On the structure of some locally nilpotent groups without contranormal subgroups

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Abstract. Following J. S. Rose, a subgroup H of a group G is said to be contranormal in G if $G = H^G$. It is well known that a finite group is nilpotent if and only if it has no proper contranormal subgroups. We study nilpotent-by-Černikov groups with no proper contranormal subgroups. Furthermore, we study the structure of groups with a finite proper contranormal subgroup.

1 Introduction

Let G be a group. A subgroup H of G is said to be *contranormal in G* if $H^G = G$, where $H^G = \langle x^{-1}hx \mid h \in H, x \in G \rangle$ is the normal closure of H in G , the smallest normal subgroup of G containing H . For example, G is contranormal in G for any group G . The term “contranormal subgroup” has been introduced by J. S. Rose in the papers [18, 19]. Contranormal subgroups have been studied for example in the papers [15, 16]. If G is a group and H is a contranormal subgroup of G , then every subgroup K containing H is contranormal in G . In particular, if H and L are contranormal subgroups of G , then the subgroup $\langle H, L \rangle$ is also contranormal in G . However, the intersection of two contranormal subgroups is not always contranormal. For example, in the group A_4 , every Sylow 3-subgroup is contranormal, but the intersection of every two Sylow 3-subgroups of A_4 is trivial, so that it is not contranormal. Notice also that if M is a maximal subgroup of G which is not normal, then clearly M is a contranormal subgroup of G . Moreover, every subgroup of a finite group G is a contranormal subgroup of a subnormal subgroup of G . As we can see by the definition, contranormal subgroups are, in a certain sense, antipode of normal and subnormal subgroups: a contranormal subgroup H of a group G is normal (respectively subnormal) if and only if $H = G$. It follows

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that groups whose subgroups are subnormal (in particular, nilpotent groups) do not contain proper contranormal subgroups. For finite groups, the converse is true.

A finite group G is nilpotent if and only if G does not have proper contranormal subgroups.

Indeed, if G is a finite group and G does not have contranormal subgroups, then every maximal subgroup of G is normal in G ; hence G is nilpotent.

On the other hand, there exist infinite non-nilpotent groups whose subgroups are subnormal (it is possible to find examples of such groups in the survey [3]). Other examples of non-nilpotent groups with no proper contranormal subgroups are non-nilpotent Sylow-nilpotent groups, where a periodic group is said to be *Sylow-nilpotent* if G is locally nilpotent and every Sylow p -subgroup of G is nilpotent for every prime p . In fact, let $G = \text{Dr}_{p \in \Pi(G)} S_p$, where S_p is a Sylow p -subgroup of G , for every $p \in \Pi(G)$. If H is a proper subgroup of G , then $H = \text{Dr}_{p \in \Pi(G)} V_p$, where V_p is a Sylow p -subgroup of H . Since $H < G$, there exists a prime $q \in \Pi(G)$ such that $V_q < S_q$. Then $V_q^G < S_q$ since S_q is nilpotent. Then $\text{Dr}_{p \in \Pi(G)} V_p^G$ is a proper normal subgroup of G containing H . Therefore, H is not contranormal in G .

Important examples of contranormal subgroups are abnormal subgroups, where a subgroup H is said to be *abnormal* in G if $g \in \langle H, H^g \rangle$ for every element $g \in G$. However, these two types of subgroups have different influence on the structure of a group G . For, an infinite locally nilpotent group does not contain proper abnormal subgroups, while there are locally nilpotent Chernikov groups having proper contranormal subgroups, as the following example shows.

