The weight of nonstrongly complete profinite groups

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Abstract. We compute the local weight of the completion of a nonstrongly complete profinite group and conclude that, if a profinite group is abstractly isomorphic to its own profinite completion, then they are equal. The local weights of all the groups in the tower of completions are computed as well.

1 Introduction

We write $H \trianglelefteq_f G$ to indicate that H is a normal subgroup of finite index in G . The profinite completion of an abstract group G is the projective limit of finite quotients

$$\hat{G} = \varprojlim_{H \trianglelefteq_f G} G/H,$$

endowed with the limit topology. There is a canonical homomorphism $i: G \rightarrow \hat{G}$. In fact, the profinite completion is a universal object in the following sense: for every homomorphism $\varphi: G \rightarrow H$, where H is a profinite group, there is a unique continuous homomorphism $\hat{\varphi}: \hat{G} \rightarrow H$ such that $\varphi = \hat{\varphi} \circ i$. The completion $\hat{G}$ itself is obviously profinite, namely, it is an inverse limit of a directed system of finite groups.

We say that a group G is *residually finite* if $\bigcap_{H \trianglelefteq_f G} H = \{e\}$. The canonical homomorphism $i: G \rightarrow \hat{G}$ is injective if and only if G is residually finite.

Let G be a profinite group. Regarding G as an abstract group, it also has a profinite completion. Moreover, every profinite group is residually finite, so $i: G \rightarrow \hat{G}$ is injective. We say that a profinite group G is *strongly complete* (or *rigid* in some papers, such as [3]) if G is equal to its own profinite completion, i.e., if the natural embedding $i: G \rightarrow \hat{G}$ is an isomorphism. One easily sees that the following conditions are equivalent for a profinite group G .

- G is strongly complete.
- Every subgroup of finite index in G is open.

- Every normal subgroup of finite index in G is open.
- Every homomorphism $\varphi: G \rightarrow H$ onto a finite group is continuous.

Strongly complete profinite groups, or more generally, the connection between a profinite group and its profinite completion, have been the focus of many papers, such as [3–5, 10]. In [7], the authors proved the following useful result.

Proposition 1. *The following conditions are equivalent for a profinite group G .*

- G is strongly complete.
- For every n , there are finitely many subgroups of index n .
- There are at most countably many subgroups of finite index.

We prove the following.

Theorem 1. *Considered as an abstract group, a nonstrongly complete profinite group is never isomorphic to its profinite completion.*

In order to prove Theorem 1, we should be familiar with an invariant of profinite groups called the *local weight*.

Definition 1. Let G be an infinite profinite group. The *local weight* of G (sometimes referred to as the *rank*), which is denoted by $\omega_0(G)$, is the cardinality of the set of all open subgroups of G .

Let us recall some properties of the local weight.

Proposition 2 ([2, Chapter 17]). *Let G be an infinite profinite group.*

- $\omega_0(G)$ is equal to the cardinality of the set of all open normal subgroups of G .
- If A is a profinite group and $\varphi: G \rightarrow A$ a continuous epimorphism, then $\omega_0(A) \leq \omega_0(G)$.
- If $H \leq_c G$ is a closed subgroup, then $\omega_0(H) \leq \omega_0(G)$. In addition, if H is open, then $\omega_0(H) = \omega_0(G)$.

There is a strong connection between the number of finite index subgroups of G and the local weight of $\hat{G}$, which arises from the following proposition.

Proposition 3 ([6, Proposition 3.2.2]). *Let G be a residually finite group. One may identify G with its image in $\hat{G}$ and assume $G \subseteq \hat{G}$. Then there is a one-to-one correspondence*

$$\{O \mid O \leq_o \hat{G}\} \begin{matrix} \xrightarrow{\Phi} \\ \xleftarrow{\Psi} \end{matrix} \{N \mid N \leq_f G\}$$

defined by

$$\Phi(O) = O \cap G \quad \text{and} \quad \Psi(N) = \bar{N},$$

where $\bar{N}$ is the topological closure of N . Moreover, for every $N \leq_f G$, we have $G/N \cong \hat{G}/\bar{N}$.

Corollary 1. *Let G be a profinite group. Then $\omega_0(\hat{G})$ is the cardinality of the set of all finite index subgroups in G .*

The paper is organized as follows. In the introduction, we give some background on profinite groups and profinite completions. In Section 2, we prove Theorem 1. In Section 3, we present the infinite tower of profinite completions and compute the number of open subgroups in each level. The computation is based on a computation of the cardinality of a certain set of ultrafilters satisfying some conditions. In Section 4, we give a simpler proof for the abelian case.

2 The weight of a profinite group

A fundamental example of a nonstrongly complete profinite group is given in [6, Chapter 4].

Example 1. Let T be a finite group and I an infinite index set. Define $G = \prod_I T$. Obviously, G is a profinite group with the product topology. Every continuous projection onto a finite quotient splits through some $\prod_J T$, where J is a finite subset of I ; therefore, $\omega_0(G) = |I|$. To every nonprincipal ultrafilter $\mathcal{F}$ on I , one can associate the subgroup $H_{\mathcal{F}} = \{(t_i) \mid \{i \in I \mid t_i = 1\} \in \mathcal{F}\}$. It is easily checked that, for every $\mathcal{F}$, $H_{\mathcal{F}}$ is a dense subgroup which satisfies $G/H_{\mathcal{F}} \cong T$, where each $t \in T$ corresponds to its diagonal image $(t)_{i \in I}$. Moreover, by [8, Section 111], there are $2^{2^{|I|}}$ ultrafilters on I . So G has at least $2^{2^{\omega_0(G)}}$ subgroups of finite index. This is actually the accurate amount of finite index subgroups, since $|G| = |\prod_I T| = 2^{|I|} = 2^{\omega_0(G)}$, and the number of finite index subgroups cannot exceed the number of maps from G to a finite set, which is equal to $2^{|G|}$.

We are now ready for the main theorem.

Theorem 2. *Let G be a nonstrongly complete profinite group with*

$$\omega_0(G) = \mathbf{m} > \aleph_0.$$

Then $\omega_0(\hat{G}) = 2^{2^{\omega_0(G)}}$.

Proof. We first prove that $\omega_0(\hat{G}) \geq 2^{2^{\omega_0(G)}}$.

Since there are only countably many finite groups, there is some finite group K such that there are $\mathbf{m}$ distinct open subgroups $H \trianglelefteq_o G$ for which $G/H \cong K$. We choose K so that no proper quotient of K has this property. Let

$$\mathcal{M} = \{H \trianglelefteq_o G \mid G/H \cong K\} \quad \text{and} \quad M = \bigcap_{H \in \mathcal{M}} H.$$

Denote by $\ell(K)$ the length of a composition series of K . We prove the claim by induction on $\ell(K)$.

