\beta \mapsto \chi_\beta$ defines an isomorphism of abelian groups $\mathfrak{g}^* \rightarrow \text{Irr}_K(\mathfrak{g})$ that is G -invariant, i.e. for $g \in G$, we have*

$$g \cdot_1 \beta \mapsto g \cdot_1 \chi_\beta \quad \text{or} \quad g \cdot_2 \beta \mapsto g \cdot_2 \chi_\beta.$$

Note that there is also an isomorphism of $\mathfrak{g} \cong N$ which is G -invariant [15, Lemma 2.3]. Thus, we can identify the characters of N with characters of $\mathfrak{g}$. Given a character χ of N , we define H_2 and H'_2 to be the stabilizer of χ in G_2 and G'_2 respectively. Thus, we have the following lemma about the stabilizer of e_χ .

Lemma 3.2. *The stabilizer H_2/N and H'_2/N of e_χ are isomorphic.*

Proof. Since $g \cdot e_\chi = e_{g \cdot \chi}$, the stabilizer of e_χ is the same as the stabilizer of χ (see [7, Section 9]). Thus, we compute the stabilizer of χ . By the discussion above, we can consider χ to be in $\text{Irr}(\mathfrak{g})$; by Lemma 3.1, we have that $\chi = \chi_\beta$ for some $\beta \in \mathfrak{g}^*$. Thus,

$$H'_2/N = \{g \in \mathbb{G}(\mathbb{F}_q) : \text{Ad}(\sigma(g))\beta = \beta\} = \sigma(H_2/N).$$

Since the map σ is an automorphism, it follows that $H_2/N \cong H'_2/N$. $\square$

4 The isomorphism of eKH_2 and eKH'_2

In this section, we prove that there is an isomorphism of block algebras eKH_2 and eKH'_2 , where $e = e_\chi$ for χ in $\text{Irr}_K(N)$ such that H_2 and H'_2 are the stabilizer of e in G_2 and G'_2 respectively. To prove this isomorphism, we take advantage of Proposition 2.2. In order to do so, we need to understand the cofactors α associated to eKH_2 and eKH'_2 . We prove that the cofactor α is trivial in both cases. For this, we introduce a projective K -representation of G with factor set α . By which, we mean a map $X: G \rightarrow \text{GL}_n(K)$ such that $X(x)X(y) = \alpha(xy)X(xy)$ for all x, y in G , where $\alpha(xy)$ is in K^* . In fact, one can check that α is in $Z^2(G; K^*)$, where we assume that G acts on K^* trivially. Similarly, one can define a projective representation as a $K^\alpha G$ -module [6]. We let H denote either H_2 or H'_2 . The projective representations of H are closely related to Clifford theory. In fact, we can use Proposition 2.2 to show that there is a projective representation V of H that extends χ . By that, we mean that V viewed as KN -module is isomorphic to the KN -module associated to χ .

Proposition 4.1. *Let $\alpha \in Z^2(H; K^*)$ be the 2-cocycle associated to the block e_χ of KN , as in Proposition 2.2. There exists a projective representation of H extending χ which has α as its associated cofactor.*

Proof. Let Y be the representation associated to χ , i.e. $Y: N \rightarrow \text{GL}_n(K)$. Let R be a set of representatives of $L = H/N$ in H and $S = KNe$. For each $x \in R$, by Proposition 2.2, there is some $s_x \in S^*$ such that $xtx^{-1} = s_xt(s_x)^{-1}$ for all $t \in S$. Note also, since e is an H -stable block of defect zero of KN , then we have $S = KNe \cong M_n(K)$. With abuse of notation, we can think of s_x as an element of $\text{GL}_n(K)$. Now define X to be the map $X: H \rightarrow \text{GL}_n(K)$ such that, for each $h = xn \in H$, we have $X(h) = s_x Y(n)$ with $x \in R$ and $n \in N$. One can check that X as defined above is a projective representation of H with factor set α in $Z^2(L; K^*)$ that extends Y . $\square$