Let D be a divisible abelian 2-group. Then D has an automorphism φ such that $\varphi(d) = d^{-1}$ for each element $d \in D$. Define the semidirect product $G = D \rtimes \langle b \rangle$ such that $d^b = \varphi(d) = d^{-1}$ for each element $d \in D$. Let a be an arbitrary element of D . Since D is divisible, there exists an element $d \in D$ such that $d^2 = a$. We have

$$[b, d] = b^{-1}d^{-1}bd = d^2 = a.$$

It follows that $[b, D] = D$. From $[b, D] \leq \langle b \rangle^G$ and $\langle b \rangle \leq \langle b \rangle^G$, we obtain that $\langle b \rangle^G = \langle b \rangle[b, D] = \langle b \rangle D = G$ so that the subgroup $\langle b \rangle$ is contranormal in G . We note that the group G is not nilpotent; however, the series

$$\langle 1 \rangle \leq \Omega_1(D) \leq \cdots \leq \Omega_n(D) \leq \Omega_{n+1}(D) \leq \cdots \leq D \leq G$$

is central so that G is a hypercentral abelian-by-finite group. Moreover, the contranormal subgroup $\langle b \rangle$ is ascendant. This group is abelian-by-finite; thus there exist hypercentral abelian-by-finite groups having proper contranormal subgroups, in particular finite contranormal subgroups.

Another example can be constructed in the following way. Let D be a Prüfer 2-group and α a non-trivial automorphism of D of infinite order; then the semi-direct product $G = D \rtimes \langle \alpha \rangle$ is a hypercentral group and $\langle \alpha \rangle$ is contranormal in G . As in the previous example, the subgroup $\langle \alpha \rangle$ is ascendant in G .

These examples motivate the following question.

What can we say about groups having no proper contranormal subgroups?

Groups without proper contranormal subgroups have been studied in the papers [14, 15, 17, 20]. In particular, a Chernikov group without proper contranormal subgroups is nilpotent; moreover, a locally finite group without proper contranormal subgroups, whose Sylow p -subgroups are Chernikov, is Sylow-nilpotent [17], soluble-by-finite groups without proper contranormal subgroups, having finite 0-rank and trivial periodic part are nilpotent [15], nilpotent-by-finite groups without proper contranormal subgroups are nilpotent [20].

In this paper, we continue the investigation of groups without proper contranormal subgroups. In Section 2, we study some relationships between this class of groups and the class of locally nilpotent groups. We prove the following result.

Proposition A. *Let G be a group, and let L, K be normal subgroups of G such that L is periodic and locally nilpotent, $L \leq K$ and K/L is finite. Suppose that G does not contain proper contranormal subgroups. Then K is locally nilpotent.*

Proposition A has many interesting corollaries. If G is a periodic group, H is a normal locally nilpotent subgroup of G and G has no proper contranormal subgroups, and if either G/H is nilpotent or G/H is a Chernikov group, then G is locally nilpotent. Another interesting consequence is that if H is a normal nilpotent subgroup of a group G with G/H nilpotent and $\Pi(H) \cap \Pi(G/H) = \emptyset$ and if G has no proper contranormal subgroups, then G is nilpotent. This result has been proved in [20] under the additional hypothesis that G/H is Chernikov.

Abelian-by-finite groups without proper contranormal subgroups are nilpotent by the result in [20]. The next natural step is to consider abelian-by-Chernikov groups. There exists a p -group H with a normal elementary abelian p -subgroup A such that H/A is a Prüfer p -group, every proper subgroup of H is subnormal in H , and $\zeta(H) = \langle 1 \rangle$ (see [9]). In the paper [20], the author studied when a nilpotent-by-Chernikov group without proper contranormal subgroups is nilpotent.

In this paper, we continue this investigation. Generalizing the main result of the paper [20], we prove the following result.

Theorem B. *Let G be a group, and let A be a normal nilpotent subgroup of G such that G/A is a Chernikov π -group. Assume that A has a finite series of G -invariant*

subgroups

$$A = A_0 \geq A_1 \geq \cdots \geq A_j \geq A_{j+1} \geq \cdots \geq A_t = \langle 1 \rangle$$

such that every factor A_j/A_{j+1} , $0 \leq j \leq t-1$, satisfies one of the following conditions:

- A_j/A_{j+1} is abelian, torsion-free and G -rationally irreducible;
- A_j/A_{j+1} is a periodic π' -group;
- A_j/A_{j+1} is a Chernikov π -group.