When $\ell(K) = 1$, K is a simple group. According to [6, Lemma 8.2.2],

$$G/M \cong \prod_{i \in I} K,$$

the product of $|I|$ copies of K for some index set I . Note that $\omega_0(G/M) \geq \mathbf{m}$ since the H_i/M for all $H_i \in \mathcal{M}$ are distinct open subgroups of G/M . On the other hand, by Proposition 2, $\omega_0(G/M) \leq \omega_0(G) = \mathbf{m}$, so $\omega_0(G/M) = \mathbf{m}$. Now, by the fundamental example, Example 1, $\omega_0(\prod_{i \in I} K) = |I|$, so $|I| = \mathbf{m}$. Thus, applying the fundamental example again, G/M has $2^{2^{\mathbf{m}}}$ subgroups of finite index. Taking their preimages, we get that G has $2^{2^{\mathbf{m}}}$ subgroups of finite index, so

$$\omega_0(\hat{G}) \geq 2^{2^{\mathbf{m}}}.$$

Assume that $\ell(K) = n > 1$ and that the claim holds for all profinite groups H which have a finite quotient A of shorter length than K , which appears $\omega_0(H)$ times. Choose a simple quotient S of K . Thus, S is a simple quotient of G as well. By minimality of K , there are less than $\mathbf{m}$ open normal subgroups $U \trianglelefteq_o G$ such that $G/U \cong S$. Hence, we can choose such a subgroup U that contains $\mathbf{m}$ different subgroups from $\mathcal{M} = \{H \trianglelefteq_o G \mid G/H \cong K\}$. This U is a profinite group, with $\omega_0(U) = \mathbf{m}$, by Proposition 2. In addition, there is some proper subgroup $K' \triangleleft K$ such that, for all $H \in \mathcal{M}$, $U/H \cong K'$. Note that $\ell(K') < \ell(K)$ since K' is a proper normal subgroup of K . Choose a quotient K'' of K' which is minimal with respect to the property that it can be realized as a quotient of U in $\mathbf{m}$ different ways. We get that $\ell(K'') \leq \ell(K') < \ell(K)$, so we can apply the induction hypothesis to the profinite group U . Thus, U has $2^{2^{\mathbf{m}}}$ finite index subgroups, and since U has finite index in G , each one of them is of finite index in G . In conclusion,

$$\omega_0(\hat{G}) \geq 2^{2^{\mathbf{m}}}.$$

We now prove that $\omega_0(\hat{G}) \leq 2^{2^{\omega_0(G)}}$. Denote $\omega_0(G) = \mathbf{m}$. By the explicit construction of the inverse limit, we have $G \subseteq \prod_{U \trianglelefteq_o G} G/U$. Since G has $\mathbf{m}$ open normal subgroups and, for each subgroup, the quotient is finite, we get $|G| \leq 2^{\mathbf{m}}$. So the number of finite index normal subgroups is equal to the number of epimorphisms from G to finite groups. Since there are countably many finite groups, we get that the number of such epimorphisms is at most $2^{|G|} \leq 2^{2^{\mathbf{m}}}$. $\square$

Theorem 1 is now a direct corollary of the above result.

Proof of Theorem 1. First, assume that $\omega_0(G) = \aleph_0$. By Proposition 1, if a profinite group has only countably many subgroups of finite index, then it is strongly complete. So G must have uncountably many subgroups of finite index. By Corollary 1, this cardinality is $\omega_0(\hat{G})$. Write $\omega_0(\hat{G}) = \mathbf{m}$. By Theorem 2, the cardinality of the set of subgroups of finite index in $\hat{G}$ is $2^{2^{\mathbf{m}}}$. We get that G and $\hat{G}$ have a different number of subgroups of finite index, so $G \not\cong \hat{G}$. Now assume $\omega_0(G) = \mathbf{m}$ and $\omega_0(G) > \aleph_0$. Then, by Theorem 2, $\omega_0(\hat{G}) = 2^{2^{\mathbf{m}}}$ and

$$\omega_0(\hat{G}) = 2^{2^{2^{\mathbf{m}}}}.$$

So, again, G and $\hat{G}$ have a different number of subgroups of finite index and thus cannot be isomorphic. $\square$

We also get a result about the connection between the cardinality of G and its local weight.

Proposition 4. *Let G be an infinite profinite group. Then $|G| = 2^{\omega_0(G)}$.*

Proof. Write $\omega_0(G) = \mathbf{m}$. We already saw that $|G| \leq 2^{\mathbf{m}}$ in the proof of Theorem 2. For the other direction, assume first that $\omega_0(G) = \mathbf{m} > \aleph_0$. As we showed in the proof of Theorem 2, G admits a subgroup U that projects onto $\prod_{i \in I} H$, where H is some finite group and I is a set of indices of cardinality $\mathbf{m}$. Thus, $|G| \geq |U| \geq |\prod_{i \in I} H| = 2^{\mathbf{m}}$.

Now assume that $\omega_0(G) = \aleph_0$. Hence, by [6, Corollary 2.6.6], we have that $G = \lim_{\leftarrow n \in \omega} G_n$ is the inverse limit of an inverse system of finite groups indexed by ω such that all the maps in the system are proper projections. Thus, for every n and every $g \in G_n$, g admits at least two preimages in G_{n+1} . So $|G| \geq 2^{\aleph_0}$. $\square$

3 The tower of profinite completions

Now we would like to generalize this result for the whole tower of completions. Let G be a nonstrongly complete profinite group. In [1], we studied the following construction. We define an ascending chain over the ordinals by setting $G_0 = G$.

- For a successor ordinal $\alpha = \beta + 1$, we have $G_\alpha = \widehat{G_\beta}$ with the natural map $\varphi_{\beta\alpha}: G_\beta \rightarrow G_\alpha$ and such that, for every $\gamma < \beta$, $\varphi_{\gamma\alpha} = \varphi_{\beta\alpha} \circ \varphi_{\gamma\beta}$.
- For a limit ordinal α , the system of maps $\{\varphi_{\gamma\beta}\}_{\gamma < \beta < \alpha}$ is compatible, so we can define $H_\alpha = \lim_{\rightarrow} \{G_\beta, \varphi_{\gamma\beta}\}$ with the natural maps $\varphi'_{\beta\alpha}: G_\beta \rightarrow H_\alpha$, and then $G_\alpha = \widehat{H_\alpha}$ with the natural map $\psi_\alpha: H_\alpha \rightarrow G_\alpha$; we also define, for every $\beta < \alpha$, $\varphi_{\beta\alpha} = \psi_\alpha \circ \varphi'_{\beta\alpha}$.

We proved that the chain never terminates and, moreover, that all the maps are injective. So we may assume that, for every $\beta < \alpha$, $G_\beta < G_\alpha$. Now we would like to calculate $\omega_0(G_\alpha)$ for all α . Theorem 2 gives the computation for the successor case.

Remark 1. For every ordinal α , $\omega_0(G_{\alpha+1}) = 2^{2^{\omega_0(G_\alpha)}}$.

We are left with finding $\omega_0(G_\alpha)$, where α is a limit ordinal. Recall that, by Proposition 3, $\omega_0(G_\alpha)$ is the number of subgroups of finite index in H_α .