Proposition 4.1 associates a cocycle α to a projective K -representation of H that extends χ ; to study this cocycle, we study a corresponding projective representation of H over a field of characteristic 0. To relate them, we need an l -modular system. By this, we mean a triple $(F; R; K)$, where F is a field of characteristic zero equipped with a discrete valuation, R is the valuation ring in F with maximal ideal (π) , and $K = R/(\pi)$ is the residue field of R , which is required to have characteristic l . If both F and K are splitting fields for G , we say that the triple is a splitting l -modular system for G . Note, we need an l -modular system so we can relate representations over a field F of characteristic zero to representations over a field K of characteristic l . The following lemma shows that, given a projective

representation over F , we can obtain a projective representation over R . Note that this lemma is just a generalization of the already known fact over group algebras that can be extended to hold over twisted group algebras. In fact, the proof is the same; see [9, Lemma 2.2.2].

Lemma 4.2. *Let G be a finite group and M an $F^\alpha G$ -module with $\alpha \in Z^2(G; R^*)$. Then M contains a lattice L that is an $R^\alpha G$ -module.*

Proof. Note, by [9, Lemma 2.2.1], to show that L is a lattice of M , it is enough to show that L is finitely generated as an R -module and L generates M as an F -vector space. Thus, pick an F basis $e_1, \dots, e_n$ of M ; then let

$$L' := Re_1 + \dots + Re_n$$

be a lattice of M . Let $L := \sum_{g \in G} gL'$; then L is an $R^\alpha G$ -module. Note that L is finitely generated as an R -module by $\{ge_i : 1 \leq i \leq n, g \in G\}$, and it also generates M as a vector space. Thus, L is a G -invariant lattice. $\square$

Before we introduce the following theorem, recall that a block of defect zero may be defined as a matrix algebra. Moreover, by Proposition 4.3, such a block is a ring summand of KG which has a projective simple module.

Proposition 4.3 ([17, Theorem 9.6.1]). *Let $(F; R; K)$ be a splitting p -modular system in which R is complete, and let G be a group of order $p^d q$, where q is prime to p . Let T be an FG -module of dimension n , containing a full RG -sublattice T_0 . The following are equivalent:*

- (1) $p^d | n$ and T is a simple FG -module.
- (2) The homomorphism $RG \rightarrow \text{End}_R(T_0)$ that gives the action of RG on T_0 identifies $\text{End}_R(T_0) \cong M_n(R)$ with a ring direct summand of RG .
- (3) T is a simple FG -module, and T_0 is a projective RG -module.
- (4) The homomorphism $KG \rightarrow \text{End}_K(T_0/\pi T_0)$ identifies

$$\text{End}_K(T_0/\pi T_0) \cong M_n(K)$$

with a ring direct summand of KG .

- (5) As a KG -module, $T_0/\pi T_0$ is simple and projective.

Most importantly, by Proposition 4.3, a block of defect zero can have only one simple module, and there is a unique ordinary simple module that reduces to it. This is used in the proof of the following theorem.

Theorem 4.4. *Given an H -stable block e_χ of defect zero of KN , where $N \trianglelefteq H$, let V be the unique ordinary simple module associated to this block. According to Proposition 4.1, let $\hat{V}$ be the projective representation of H that extends V , with cofactor $\hat{\alpha}$ in $Z^2(H/N; R^*)$. Any H -invariant lattice of $\hat{V}$, call it L , gives rise to a $K^\alpha H$ -module that extends the simple projective KN -module associated to e_χ , where $\alpha = \hat{\alpha} \bmod (\pi)$.*

Proof. Let $\hat{V}$ be as above, so $\hat{V}$ is an $F^{\hat{\alpha}}H$ -module such that $\text{Res}_N^H(\hat{V}) \simeq V$ as an FN -module. Now, by Lemma 4.2, take L to be an H -invariant lattice of $\hat{V}$; then consider its reduction to a $K^\alpha H$ -module, call it $\bar{L} := K \otimes_R L$. Note that L is also an N -invariant lattice of V , the unique simple ordinary module associated to the block e_χ . Thus, by Proposition 4.3, the reduction of L is a simple module of KN . Therefore, $\text{Res}_N^H(\bar{L})$ is isomorphic to the simple module associated to this block. One can conclude that $\bar{L}$ is a $K^\alpha H$ -module that extends the simple projective KN -module associated to e_χ , where $\alpha = \hat{\alpha} \bmod (\pi)$. $\square$