Suppose that G does not contain proper contranormal subgroups. Then G is nilpotent.

As we proved before, there exist hypercentral abelian-by-finite groups with a proper contranormal subgroup and also with a finite contranormal subgroup. Therefore, we can ask about the structure of locally nilpotent abelian-by-finite groups having proper contranormal subgroups. We show here the following result.

Proposition C. *Let G be a locally nilpotent group and A a normal abelian subgroup of G with G/A finite. Suppose that G has a proper contranormal subgroup C . Then $C = BK$, where $B \leq A$ is normal in G , K is a finitely generated subgroup such that $G = AK$, and $A = B[K, A]$. In particular, the factor group G/B has the finite contranormal subgroup KB/B .*

Thus it is natural to ask about the structure of locally nilpotent groups with a finite contranormal subgroup. Our final result is a description of hypercentral groups which contain a finite contranormal subgroup.

Theorem D. *Let G be a hypercentral group. If G contains a finite contranormal subgroup, then G satisfies the following conditions:*

- (i) $G = VC$, where V is a normal divisible abelian subgroup and C is a finite contranormal subgroup of G ;
- (ii) $\Pi(G) = \Pi(C)$; in particular, the set $\Pi(G)$ is finite;
- (iii) V has a family of G -invariant G -quasifinite subgroups $\{D_\mu \mid \mu \in M\}$ such that $V = \langle D_\mu \mid \mu \in M \rangle$;
- (iv) $[D_\mu, C] = D_\mu$ for all $\mu \in M$; in particular, $[V, C] = V$.

Here an infinite normal abelian subgroup A of a group G is called G -quasifinite if every proper G -invariant subgroup of A is finite.

2 Periodic groups without proper contranormal subgroups

We start our investigation with this easy and very useful lemma.

Lemma 2.1. *Let G be a group.*

- (i) *If C is a contranormal subgroup of G and H is a normal subgroup of G , then CH/H is a contranormal subgroup of G/H .*
- (ii) *If H is a normal subgroup of G and C is a subgroup of G such that $H \leq C$ and C/H is a contranormal subgroup of G/H , then C is a contranormal subgroup of G .*
- (iii) *If C is a contranormal subgroup of G and D is a contranormal subgroup of C , then D is a contranormal subgroup of G .*

Proof. These assertions are obvious. □

Now we prove Proposition A.

Proof of Proposition A. First suppose that K/L is a minimal normal subgroup of G/L . Then either

$$(K/L) \cap C_{G/L}(K/L) = \langle 1 \rangle \quad \text{or} \quad (K/L) \cap C_{G/L}(K/L) = K/L.$$

Assume that the first holds. As K/L is finite, the factor group $(G/L)/C_{G/L}(K/L)$ is finite. This factor group does not contain proper contranormal subgroups by Lemma 2.1. But a finite group which does not contain proper contranormal subgroups is nilpotent; thus K/L is abelian. Then $K/L \leq C_{G/L}(K/L)$, and we obtain a contradiction. Then $(K/L) \cap C_{G/L}(K/L) = K/L$, and K/L is a finite abelian p -subgroup. Suppose that K is not locally nilpotent. Then K is a not a p -subgroup; hence L is a not a p -group. Let P be the Sylow p -subgroup of L . Then P is G -invariant, and we have $K/P = (L/P) \rtimes S/P$ (see, for example, [5, Theorem 2.4.5]), where S/P is a finite Sylow p -subgroup of K/P . If S/P is G -invariant, then S is a G -invariant Sylow p -subgroup of K . Then S is locally nilpotent, and $K = LS$ is locally nilpotent since it is a product of two normal locally nilpotent subgroups. Therefore, S/P is not G -invariant. Then $N_{G/P}(S/P)$ is a proper subgroup of G/P . Let g be an element of G . Since K is normal in G , $(S/P)^{gP} \leq K/P$; moreover, $(S/P)^{gP}$ is a Sylow p -subgroup of K/P . Let $U/P = \langle S/P, (S/P)^{gP} \rangle$. Clearly, K is locally finite, so the subgroup U/P is finite. Therefore, the subgroups S/P and $(S/P)^{gP}$ are conjugate in U/P . Thus the subgroup S/P is pronormal in G/P . But, in this case, the subgroup $N_{G/P}(S/P)$ is abnormal in G/P (see, for example, [10, Proposition 2.6]). Since every proper