Lemma 1. Let α be a limit ordinal. Then

$$\omega_0(G_\alpha) \leq \prod_{\beta < \alpha} \omega_0(G_\beta).$$

Proof. For any group A , we denote by $FI(A)$ the set of finite index normal subgroups in A . As we pointed out, if α is a limit ordinal, then $\omega_0(G_\alpha) = |FI(H_\alpha)|$. Now consider the map $\theta: FI(H_\alpha) \rightarrow \prod_{\beta < \alpha} FI(G_\beta)$ defined by

$$\theta(U) = \prod_{\beta < \alpha} (U \cap G_\beta).$$

This map is obviously one-to-one, so

$$\begin{aligned} \omega_0(G_\alpha) &\leq \prod_{\beta < \alpha} |FI(G_\beta)| = \prod_{\beta < \alpha} 2^{2^{\omega_0(G_\beta)}} \\ &= \prod_{\beta < \alpha} \omega_0(G_{\beta+1}) = \prod_{\beta < \alpha} \omega_0(G_\beta). \end{aligned}$$

The second equality is due to Corollary 1 and Theorem 2. □

Now we are going to prove that, for every nonstrongly complete profinite group,

$$\omega_0(G_\alpha) = \prod_{\beta < \alpha} \omega_0(G_\beta).$$

We will do so in several steps.

Lemma 2. Let G be a nonstrongly complete profinite group and $U \leq_o G$. Then U is nonstrongly complete too, and $\omega_0(G_\alpha) = \omega_0(U_\alpha)$ for every ordinal α .

Proof. First we show that U is nonstrongly complete. Let $H \leq_f G$ be a finite-index subgroup that is not open. Then $H \cap U \leq_f U$. In addition, $U \cap H$ is not open in G since otherwise $\text{int}(H) \neq \emptyset$ implies that H is open. Consequently, since U is open in G , $H \cap U$ is not open in U . In order to prove the lemma, by Proposition 2, it is enough to prove that, for all α , $U_\alpha \leq_o G_\alpha$. We prove this by transfinite induction. Successor ordinal: Assume that $U_\alpha \leq_o G_\alpha$. Then $U_{\alpha+1} = \widehat{U_\alpha}$, $G_{\alpha+1} = \widehat{G_\alpha}$. By [6, Proposition 3.2.2],

$$\overline{U_\alpha} \leq_o G_{\alpha+1} \quad \text{and} \quad [G_\alpha : U_\alpha] = [G_{\alpha+1} : \overline{U_\alpha}].$$

By [6, Lemma 3.2.4], $\overline{U_\alpha} = \hat{i}[U_\alpha]$, where $\hat{i}$ is the lifting of the inclusion map $i: U_\alpha \rightarrow G_\alpha$. Finally, note that, since by the induction hypothesis $U_\alpha \leq_o G_\alpha$, then every finite-index subgroup of U_α is also a finite-index subgroup of G_α , so the profinite topology of G_α induces on U_α its full profinite topology, and thus, by [6, Lemma 3.6.2], $\hat{i}[U_\alpha] = \widehat{U_\alpha}$. In conclusion,

$$U_{\alpha+1} \leq_o G_{\alpha+1} \quad \text{and} \quad [G_\alpha : U_\alpha] = [G_{\alpha+1} : U_{\alpha+1}].$$

Limit ordinal: Now assume that β is a limit ordinal such that, for every $\gamma < \alpha < \beta$, $U_\alpha \leq_o G_\alpha$ and, in addition, $[G_\alpha : U_\alpha] = [G_\gamma : U_\gamma]$. Then $\lim_{\rightarrow \alpha < \beta} U_\alpha$ is a subgroup of $\lim_{\rightarrow \alpha < \beta} G_\alpha$ of the same index, and thus, by [6, Proposition 3.2.2],

$$U_\beta = \widehat{\lim_{\rightarrow \alpha < \beta} U_\alpha} \leq_o \widehat{\lim_{\rightarrow \alpha < \beta} G_\alpha} = G_\beta$$

of the same index. □

Remark 2. Let G be a nonstrongly complete profinite group. In order to compute $\omega_0(G_\alpha)$ for a limit ordinal α , we may assume G has a quotient of the form $\prod_{\omega_0(G)} S$ for some finite simple group S .

Proof. By the proof of Theorem 2, every nonstrongly complete profinite group of $\omega_0(G) = \mathbf{m} > \aleph_0$ has an open subgroup U with a quotient of the form $\prod_{\mathbf{m}} S$, where S is some finite simple group. By Lemma 2, we can replace G by U . Note that, in order to compute $\omega_0(G_\alpha)$, where α is a limit ordinal, we may assume that the tower begins with $\hat{G}$. Thus, in the case $\omega_0(G) = \aleph_0$, we can replace G by $\hat{G}$ and get that $\omega_0(G) = \mathbf{m} > \aleph_0$. □

Proposition 5. Let $\phi: G \rightarrow H$ be an epimorphism of nonstrongly complete profinite groups. Then, for every ordinal α , there are epimorphisms $\phi_\alpha: G_\alpha \rightarrow H_\alpha$ which are compatible with $\varphi_{\alpha\beta}$.

Proof. We will prove this by transfinite induction. Let α be an ordinal such that, for each $\beta < \alpha$, the claim holds. First case: $\alpha = \gamma + 1$. Then there is an epimorphism $\phi_\gamma: G_\gamma \rightarrow H_\gamma$. By [6, Proposition 3.2.5], there is an epimorphism

$$\hat{\phi}: G_{\gamma+1} = \widehat{G_\gamma} \rightarrow \widehat{H_\gamma} = H_{\gamma+1}$$

which makes the following diagram commutative:

$$\begin{array}{ccc} G_{\gamma+1} = \widehat{G_\gamma} & \xrightarrow{\hat{\phi}_\gamma} & \widehat{H_\gamma} = H_{\gamma+1} \\ \uparrow & & \uparrow \\ G_\gamma & \xrightarrow{\phi_\gamma} & H_\gamma \end{array}$$

Define $\phi_{\gamma+1} = \hat{\phi}_\gamma$. Second case: α is a limit ordinal. We can define ϕ_α to be the direct limit $\lim_{\rightarrow \beta < \alpha} \phi_\beta$. $\square$

Corollary 2. *Let $G \rightarrow H$ be an epimorphism of nonstrongly complete profinite groups. Then, for every ordinal α , $\omega_0(G_\alpha) \geq \omega_0(H_\alpha)$.*

Proof. By Proposition 2, it is enough to prove that H_α is a quotient of G_α . So, by Proposition 5, we are done. $\square$

Corollary 3. *It is enough to prove the computation for nonstrongly complete profinite groups of the form $\prod_{\mathbf{m}} S$ for a finite simple group S .*

Proof. By Remark 2, we may assume G has a quotient of the form $H = \prod_{\mathbf{m}} S$, where $\mathbf{m} = \omega_0(G)$. Recall that, by Example 1, $\omega_0(H) = \mathbf{m}$. Assume we have proved that, for every limit ordinal α ,

$$\omega_0(H_\alpha) = \prod_{\beta < \alpha} \omega_0(H_\beta).$$

We will prove by transfinite induction that, under this assumption, for every ordinal α , $\omega_0(G_\alpha) = \omega_0(H_\alpha)$.