Proposition 4.5. *The α obtained from Theorem 4.4 is trivial.*

Proof. Note, by work of Stasinski and Vera-Gajardo, it was proven that any χ element of $\text{Irr}_F(N)$ extends to its stabilizer H (see [15, Proposition 4.5]). Thus, by Proposition 4.1, the cofactor associated to the extension of χ is trivial over F , i.e. $\hat{\alpha} = 1$. By Theorem 4.4, there is a $K^\alpha H$ -module $\bar{L}$ that extends the simple projective module associated to e_χ , where $\alpha = \hat{\alpha} \bmod (\pi) = 1$. Thus, α is trivial. $\square$

We recall the following result from Lemma 3.2 that the stabilizer H_2 and H'_2 of χ are isomorphic mod N , i.e. $H_2/N \simeq H'_2/N$. We will denote this quotient by $L := H_2/N \simeq H'_2/N$.

Theorem 4.6. *There exists an isomorphism of block algebras between eKH_2 and eKH'_2 , where $e = e_\chi$ for χ in $\text{Irr}_K(N)$ and H_2 and H'_2 are the stabilizer of e in G_2 and G'_2 respectively.*

Proof. We can apply Proposition 2.2 since e is H -stable block of defect zero of KN . Thus, we have an isomorphism of K -algebras

$$\Phi : eKH_2 \cong eKN \otimes_K KL \cong eKH'_2$$

since, by Proposition 4.5, α^{-1} is trivial. $\square$

5 The isomorphism of KG_2 and KG'_2

The following results about interior G -algebras will be useful in order to prove the isomorphism of those two group algebras. Note first that a G -algebra over a field K is a K -algebra A together with an action of G on A by K -algebra automorphisms. An interior G -algebra is a G -algebra where the action of G is given by inner automorphism induced from a group homomorphism $G \rightarrow A^*$, where A^* denotes the group of units of A . The examples to keep in mind are that KG and bKG are interior G -algebras with G acting by conjugation, where $b \in \text{Bl}(G)$.

Given H a subgroup of G and B an interior H -algebra, we define $\text{Ind}_H^G(B)$ to be the K -module $KG \otimes_{KH} B \otimes_{KH} KG$, and one can put an interior G -algebra structure on $\text{Ind}_H^G(B)$. For more details on the interior G -algebra structure on $\text{Ind}_H^G(B)$, one can consult Thévenaz's book on G -algebras and modular representation theory [16]. In fact, the following lemma shows that, as a K -algebra, one can think of $\text{Ind}_H^G(B)$ as a matrix algebra over B .

Lemma 5.1 ([16, Lemma 16.1]). *Let H be a subgroup of G of index n , and B an interior H -algebra. Then we have $\text{Ind}_H^G(B) \cong M_n(B)$ as K -algebras.*

In the following proposition, we will see that, given certain conditions, there is a way of relating the algebra obtained by the induction from H to G of a certain H -algebra with the algebra obtained by the idempotent $\text{Tr}_H^G(i)$, where i is an idempotent fixed by H . Given A an interior G -algebra, we will denote the set of elements of A fixed by H as A^H and 1_A as the multiplicative identity of A .

Proposition 5.2 ([16, Proposition 16.6]). *Let A be an interior G -algebra, and let H be a subgroup of G . Assume that there exists an idempotent $i \in A^H$ such that $1_A = \text{Tr}_H^G(i)$ and $i^g i = 0$ for all $g \in G - H$. Then there is an isomorphism of interior G -algebras*

$$F : \text{Ind}_H^G(iAi) \simeq A$$

given by $x \otimes b \otimes y \mapsto x \cdot b \cdot y$ ($x, y \in G$, $b \in iAi$).

We use Proposition 5.2 to prove the following.