abnormal subgroup is contranormal, we obtain that $N_{G/P}(S/P)$ is contranormal in G/P , a contradiction since, by Lemma 2.1, G/P has no proper contranormal subgroups. This contradiction proves that K is locally nilpotent.

Suppose now that K/L is not a minimal normal subgroup of G/L . Then there exists a series

$$L = L_0 \leq L_1 \leq \cdots \leq L_j \leq L_{j+1} \leq \cdots \leq L_n = K$$

of G -invariant subgroups such that L_{j+1}/L_j is a minimal normal subgroup of G/L_j , $0 \leq j \leq n-1$. Then, by induction, K is locally nilpotent, as required. $\square$

Recall that a group G is called hyperfinite if G has an ascending series of normal subgroups whose factors are finite. More generally, let G be a group and H a normal subgroup of G ; we say that H is G -hyperfinite if H has an ascending series of G -invariant subgroups whose factors are finite.

Proposition A has many interesting corollaries.

Corollary 2.2. *Let G be a group, and let H, K be normal subgroups of G such that H is periodic and locally nilpotent, $H \leq K$, and K/H is G -hyperfinite. Suppose that G does not contain proper contranormal subgroups. Then K is locally nilpotent. In particular, if G/H is hyperfinite, then G is locally nilpotent.*

Proof. Denote by L the locally nilpotent radical of K , and assume that $L < K$. Since K/L is hyperfinite, there exists a non-trivial finite G -invariant subgroup F/L of K/L . Then F is locally nilpotent by Proposition A; thus $F \leq L < F$. This contradiction proves the result. $\square$

Corollary 2.3. *Let G be a periodic group, and let H be a normal locally nilpotent subgroup of G such that G/H is nilpotent. Suppose that G does not contain proper contranormal subgroups. Then G is locally nilpotent.*

Proof. Every periodic nilpotent group is hyperfinite, and we can apply Corollary 2.2. $\square$

Corollary 2.4. *Let G be a periodic group, and let H be a normal nilpotent subgroup of G such that G/H is nilpotent and $\pi(H) \cap \pi(G/H) = \emptyset$. Suppose that G does not contain proper contranormal subgroups. Then G is nilpotent.*

Proof. G is locally nilpotent by Corollary 2.3. H is the Sylow $\pi(H)$ -subgroup of G ; hence $G = H \times S$, where S is the Sylow $\pi(G/H)$ -subgroup of G . Therefore, $S \simeq G/H$ implies that S is nilpotent, and G is nilpotent, as required. $\square$

Corollary 2.5. *Let G be a periodic group, and let H be a normal locally nilpotent subgroup such that G/H is a Chernikov group. Suppose that G does not contain proper contranormal subgroups. Then G is locally nilpotent.*

Corollary 2.6. *Let G be a locally finite group, and let H be a normal locally nilpotent subgroup such that the Sylow p -subgroups of G/H are Chernikov groups, for every prime p . Suppose that G does not contain proper contranormal subgroups. Then G is locally nilpotent.*

Proof. A locally finite group without proper contranormal subgroups, whose Sylow p -subgroups are Chernikov groups, is Sylow-nilpotent (see, for example, [17, Corollary C3]). In particular, it is hyperfinite, and we can apply Corollary 2.2. $\square$

Corollary 2.7. *Let G be a hyperfinite group. Suppose that G does not contain proper contranormal subgroups. Then G is hypercentral.*

Proof. G is locally nilpotent by Corollary 2.3. Then G is hypercentral since it is hyperfinite. $\square$

3 Nilpotent-by-finite groups without proper contranormal subgroups

In this section, we prove Theorem B. In the proof, the following two lemmas will be used.