- $\alpha = 0$: $\omega_0(H) = \mathbf{m} = \omega_0(G)$.
- $\alpha = \beta + 1$: $\omega_0(H_{\beta+1}) = 2^{2^{\omega_0(H_\beta)}} = 2^{2^{\omega_0(G_\beta)}} = \omega_0(G_{\beta+1})$.
- α is a limit ordinal: By Proposition 5, $\omega_0(G_\alpha) \geq \omega_0(H_\alpha)$. But, by the assumption, $\omega_0(H_\alpha) = \prod_{\beta < \alpha} \omega_0(H_\beta)$. By induction hypothesis, $\omega_0(H_\beta) = \omega_0(G_\beta)$ for all $\beta < \alpha$. So

$$\omega_0(G_\alpha) \geq \omega_0(H_\alpha) = \prod_{\beta < \alpha} \omega_0(H_\beta) = \prod_{\beta < \alpha} \omega_0(G_\beta) \geq \omega_0(G_\alpha). \quad \square$$

In order to prove our claim, we need some propositions about ultrafilters and the subgroups they define.

Proposition 6 ([9, Theorem 7.6]). *Let X be a set of cardinality $\mathbf{m}$. Denote by $\mathcal{A}$ the set of all subsets $Y \subseteq X$ for which $|Y^c| < \mathbf{m}$. It has the finite intersection property and thus is contained in a filter. The number of ultrafilters containing $\mathcal{A}$ is equal to $2^{2^{\mathbf{m}}}$.*

Let I be a set. We denote the set of all ultrafilters on I by βI .

Lemma 3. *Let S be a finite simple group, I an infinite set of cardinality $\mathbf{m}$, and let $\mathcal{C}$ be the set of all subgroups of $\prod_{\mathbf{m}} S$, of the form $H_{\mathcal{F}}$ for $\mathcal{F} \in \beta I$ as defined in Example 1. Then $\mathcal{C}$ forms a subbase for a topology on G , contained in the profinite topology. Denote the completion of G with respect to this topology by $G\hat{\mathcal{C}}$. Then $G\hat{\mathcal{C}} \cong \prod_{\beta I} S$ is the product of $2^{2^{\mathbf{m}}}$ copies of S .*

Proof. By results of [8, Example 111], there are $2^{2^{\mathbf{m}}}$ nonprincipal ultrafilters on a set of cardinality $\mathbf{m}$. The only thing we need to show is that, for finitely many ultrafilters $\mathcal{F}_1, \dots, \mathcal{F}_n$, $G/(H_{\mathcal{F}_1} \cap \dots \cap H_{\mathcal{F}_n}) \cong \prod_n S$. We prove this by induction on n . Assume the claim holds for n , and consider $G/(H_{\mathcal{F}_1} \cap \dots \cap H_{\mathcal{F}_{n+1}})$. Clearly, the natural morphism

$$G/(H_{\mathcal{F}_1} \cap \dots \cap H_{\mathcal{F}_n}) \rightarrow G/(H_{\mathcal{F}_1} \cap \dots \cap H_{\mathcal{F}_n}) \times G/H_{\mathcal{F}_{n+1}} \cong \left(\prod_n S \right) \times S$$

is injective. So the only thing left to show is that it is onto. Since $G/H_{\mathcal{F}_{n+1}}$ is simple, it is enough to show that $H_{\mathcal{F}_1} \cap \dots \cap H_{\mathcal{F}_{n+1}}$ is a proper subset of $H_{\mathcal{F}_1} \cap \dots \cap H_{\mathcal{F}_n}$. Indeed, since $\mathcal{F}_{n+1}$ is different from $\mathcal{F}_1, \dots, \mathcal{F}_n$, for every $i \in \{1, \dots, n\}$, there is some $B_i \subseteq I$ such that $B_i \in \mathcal{F}_i$ and $B_i^c \in \mathcal{F}_{n+1}$. So

$$B = B_1 \cup \dots \cup B_n \in \mathcal{F}_1 \cap \dots \cap \mathcal{F}_n,$$

while $B^c \in \mathcal{F}_{n+1}$. Hence, taking an element $g \in G$ which satisfies $g_i = e$ (for all $i \in B$) and $g_j \neq e$ (for all $j \in B^c$), we get that $g \in H_{\mathcal{F}_1} \cap \dots \cap H_{\mathcal{F}_n}$, while $g \notin H_{\mathcal{F}_{n+1}}$. In conclusion, since the epimorphisms between the quotient groups are precisely the natural epimorphisms $\prod_{I'} S \rightarrow \prod_{J'} S$, where $J' \subset I'$, we get by [6, Exercise 1.1.4] that $G\hat{\mathcal{C}} \cong \prod_{2^{2^{\mathbf{m}}}} S$. $\square$

Proposition 7. *Let X be a set of cardinality $2^{\mathbf{n}}$, and let $Y \subseteq P(X)$ be a set of cardinality $\mathbf{n}$, closed under finite intersections, such that, for each $B \in Y$, $|B| = 2^{\mathbf{n}}$. Then there are $2^{2^{\mathbf{n}}}$ ultrafilters $\mathcal{F}$ on X consisting of sets of cardinality $2^{\mathbf{n}}$ which contains Y .*

Proof. We construct a set of subsets of X , $\{C_i, D_i\}_{i < 2^n}$ such that, for every $i < 2^n$, $C_i \cap D_i = \emptyset$ and, for every disjoint nonempty finite subsets $I, J \subseteq 2^n$,

$$\left(\bigcap_{i \in I} C_i\right) \cap \left(\bigcap_{j \in J} D_j\right) \cap B$$

is of cardinality 2^n for every $B \in Y$. Let us look at the following set of quarters (B, I, J, α) such that $B \in Y$, I and J are disjoint nonempty finite subsets of 2^n , and $\alpha < 2^n$ is an ordinal. Since this set is of cardinality 2^n , we can index it by $\beta < 2^n$. We construct the required subsets by recursion. We build subsets $\{C_{i,\beta}, D_{i,\beta}\}_{i < 2^n, \beta < 2^n}$ and then define for all $i < 2^n$,

$$C_i = \bigcup_{\beta < 2^n} C_{i,\beta} \quad \text{and} \quad D_i = \bigcup_{\beta < 2^n} D_{i,\beta}.$$

Suppose $C_{i,\gamma}$ and $D_{i,\gamma}$ were defined for every $\gamma < \beta$, and their cardinality is less than or equal to γ , and only less than 2^n of them are not empty. Now look at the quarter (B, I, J, α) corresponding to the ordinal β . Choose

$$x \in B \setminus \left(\left(\bigcup_{i < 2^n, \gamma < \beta} C_{i,\gamma} \right) \cup \left(\bigcup_{i < 2^n, \gamma < \beta} D_{i,\gamma} \right) \right).$$