Proposition 5.3. *Given a block b of KG_2 , there is a block $\hat{\Phi}(b)$ of KG'_2 such that the K -algebras bKG_2 and $\hat{\Phi}(b)KG'_2$ are isomorphic.*

Proof. Fix $\chi \in \text{Irr}(N)$ up to conjugation by G , and let $e = e_\chi$ be a primitive idempotent of KN ; now fix a block $b \in \text{Bl}(G_2|e)$ such that, by Clifford's theorem (Theorem 2.1), we have $b = \text{Tr}_{H_2}^{G_2}(d)$ for some primitive idempotent d of KH_2 , where H_2 is the stabilizer of e in G_2 . Note that, by Theorem 4.6, we have

that $\Phi : eKH_2 \cong eKH'_2$ as K -algebras. Now the map Φ gives a bijection between $\text{Bl}(H_2|e)$ and $\text{Bl}(H'_2|e)$ such that $\Phi : dKH_2 \cong \Phi(d)KH'_2$ for each $d \in \text{Bl}(H_2|e)$. Moreover, Theorem 2.1 tells us there is a bijection between $\text{Bl}(G|e)$ and $\text{Bl}(H|e)$. Thus, we obtain a bijection between $\text{Bl}(G_2|e)$ and $\text{Bl}(G'_2|e)$ by the map

$$b \mapsto \hat{\Phi}(b) = \text{Tr}_{H'_2}^{G'_2}(\Phi(d)).$$

Let $A = bKG_2$, so $1_A = b = \text{Tr}_{H_2}^{G_2}(d)$. Note that $d \cdot b = d$, so $d \in A^{H_2}$ and $d^g d = 0$ for $g \in G_2 - H_2$ (see [5, proof of Lemma 6.8.4]). Thus, we have that

$$dAd = dbKG_2d = dKG_2d = dKH_2d = dKH_2.$$

Thus, by Proposition 5.2, there is an isomorphism of interior G_2 -algebras:

$$\text{Ind}_{H_2}^{G_2}(dKH_2) \cong bKG_2.$$

Note by Lemma 5.1 that $\text{Ind}_{H_2}^{G_2}(dKH_2) \cong M_n(dKH_2)$ as K -algebras, with $n = |G_2 : H_2| = |G'_2 : H'_2|$. Since $\Phi : dKH_2 \cong \Phi(d)KH'_2$, we conclude that

$$\hat{\Phi} : bKG_2 \cong \text{Ind}_{H_2}^{G_2}(dKH_2) \cong \text{Ind}_{H'_2}^{G'_2}(\Phi(d)KH'_2) \cong \hat{\Phi}(b)KG'_2.$$

Thus, we have that bKG_2 and $\hat{\Phi}(b)KG'_2$ are isomorphic as K -algebras. $\square$

With the above K -algebra isomorphism, we define a map $\Psi : KG_2 \rightarrow KG'_2$ such that $\Psi(a) = \sum_{b \in \text{Bl}(G_2)} \hat{\Phi}(a \cdot b)$ for $a \in KG_2$ and show that this map is an isomorphism of K -algebras.

Theorem 5.4. *Let $G_2 = \mathbb{G}(\mathcal{O}_2)$ and $G'_2 = \mathbb{G}(\mathcal{O}'_2)$ be the group of points of any reductive group scheme $\mathbb{G}$ over $\mathbb{Z}$ such that p is very good for $\mathbb{G} \times \mathbb{F}_q$ or $\mathbb{G} = \text{GL}_n$. There exists an isomorphism of group algebra $K\mathbb{G}(\mathcal{O}_2) \cong K\mathbb{G}(\mathcal{O}'_2)$, where K is a sufficiently large field of characteristic different from p .*

Proof. Define the map $\Psi : KG_2 \rightarrow KG'_2$ by

$$\Psi(x) = \sum_{b \in \text{Bl}(G_2)} \hat{\Phi}(xb) \quad \text{for } x \in KG_2.$$