Lemma 3.1. *Let G be a group, and let A be a normal abelian p -subgroup of G such that G/A is nilpotent and $G/C_G(A)$ is periodic. Suppose that G does not contain proper contranormal subgroups. Then $G/C_G(A)$ is a p -group.*

Proof. Assume that the factor group $G/C_G(A)$ is not a p -group. Then we have

$$G/C_G(A) = P/C_G(A) \times Q/C_G(A),$$

where $P/C_G(A)$ is a p -group and $Q/C_G(A)$ is a non-trivial p' -group. Then the intersection $Q/C_G(A) \cap \zeta(G/C_G(A))$ is non-trivial. Let $g \in Q/C_G(A)$ be a non-trivial element of $Q/C_G(A) \cap \zeta(G/C_G(A))$. Then $A = C_A(g) \times [g, A]$ (see, for example, [1, Proposition 2.12]), where $[g, A]$ is non-trivial since $g \notin C_G(A)$. Since $g \in \zeta(G/C_G(A))$, the subgroup $C_A(g)$ is G -invariant; then we can consider the factor group $G/C_A(g)$. Therefore, without loss of generality, we may assume that $C_A(g)$ is trivial. Thus $A = [g, A]$ and $C_A(g) = \langle 1 \rangle$. Hence the subgroup A has a complement S in G (see, for example, [6, Theorem 8.2.7]), and

$G = A \rtimes S$. Hence S is a proper subgroup of G since A is non-trivial. Then, from $A = [g, A]$, we obtain $A = [S, A]$. Obviously, $[S, A] \leq S^G$, and therefore $G = AS \leq S^G$. Thus S is a proper contranormal subgroup of G . By this contradiction, $G/C_G(A)$ is a p -group. $\square$

A torsion-free abelian normal subgroup R of a group G is G -rationally irreducible if R/S is periodic for every non-trivial G -invariant subgroup S of R .

Lemma 3.2. *Let G be a group and A a normal nilpotent subgroup of G such that G/A is a Chernikov group. Assume that B and C are G -invariant subgroups of A with $B \leq C$, C/B abelian, torsion-free and G -rationally irreducible, G/C nilpotent. Suppose that G does not contain proper contranormal subgroups. Then $C/B \leq \zeta(G/B)$.*

Proof. Assume $B = \langle 1 \rangle$ without loss of generality. First notice that $A \leq C_G(C)$. In fact, $C \cap \zeta(A)$ is a non-trivial G -invariant subgroup of C ; thus $C/(C \cap \zeta(A))$ is periodic. Let $c \in C, a \in A$; then $[c^s, a] = 1$ for a suitable positive integer s ; then $1 = [c^s, a] = [c, a]^s$ since C is abelian and G -invariant; hence $[c, a] = 1$ since C is torsion-free. That holds for each $a \in A$, so $C \leq \zeta(A)$ and $A \leq C_G(C)$.