Define

$$C_{i,\beta} = \begin{cases} (\bigcup_{\gamma < \beta} C_{i,\gamma}) \cup \{x\} & \text{if } i \in I, \\ \bigcup_{\gamma < \beta} C_{i,\gamma} & \text{otherwise.} \end{cases}$$

Likewise, define

$$D_{i,\beta} = \begin{cases} (\bigcup_{\gamma < \beta} D_{i,\gamma}) \cup \{x\} & \text{if } i \in J, \\ \bigcup_{\gamma < \beta} D_{i,\gamma} & \text{otherwise.} \end{cases}$$

Note that, since the cardinality of each one of the subsets $C_{i,\gamma}, D_{i,\gamma}$ for all $\gamma < \beta$ is less than or equal to γ , then $(\bigcup_{i < 2^n, \gamma < \beta} C_{i,\gamma}) \cup (\bigcup_{i < 2^n, \gamma < \beta} D_{i,\gamma})$ is of cardinality less than β which in turn is less than 2^n . Thus, using the fact that $|B| = 2^n$, it is possible to find such an element $x \in B$. In addition, it is easy to see that $C_{i,\beta}$ and $D_{i,\beta}$ satisfy the properties in the recursion hypothesis. Define $C_i = \bigcup_{\beta < 2^n} C_{i,\beta}$ and $D_i = \bigcup_{\beta < 2^n} D_{i,\beta}$. By the construction, they satisfy the required properties.

Now denote by $\mathcal{A}$ the set of all subsets $B \subseteq X$ for which $|B^c| < 2^n$. For each $I \subseteq 2^n$ (not necessarily finite), define the following sets:

$$A_I = Y \cup \{C_i\}_{i \in I} \cup \{D_j\}_{j \notin I} \cup \mathcal{A}.$$

Since all the finite intersections of subsets from Y , $\{C_i\}_{i \in I}$ and $\{D_j\}_{j \notin I}$ are of cardinality 2^n , then so are the finite intersections with elements from $\mathcal{A}$. Thus, for

all $I \subseteq 2^n$, A_I is contained in some ultrafilter. In addition, if $I \neq J$, then without loss of generality, there is some $i \in I \setminus J$. Thus, $C_i \in A_I$ and $D_i \in A_J$ are disjoint, so A_I and A_J cannot be contained in the same ultrafilter. In conclusion, there are 2^{2^n} different ultrafilters containing Y , each of them consisting of subsets of maximal cardinality. $\square$

Theorem 3. *Let S be a finite simple group, and let $G = \prod_{\mathbf{m}} S$. Denote by $\{G_\alpha\}_\alpha$ the infinite tower of profinite completions of G . Then there exists an infinite chain of profinite groups $\{K_\alpha\}_\alpha$, with compatible morphisms $\epsilon_{\beta\alpha}: K_\beta \rightarrow K_\alpha$ for all $\beta < \alpha$, which satisfies the following conditions.*

- (i) $K_0 = G$.
- (ii) For every ordinal β , $K_{\beta+1} = \prod_{2^{2^{\omega_0(K_\beta)}}} S$
- (iii) For every limit ordinal α , $K_\alpha = \prod_{\mathbf{n}} S$, where $\mathbf{n} = \prod_{\beta < \alpha} \omega_0(K_\beta)$.
- (iv) For every ordinal α , there are epimorphisms $\eta_\alpha: G_\alpha \rightarrow K_\alpha$ which are compatible with the homomorphisms $\varphi_{\beta\alpha}: G_\beta \rightarrow G_\alpha$ and $\epsilon_{\beta\alpha}: K_\beta \rightarrow K_\alpha$.
- (v) Every K_α admits a set of normal subgroups $\mathbb{A}_\alpha$ such that
 - $|\mathbb{A}_0| = 2^{2^{\mathbf{m}}}$,
 - for every $N \in \mathbb{A}_\alpha$, N is a subgroup of the form $H_{\mathcal{F}}$, where $\mathcal{F}$ is an ultrafilter on $\omega_0(K_\alpha)$, all of whose elements have maximal cardinality,
 - for every $N \in \mathbb{A}_\alpha$, $K_\alpha/N \cong S$,
 - for every $\beta < \alpha$ and $N \in \mathbb{A}_\alpha$, $N \cap K_\beta \in \mathbb{A}_\beta$,
 - for every ordinal α and $N \in \mathbb{A}_\alpha$, there are $2^{\omega_0(K_{\alpha+1})}$ subgroups N' in $\mathbb{A}_{\alpha+1}$ which satisfy $N' \cap K_\alpha = N$,
 - for every limit ordinal α and every series $(N_\beta)_{\beta < \alpha}$ of normal subgroups, $N_\beta \in \mathbb{A}_\beta$ such that $N_\beta \cap K_\gamma = N_\gamma$ if $\gamma < \beta$, there are $\prod_{\beta < \alpha} \omega_0(K_\beta)$ subgroups $N_\alpha \in \mathbb{A}_\alpha$ whose intersection with each K_β , $\beta < \alpha$, is equal to N_β .

Proof. Let α be an ordinal, and assume that, for all $\beta < \alpha$, such groups exist. First case: $\alpha = 0$. Define $\mathbb{A}_0$ to be set of all subgroups of the form $H_{\mathcal{F}}$ for $\mathcal{F}$ an ultrafilter on $\mathbf{m}$, all of whose elements have cardinality $\mathbf{m}$. By Proposition 6, $|\mathbb{A}_0| = 2^{2^{\mathbf{m}}}$.

Second case: $\alpha = \beta + 1$ for some ordinal β . By the induction hypothesis, we have $K_\beta \cong \prod_{\mathbf{n}} S$ for some cardinal $\mathbf{n}$. Let $\mathcal{C}$ be the set of all subgroups of the form $H_{\mathcal{F}}$, where $\mathcal{F}$ is an ultrafilter on $\mathbf{n} = \omega_0(K_\beta)$. Define $K_{\beta+1} = K_\beta \cdot \mathcal{C}$. Then,

by Lemma 3, $K_{\beta+1} \cong \prod_{2^{2^n}} S$. By definition of the completion with respect to some directed system of subgroups, there is a natural homomorphism

$$\epsilon_{\beta,\alpha}: K_\beta \rightarrow K_{\beta,\hat{\mathcal{C}}} = K_\alpha.$$

Define $\epsilon_{\gamma,\alpha}$ for all $\gamma < \beta$ to be $\epsilon_{\beta,\alpha} \circ \epsilon_{\gamma,\beta}$.