First, we show that Ψ is an algebra homomorphism. Note that, by definition, it is a linear map since each of the $\hat{\Phi}$ from Proposition 5.3 are. Now, given x and y both in KG_2 , we want to show that $\Psi(xy) = \Psi(x)\Psi(y)$. To prove this, it is sufficient to show that different blocks of G_2 are mapped to different blocks of G'_2 . Because if they are, we have that

$$\Psi(x)\Psi(y) = \sum_{b \in \text{Bl}(G_2)} \hat{\Phi}(xb) \cdot \sum_{b \in \text{Bl}(G_2)} \hat{\Phi}(yb) = \sum_{b \in \text{Bl}(G_2)} \hat{\Phi}(xb) \cdot \hat{\Phi}(yb)$$

since the products of elements of different blocks are zero. Moreover, by $\hat{\Phi}$ being a K -algebra homomorphism for each block b , we have

$$\begin{aligned}\Psi(x)\Psi(y) &= \sum_{b \in \text{BI}(G_2)} \hat{\Phi}(xb) \cdot \hat{\Phi}(yb) = \sum_{b \in \text{BI}(G_2)} \hat{\Phi}(xbyb) \\ &= \sum_{b \in \text{BI}(G_2)} \hat{\Phi}(xyb) = \Psi(xy).\end{aligned}$$

Now, to show that different blocks of G_2 are mapped to different blocks of G'_2 , we look at two cases. Let b and c be two different blocks of G_2 .

Case 1: Assume b and c cover the same block e of KN . Thus, by Clifford's theorem (Theorem 2.1), $b = \text{Tr}_{H_2}^{G_2}(d_1)$ and $c = \text{Tr}_{H_2}^{G_2}(d_2)$ such that $d_1 \cdot d_2 = 0$. Recall by Theorem 4.6 that the map $\Phi : eKH_2 \cong e\bar{K}H'_2$ is a K -algebra isomorphism, and thus $\Phi(d_1)\Phi(d_2) = 0$. Applying Theorem 2.1 for the blocks of KG'_2 over e , we have that

$$\text{Tr}_{H'_2}^{G'_2}(\Phi(d_1)) \neq \text{Tr}_{H'_2}^{G'_2}(\Phi(d_2))$$

since $\Phi(d_1) \neq \Phi(d_2)$. Thus, $\hat{\Phi}(b) \neq \hat{\Phi}(c)$.

Case 2: Assume b and c cover different blocks e_1 and e_2 of KN respectively. Note, to prove that $\hat{\Phi}(b) \neq \hat{\Phi}(c)$, it is enough to show that they cover different blocks of KN as blocks of G'_2 . Assume the opposite. By definition of the map $\hat{\Phi}$, we also have that $\hat{\Phi}(b)$ and $\hat{\Phi}(c)$ cover the blocks e_1 and e_2 of KN respectively. Thus, by [5, Proposition 6.8.2], we have that e_1 and e_2 are G'_2 -conjugated blocks. Since KN is a semisimple algebra, i.e. all the blocks have defect zero, e_1 and e_2 are defined by unique irreducible characters χ and τ of KN . Thus, e_1 and e_2 are G'_2 -conjugated blocks if and only if χ and τ are G'_2 -conjugated characters. If ${}^g\chi = \tau$ for some $g \in G'_2$, then for any $h \in G_2$ such that $h = \sigma(\bar{g})$, we have ${}^h\chi = \tau$, where the bars denote reductions to the residue field $\mathbb{F}_q$. Thus, if χ and τ are G'_2 -conjugated characters, then χ and τ are also G_2 -conjugated. Thus, e_1 and e_2 are G_2 -conjugated. This is a contradiction with the fact that b and c cover different blocks e_1 and e_2 of KN .

Therefore, we can conclude that different blocks of G_2 are mapped to different blocks of G'_2 . Hence, we have an algebra homomorphism from $\Psi: KG_2 \rightarrow KG'_2$, and since each $\hat{\Phi}$ is injective and maps different blocks to different blocks, we can conclude Ψ is an injective map. Thus, by dimension reasons, Ψ is also surjective. Therefore, $KG_2 \cong KG'_2$ as K -algebras. $\square$

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