Suppose $G/C_G(C) \neq \langle 1 \rangle$. Let $e \in C \setminus \langle 1 \rangle$, and write $E = \langle e \rangle^G$. Then the factor group C/E is periodic. There exists a prime q such that $E \neq E^q = K$ and $q \notin \pi(G/A)$ (see, for example, [11, Theorem 1.15]). Then the subgroup K is G -invariant, the factor group C/K is periodic and its Sylow q -subgroup Q/K is non-trivial. We have $C/K = Q/K \times R/K$, where R/K is the Sylow q' -subgroup of C/K . By Lemma 3.1, $G/C_G(C/R)$ is a q -group. But $A \leq C_G(C/R)$ and G/A is a q' -group. Thus $G = C_G(C/R)$. Then the subgroup $L = [G, C] \leq R$, and $L \neq C$. Obviously, the subgroup $L = [G, C]$ is G -invariant and non-trivial. The inclusion $C/L \leq \zeta(G/L)$ and the nilpotence of G/C imply that G/L is nilpotent. Since C is G -rationally irreducible, the factor C/L is periodic. Choose in L a non-trivial element v , and put $V = \langle v \rangle^G$. Then the factor C/V is periodic, and L/V is also periodic. Suppose that the set $\Pi(L/V)$ contains a prime p such that $p \notin \Pi(G/A)$. Let S/V be the Sylow p' -subgroup of L/V ; then L/S is a non-trivial p -factor of G . Using again Lemma 3.1, we obtain that $G = C_G(L/S)$. Let $c \in C, g \in G$. Then $(cS)^{g^S} = cSuS$ for some element $uS \in L/S$. Suppose that $uS \neq S$. We have $g^t \in A \leq C_G(C/S)$ for some p' number t . It follows that $(uS)^t = S$. On the other hand, uS is a p -element, and we obtain a contradiction. This contradiction shows that $(cS)^{g^S} = cS$ so that the factor C/S is G -central, and we obtain a contradiction. Suppose now that $\Pi(L/V) \subseteq \Pi(G/A)$. In this case, we can find a prime r such that $V \neq V^r = U$ and $r \notin \Pi(G/A)$ (see, for example, [11, Theorem 1.15]). Then the factor L/U is periodic, and the set $\Pi(L/U)$

contains a prime r such that $r \notin \pi(G/A)$. Now we can repeat the above arguments, and again, we obtain a contradiction. This contradiction shows that $[G, C]$ is trivial. $\square$

Lemma 3.2 has the following corollary.

Corollary 3.3. *Let G be a group, and let A be a normal nilpotent subgroup of G such that G/A is a Chernikov group. Suppose that A contains G -invariant subgroups B and C , $B \leq C$, such that C/B is abelian, torsion-free and has finite 0-rank, and G/C is nilpotent. Suppose that G does not contain proper contranormal subgroups. Then the factor C/B is G -central.*

Proof. Since C/B has finite 0-rank, there exists a finite series

$$B = C_0 \leq C_1 \leq \cdots \leq C_j \leq C_{j+1} \leq \cdots \leq C_n = C$$

of G -invariant subgroups whose factors C_{j+1}/C_j are torsion-free and G -rationally irreducible, $0 \leq j \leq n-1$. Using Lemma 3.2, we obtain that the factor C_n/C_{n-1} is G -central. Since G/C is nilpotent, G/C_{n-1} is also nilpotent. Therefore, we can apply Lemma 3.2 to the factor C_{n-1}/C_{n-2} , and we get that this factor is G -central. By induction, we obtain that every factor C_{j+1}/C_j is G -central, $0 \leq j \leq n-1$. Then G/B is nilpotent, and $G/C_G(C/B)$ periodic implies that C/B is G -central, as required. $\square$

The next result is an important step in the task of establishing the nilpotence of a nilpotent-by-Chernikov group without proper contranormal subgroups.

Proposition 3.4. *Let G be a group, and let A be a normal abelian subgroup of G such that G/A is a Chernikov π -group. Suppose that A has a finite series of G -invariant subgroups*

$$A = A_0 \geq A_1 \geq \cdots \geq A_j \geq A_{j+1} \geq \cdots \geq A_t = \langle 1 \rangle$$

such that every factor A_j/A_{j+1} , $0 \leq j \leq t-1$, satisfies one of the following conditions:

- A_j/A_{j+1} is torsion-free and G -rationally irreducible;
- A_j/A_{j+1} is a periodic π' -group;
- A_j/A_{j+1} is a Chernikov π -group.

Suppose that G does not contain proper contranormal subgroups. Then G is nilpotent.