Since $\mathcal{C}$ is a subtopology of the profinite topology, then there is a natural epimorphism $\xi_\beta: \widehat{K_\beta} \rightarrow K_{\beta,\hat{\mathcal{C}}}$ which commutes with the natural homomorphisms $K_\beta \rightarrow \widehat{K_\beta}$, $K_\beta \rightarrow K_{\beta,\hat{\mathcal{C}}}$. By the induction hypothesis, there is a natural epimorphism $\eta: G_\beta \rightarrow K_\beta$. By [6, Proposition 3.2.5], it can be lifted to a compatible epimorphism $\widehat{\eta}_\beta: \widehat{G_\beta} \rightarrow \widehat{K_\beta}$. So define $\eta_{\beta+1}$ to be $\xi_\beta \circ \widehat{\eta}_\beta$. Let $N \in \mathcal{A}_\beta$. By the induction hypothesis, N has the form $H_{\mathcal{F}}$ for some ultrafilter on $\mathbf{n}$ such that all of its elements have cardinality $\mathbf{n}$. Recall that, for each ultrafilter $\mathcal{F}$, there is a given isomorphism $\prod_{\mathbf{n}} S / H_{\mathcal{F}} \cong S$. We use this to identify the quotient with S . Use this isomorphisms to identify all the quotient groups with the same group. For all $g \in K_\beta$, define A_g to be the set of all ultrafilters $\mathcal{F}'$ for which the image of g in the natural epimorphism $\prod_{\mathbf{n}} S \rightarrow \prod_{\mathbf{n}} S / H_{\mathcal{F}'}$ is equal to the image of g in the natural epimorphism $\prod_{\mathbf{n}} S \rightarrow \prod_{\mathbf{n}} S / H_{\mathcal{F}}$. There is a set of indices $A \in \mathcal{F}$ determining the image of g in $\prod_{\mathbf{n}} S / H_{\mathcal{F}}$, so in fact $A_g = \{\mathcal{F}' \mid A \in \mathcal{F}'\}$. Note that, since $\mathcal{F}$ contains only sets of maximal cardinality, $|A| = \mathbf{n}$, so the number of ultrafilters containing A is equal to the number of ultrafilters on A , which is 2^{2^n} . In addition, for each $g_1 \neq g_2 \in K_\beta$, there is some $g_3 \in K_\beta$ for which $A_{g_1} \cap A_{g_2} = A_{g_3}$. Indeed, let A and B be the subsets determining the images of g_1 and g_2 correspondingly. Then $A_{g_1} \cap A_{g_2}$ is equal to the set of all ultrafilters containing $A \cap B$. So define g_3 to be e on $A \cap B$ and x on $(A \cap B)^c$ for some $e \neq x \in S$.

Let $\beta\mathbf{n}$ denote the set of all ultrafilters on $\mathbf{n}$. Then the set of the subsets of the form A_g for $g \in K_\beta$ is a set of 2^n subsets of the maximal cardinality, closed under finite intersections. Thus, by Proposition 7, there are $2^{\omega_0(K_\alpha)}$ ultrafilters containing all these sets, which consist of subsets of maximal cardinality. Denote this collection by $\mathbb{B}_{\mathcal{F},\alpha}$. For each $\mathcal{G} \in \mathbb{B}_{\mathcal{F},\alpha}$, the subgroup $H_{\mathcal{G}}$ of K_α has the property that its intersection with K_β is equal to $H_{\mathcal{F}}$ and $K_\alpha / H_{\mathcal{G}} \cong S$. Define $\mathbb{B}_\alpha = \bigcup_{\{\mathcal{F} \mid H_{\mathcal{F}} \in \mathbb{A}_\beta\}} \mathbb{B}_{\mathcal{F},\alpha}$ and $\mathbb{A}_\alpha = \{H_{\mathcal{F}} \mid \mathcal{F} \in \mathbb{B}_\alpha\}$.

Third case: Let α be a limit ordinal, and define K'_α to be the direct limit of $\{K_\beta, \epsilon_{\gamma,\beta}\}$. Denote by $\mathcal{C}_\beta$ the set of all subgroups N of the form $\lim_{\rightarrow} N_\beta$, where $\{N_\beta\}_{\beta < \alpha}$ is a compatible system of normal subgroups as described in condition 6. For every $\beta < \alpha$ and N_β in the system, $K_\beta / N_\beta \cong S$, and those isomorphisms are compatible, so we get that $K'_\alpha / N \cong S$. In addition, if we take $N_1, \dots, N_l$ different subgroups of this form, then they are direct limits of different systems $N_i = \lim_{\rightarrow} N_{i,\beta}$. So there is some β from which, for all $\beta < \gamma < \alpha$, the subgroups

$N_{i,\gamma}$ are different. Thus, for every $\beta < \gamma < \alpha$, $K_\gamma/N_{1,\gamma} \cap \cdots \cap N_{l,\gamma} \cong \prod_l S$. Hence, $K'_\alpha/N_1 \cap \cdots \cap N_l \cong \prod_l S$. Define K_α to be the completion of K'_α with respect to this set of subgroups. Then $K_\alpha = \prod_{\mathbf{n}} S$ when $\mathbf{n} = \prod_{\beta < \alpha} \omega_0(K_\beta)$. Define $\epsilon_{\beta,\alpha}$ to be the compositions of the natural maps to the direct limit, and from the direct limit to its completion,

$$\epsilon_{\beta,\alpha}: K_\beta \rightarrow \lim_{\rightarrow \beta < \alpha} K_\beta \rightarrow \widehat{\lim_{\rightarrow \beta < \alpha} K_\beta} = K_\alpha,$$

and define η_α to be the lifting of $\lim_{\rightarrow \beta < \alpha} \eta_\beta$ to the corresponding completions.

$$\begin{array}{ccc} G_\alpha = \widehat{\lim_{\rightarrow \beta < \alpha} G_\beta} & \xrightarrow{\eta_\alpha = \widehat{\lim_{\rightarrow \beta < \alpha} \eta_\beta}} & \widehat{\lim_{\rightarrow \beta < \alpha} K_\beta} = K_\alpha \\ \uparrow & & \uparrow \\ \lim_{\rightarrow \beta < \alpha} G_\beta & \xrightarrow{\lim_{\rightarrow \beta < \alpha} \eta_\beta} & \lim_{\rightarrow \beta < \alpha} K_\beta \\ \uparrow & & \uparrow \\ \{G_\beta\}_{\beta < \alpha} & \xrightarrow{\{\eta_\beta\}_{\beta < \alpha}} & \{K_\beta\}_{\beta < \alpha}. \end{array}$$

For each $g \in K'_\alpha$ and a subgroup N of this form, let A_g be the set of all such subgroups N' such that the image of g in K'_α/N' is equal to its image in K'_α/N . Since g is an element in the direct limit, it comes from some K_β . In addition, N is the direct limit of a compatible chain $\{N_\beta\}_{\beta < \alpha}$. Thus, $N' = \lim_{\rightarrow \beta < \alpha} N'_\beta$ satisfies the condition if and only if $N'_{\beta+1}$ is a lifting of N_β in $K_{\beta+1}$. So the number of such subgroups is equal to $\omega_0(K_\alpha)$. In addition, by the same argument, we can show that the set of all A_g is closed under finite intersections. Thus, in a similar way to the successor case, we can show there are $2^{\omega_0(K_\alpha)}$ subgroups extending any such $N \trianglelefteq K'_\alpha$ and satisfying all the conditions. Define $\mathbb{A}_\alpha$ to be the set of all such extensions. $\square$