Proof. G/A is nilpotent (see [17, Lemma 4.9]). Consider the factor group G/A_1 . If A/A_1 is torsion-free and G -rationally irreducible, then Lemma 3.2 implies that it is G -central. Hence G/A_1 is nilpotent. If the factor A/A_1 is a periodic π' -group, then Corollary 2.4 implies that G/A_1 is nilpotent (see also [20, Theorem 1]). Finally, suppose that A/A_1 is a Chernikov π -group. Then G/A_1 is a Chernikov group, and therefore it is nilpotent (see [17, Lemma 4.9]). Suppose that the factor group G/A_j is nilpotent, and consider the factor G/A_{j+1} . If A_j/A_{j+1} is torsion-free and G -rationally irreducible, then Lemma 3.2 implies that this factor is G -central. Hence G/A_{j+1} is nilpotent. If A_j/A_{j+1} is a periodic π' -group, then Lemma 3.1 implies that it is G -central; thus G/A_{j+1} is nilpotent. Finally, assume that A_j/A_{j+1} is a Chernikov π -group. Let D/A be the divisible part of G/A . Since A_j/A_{j+1} is union of G -invariant finite subgroups, then $D \leq C_G(A_j/A_{j+1})$. Since G/A_j is nilpotent, we get that D/A_{j+1} is nilpotent. Then G/A_{j+1} is nilpotent-by-finite, and thus it is nilpotent (see [20, Theorem 1]). For $j = t - 1$, we obtain the result. $\square$

Now we can prove Theorem B.

Proof of Theorem B. Let K be a nilpotent normal subgroup of G such that G/K is a Chernikov group. Suppose that G/K is a π -group and that K has a finite series as in the hypothesis. We argue by induction on t .

First assume that $t = 1$. In this case, K is either a torsion-free abelian G -rationally irreducible group, or a periodic π' -group, or a Chernikov π -group. Write $S = [K, K]$. Lemma 2.1 implies that the factor group G/S does not contain proper contranormal subgroups. Moreover, G/S is abelian-by-Chernikov. If K is abelian, then $S = \{1\}$; in the other two cases, K/S is either a periodic π' -group, or a Chernikov π -group. Then Proposition 3.4 implies that G/S is nilpotent. Using now [8, Theorem 7], we obtain that G is nilpotent, as required.

Now suppose $t > 1$ and, by induction, G/A_{t-1} nilpotent. If A_{t-1} is an abelian torsion-free G -rationally irreducible group, then Lemma 3.2 implies that this subgroup is G -central, and then G is nilpotent.

Now suppose that A_{t-1} is periodic. For every positive integer h , write B_h for the intersection of A_{t-1} with the h -center of K . Then B_h is normal in G for every h , $A_{t-1} = B_k$ for a suitable k , and $B_v = \{1\}$ for some $v \leq k$. Moreover, $K/B_{h+1} \leq C_{G/B_{h+1}}(B_h/B_{h+1})$. We show that G/B_h is nilpotent for every h , and thus $G/B_v = G$ is nilpotent. Suppose, by induction, G/B_{h+1} nilpotent. If A_{t-1}/B_{h+1} is a π' -group, then B_h/B_{h+1} is a π' -group, and Lemma 3.1 implies that $B_h/B_{h+1} \leq \zeta(G/B_{h+1})$; therefore G/B_h is nilpotent. If A_{t-1} is a Chernikov π -group, then, if D/K is the divisible part of G/K , arguing as in the proof of Proposition 3.4, we obtain that D/B_h is nilpotent. Then G/B_h is nilpotent-by-finite, and therefore it is nilpotent (see [20, Theorem 1]). $\square$

Note that if A is a torsion-free abelian subgroup of finite 0-rank of a group G , then, arguing as in Corollary 3.3, it is possible to construct a finite series of G -invariant subgroups whose factors are torsion-free and rationally irreducible. Hence Theorem B generalizes [20, Theorem 1].

4 Locally nilpotent abelian-by-finite groups with a finite contranormal subgroup

We start this section by proving Proposition C.