Theorem 4. *Let G be a nonstrongly complete profinite group. Then, for every limit ordinal $\alpha \neq 0$, $\omega_0(G_\alpha) = \prod_{\beta < \alpha} \omega_0(G_\beta)$.*

Proof. By Corollary 3, it is enough to prove the claim for $G = \prod_{\mathbf{m}} S$. By Theorem 3, there is a series of subgroups K_α as described in the theorem. We will prove by transfinite induction that $\omega_0(G_\alpha) = \omega_0(K_\alpha)$. So assume this equality holds for every $\beta < \alpha$. For $\alpha = 0$, the claim is clear. For $\alpha = \beta + 1$, $\omega_0(G_\alpha) = 2^{2^{\omega_0(G_\beta)}}$. Since $\mathcal{C}$ has $2^{2^{\omega_0(K_\beta)}}$ subgroups, we have that

$$\omega_0(K_\alpha) = 2^{2^{\omega_0(K_\beta)}} = 2^{2^{\omega_0(G_\beta)}} = \omega_0(G_\alpha).$$

For α a limit ordinal, we already know that $\omega_0(G_\alpha) \leq \prod_{\beta < \alpha} \omega_0(G_\beta)$. By Proposition 5, we get that $\omega_0(G_\alpha) \geq \omega_0(K_\alpha)$. Since K_α is defined as the completion of K'_α with respect to the set $\mathcal{C}$ of subgroups, $\omega_0(K_\alpha) = |\mathcal{C}|$. By the construction of $\mathcal{C}$,

$$\begin{aligned} |\mathcal{C}| &= 2^{2^m} \prod_{0 \neq \beta < \alpha} 2^{\omega_0(K_\beta)} = 2^{2^m} \prod_{0 \neq \beta < \alpha} 2^{\omega_0(G_\beta)} = \prod_{\beta < \alpha} 2^{2^{\omega_0(G_\beta)}} \\ &= \prod_{\beta < \alpha} \omega_0(G_{\beta+1}) = \prod_{\beta < \alpha} \omega_0(G_\beta). \end{aligned} \quad \square$$

In fact, we can calculate $\prod_{\beta < \alpha} \omega_0(G_\beta)$ explicitly.

Lemma 4 ([9, Lemma 5.9]). *If λ is an infinite cardinal and $\langle \kappa_\beta : \beta < \lambda \rangle$ is a non-decreasing sequence of nonzero cardinals, then*

$$\prod_{\beta < \lambda} \kappa_\beta = \sup_{\beta} \{\kappa_\beta\}^\lambda.$$

Definition 2. Let κ be a cardinal. We say that κ is a strong limit cardinal if, for all $\lambda < \kappa$, $2^\lambda < \kappa$.

In our case, the limit ordinal α is not necessarily a cardinal, but we may replace it by its cofinality. So one may take $\lambda = \text{cof}(\alpha)$, and $\kappa_\beta = \omega_0(G_{\psi(\beta)})$, where $\psi: \lambda \rightarrow \alpha$ is an order-preserving cofinal map. Note that $\sup_{\beta} \kappa_\beta$ is a strong limit cardinal with cofinality $\text{cof}(\alpha) = \lambda$. Thus, we can use the following proposition.

Proposition 8 ([9, p. 58, equation 5.23]). *Let κ be a strong limit cardinal. Then $\kappa^{\text{cof}(\kappa)} = 2^\kappa$.*

Corollary 4. *We have that, for a limit ordinal α ,*

$$\omega_0(G_\alpha) = \prod_{\beta < \alpha} \omega_0(G_\beta) = 2^{\sup_{\beta < \alpha} \{\omega_0(G_\beta)\}}.$$

Remark 3. Let G and G' be two nonstrongly complete profinite groups. Then there exists some ordinal α such that, for all $\beta \geq \alpha$, $\omega_0(G_\beta) = \omega_0(G'_\beta)$.

Proof. Write $\mathbf{m}_0 = \omega_0(G)$ and $\mathbf{n}_0 = \omega_0(G')$. It is enough to prove that there exists α such that $\omega_0(G_\alpha) = \omega_0(G'_\alpha)$, since each successor level depends only on the previous level, and any limit level does not depend on initial segments. Assume $\mathbf{n}_0 > \mathbf{m}_0$. Since the series of local weights is a strongly ascending series over the ordinals, it is not bounded. So denote by α_0 the minimal ordinal for

which $\omega_0(G_{\alpha_0}) \geq \mathbf{n}_0$. Now continue by recursion. Assume α_n is defined. Define α_{n+1} to be the minimal ordinal for which $\omega_0(G_{\alpha_{n+1}}) \geq \omega_0(G'_{\alpha_n})$. Eventually, $\alpha = \sup\{\alpha_n\}_{n \in \omega}$ is the desired ordinal. $\square$

4 The abelian case

If G is a nonstrongly complete profinite abelian group, then we can offer a simpler derivation for the local weight of G_α , where α is a limit ordinal.

By Remark 2, we may assume G has a quotient of the form $\prod_{\omega_0(G)} S$ for some finite simple group. Since G is abelian, then so is S .

In conclusion, there is a finite simple abelian group S onto which G projects continuously in $\omega_0(G)$ different ways.

Lemma 5. *Let G be a nonstrongly complete profinite group and S a finite simple group which appears as a quotient of G in $\mathbf{n}$ different ways for some infinite cardinality $\mathbf{n}$. Then G has 2^{2^n} dense normal subgroups H_i such that $G/H_i \cong S$.*

Proof. By the proof of Theorem 2, since G has $\mathbf{n}$ projections onto S , then G projects onto $\prod_{\mathbf{n}} S$. By Example 1, $\prod_{\mathbf{n}} S$ has 2^{2^n} noncontinuous epimorphisms on S . Compose them with the epimorphism from G to $\prod_{\mathbf{n}} S$ to get 2^{2^n} noncontinuous epimorphisms from G to S . For any such epimorphism, the kernel is a subgroup H of finite index which is not open, and thus not closed, and for which $G/H \cong S$. Thus, there are no normal subgroups between H and G . But, since H is normal, so is $\overline{H}$. So $\overline{H} = G$. That is, H is dense. $\square$

Lemma 6. *Let G be a nonstrongly complete profinite group. Then G is a semi-directed component of $\hat{G}$.*

Proof. Look at the identity map $G \rightarrow G$. By the universal property of profinite completion, we get the following diagram:

$$\begin{array}{ccc} \hat{G} & & \\ i \uparrow & \searrow \hat{id} & \\ G & \xrightarrow{id} & G. \end{array}$$

Thus, $\hat{G} \cong G \ltimes K$ for $K = \ker \hat{id}$. $\square$

Remark 4. If G is abelian then so are all its finite quotients, and thus $\hat{G}$ is abelian as an inverse limit of abelian groups.

Corollary 5. *If G is abelian, then $\hat{G} \cong G \times K$ for $K = \ker \hat{id}$.*

Remark 5. Let G be a nonstrongly complete profinite group. Write $K = \ker \hat{id}$. Then, since K is the kernel of the natural epimorphism

$$\lim_{\leftarrow U \trianglelefteq_f G} G/U \rightarrow \lim_{\leftarrow U \trianglelefteq_o G} G/U$$

and since, for every $U \trianglelefteq_f G$, the closure $\overline{U}$ satisfies $\overline{U} \trianglelefteq_o G$, we get that

$$K = \lim_{\leftarrow U \trianglelefteq G} \overline{U}/U.$$

From now on, we write K for $\ker(\hat{id}: \hat{G} \rightarrow G)$, and K_α for $\ker(\hat{id}: \hat{G}_\alpha \rightarrow G_\alpha)$.

Corollary 6. *If H is a normal dense subgroup of finite index of G , then K projects naturally on G/H . Moreover, if $H_1 \neq H_2$, the natural projections $K \rightarrow G/H_1$, $K \rightarrow G/H_2$ are different.*

Proof. Being the inverse limit of the directed system of finite groups

$$\{\overline{U}/U\}_{U \trianglelefteq_f G},$$

every group in the system corresponds to a continuous quotient of K . So each dense normal subgroup H of G corresponds to a continuous epimorphism

$$K \rightarrow \overline{H}/H = G/H. \quad \square$$

Proposition 9. *Let $\mathfrak{n}$ be an infinite cardinality, and let G be a nonstrongly complete profinite abelian group, which projects onto some finite simple group S in $\mathfrak{n}$ different ways, for which the kernel is a dense subgroup. Then each projection $G \rightarrow S$ has $2^{2^\mathfrak{n}}$ different liftings to $\hat{G}$.*

Proof. By Corollary 6, K has $\mathfrak{n}$ different projections onto S . So, by Lemma 5, K has $2^{2^\mathfrak{n}}$ abstract epimorphisms $\psi_i: K \rightarrow S$.

Let $f: G \rightarrow S$ be some projection. Recall that, since G is abelian, then so is S . By Corollary 5, $\hat{G} = G \times K$. Since S is abelian, we can define epimorphisms $(f, \psi_i): G \times K \rightarrow S$, which clearly extend f . $\square$

Proposition 10. *Let G be nonstrongly complete profinite abelian group which has $\omega_0(G)$ projections onto some finite simple group S . Consider $\{G_\alpha\}$ to be the chain of profinite completions as described in the beginning of the previous section. Then*

$$\omega_0(G_\alpha) = \prod_{\beta < \gamma < \alpha} 2^{\omega_0(G_\gamma)} \quad \text{for all limit ordinals } \alpha.$$

For each α , there is a set $\mathbb{A}_\alpha$ of not necessarily open subgroups H_i which satisfy the following conditions.

- (i) For each $H_i \in \mathbb{A}_\alpha$, $G_\alpha/H_i \cong S$
- (ii) For each $\beta < \alpha$ and $H_i \in \mathbb{A}_\alpha$, $H_i \cap G_\beta \in \mathbb{A}_\beta$.
- (iii) For each limit ordinal α , all the subgroups in $\mathbb{A}_\alpha$ are open.
- (iv) For each $\beta < \alpha$ and $H \in \mathbb{A}_\beta$, there are $\mathbf{n}_\alpha$ subgroups $H' \in \mathbb{A}_\alpha$ for which $H' \cap G_\beta = H$, where

$$\mathbf{n}_\alpha = \begin{cases} 2^{2^{\omega_0(G_\alpha)}}, & \alpha \text{ is a successor ordinal,} \\ \omega_0(G_\alpha), & \alpha \text{ is a limit ordinal.} \end{cases}$$

- (v) For each α , $|\mathbb{A}_\alpha| = \mathbf{n}_\alpha$.

Proof. We prove the claim by transfinite induction.

For $\alpha = 0$, take $\mathbb{A}_0 = \{\ker f_i \mid f_i: G \rightarrow S \text{ is an epimorphism}\}$. By assumption, G has $\omega_0(G)$ different continuous projections onto S , so by Lemma 5,

$$|\mathbb{A}_0| = 2^{2^{\omega_0(G)}}.$$

Let $\alpha = \beta + 1$, and assume $\mathbb{A}_\beta$ is defined. By induction hypothesis,

$$|\mathbb{A}_\beta| = 2^{2^{\omega_0(G_\beta)}} \text{ or } \omega_0(G_\beta).$$

In the first case, since the number of open subgroups of G_β is equal to $\omega_0(G_\beta)$, by definition, then $2^{2^{\omega_0(G_\beta)}}$ of the subgroups in $\mathbb{A}_\beta$ are not open, and thus, since S is simple, they are dense. In the second case, we can use Lemma 5 to conclude that G_β has $2^{2^{\omega_0(G_\beta)}}$ dense subgroups H for which $G/H \cong S$. Thus, by Proposition 9, each subgroup $H \in \mathbb{A}_\beta$ has a set $\mathbb{A}_{\beta+1,H}$ of cardinality

$$2^{2^{2^{\omega_0(G_\beta)}}} = 2^{2^{\omega_0(G_\alpha)}}$$

consisting of normal subgroups $H' \trianglelefteq G_\alpha$ for which $G/H' \cong S$ and $H' \cap G_\beta = H$. Define $\mathbb{A}_\alpha = \bigcup_{H \in \mathbb{A}_\beta} \mathbb{A}_{\beta+1,H}$.

Let α be a limit ordinal. For each series $(H_\beta)_{\beta < \alpha}$ of normal subgroups from $\mathbb{A}_\beta$ such that, for every $\gamma < \beta$, $H_\beta \cap G_\gamma = H_\gamma$, the union is a normal subgroup $H \trianglelefteq \lim_{\rightarrow \beta < \alpha} G_\beta$ for which the quotient is isomorphic to S and the intersection with G_β is equal to H_β . By induction hypothesis, there are $\prod_{\beta < \gamma < \alpha} 2^{2^{\omega_0(G_\gamma)}}$ such series. Define $\mathbb{A}_\alpha$ to be the set of closures of all these subgroups in

$$\widehat{\lim_{\rightarrow \beta < \alpha} G_\beta} = G_\alpha.$$

We get that G_α has at least $\lim_{\rightarrow \beta < \alpha} G_\beta$ open subgroups. On the other hand, by Lemma 1, $\omega_0(G_\beta) \leq \prod_{\gamma < \beta} \omega_0(G_\gamma)$, so $\omega_0(G_\beta) = \prod_{\gamma < \beta} \omega_0(G_\gamma)$. $\square$

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