Proof of Proposition C. Suppose $AC \neq G$. Then Lemma 2.1 implies that CA/A is a proper contranormal subgroup of the finite nilpotent group G/A . But a nilpotent group does not contain a proper contranormal subgroup. Hence $AC = G$. Choose in C a finitely generated subgroup K such that $AK = G$; then $C = BK$, where $B = C \cap A$. Since A is normal in G , $[K, B] \leq A$. On the other hand, $[K, B] \leq C$ so that $[K, B] \leq C \cap A = B$. Therefore, the subgroup B is K -invariant. B is also A -invariant since A is abelian; thus, from $G = AK$, we get that B is G -invariant. The intersection $K \cap A$ is normal in G . Considering the factor group $G/(K \cap A)$, without loss of generality, we may assume that $K \cap A$ is trivial. Then the subgroup K is finite. Since $G = AK$, with A normal in G , it follows that $[K, A]$ is normal in G and $[G, G] = [K, A][K, K] \leq K[A, K]$. Thus $G/(K[K, A])$ is abelian. By Lemma 2.1, $C[K, A]/(K[K, A])$ is contranormal in $G/(K[K, A])$. It follows that $C[K, A]/(K[K, A]) = G/(K[K, A])$. Therefore, we have

$$G = C[K, A] = BK[K, A] = B[K, A] \rtimes K.$$

In particular, we obtain that $A = B[K, A]$. The subgroup B is normal in G . Then we obtain that

$$G/B = A/B \rtimes KB/B = [K, A]B/B \rtimes KB/B = [KB/B, A/B] \rtimes KB/B.$$

It follows that $G/B = (KB/B)^{G/B}$; hence KB/B is contranormal in G/B . $\square$

We start our investigation assuming that G is a p -group, p a prime.

Proposition 4.1. *Let G be an abelian-by-finite p -group, p a prime. If G contains a finite contranormal subgroup, then G satisfies the following conditions:*

- (i) $G = VC$, where V is a normal divisible abelian subgroup and C is a finite contranormal subgroup of G ;
- (ii) V has a family of G -invariant G -quasifinite subgroups $\{D_\mu \mid \mu \in M\}$ such that $V = \langle D_\mu \mid \mu \in M \rangle$;
- (iii) $[D_\mu, C] = D_\mu$ for all $\mu \in M$; in particular, $[V, C] = V$.

Proof. Let A be a normal abelian subgroup of G having finite index, and let C be a finite contranormal subgroup of G . By Lemma 2.1, CA/A is contranormal in G/A . Since G/A is a finite p -group, it is nilpotent. The fact that a nilpotent group does not include proper contranormal subgroups implies that $CA/A = G/A$, so $G = CA$. If $A = A^p$, then A is divisible and (i) holds. Suppose $B = A^p \neq A$. Then B is normal in G and G/B is an extension of an elementary abelian p -subgroup by a finite p -group. Such groups are nilpotent [2]. On the other hand, Lemma 2.1 shows that CB/B is a contranormal subgroup of G/B . The fact that a nilpotent group does not include a proper contranormal subgroup implies that $CB/B = G/B$. It follows that A/B is finite. The finiteness of A/A^p implies that $A = F \times V$, where V is a divisible subgroup and F is a finite subgroup (see, for example, [12, Lemma 3]). Clearly, the subgroup V is G -invariant. Being a finite p -group, the factor group G/V is nilpotent. As above, it follows that $CV/V = G/V$, so $G = VC$, and again, (i) holds. Now suppose $G = VC$, where V is divisible, abelian and normal in G . Since V is an abelian divisible p -subgroup we have $V = \times_{\lambda \in \Lambda} P_\lambda$, where P_λ is a Prüfer p -subgroup for all $\lambda \in \Lambda$ (see, for example, [7, Theorem 23.1]). Let Q_1 be a Prüfer p -subgroup of V . Since G/V is finite, Q_1 has only finitely many conjugates so that $Y = Q_1^G$ is a divisible Chernikov subgroup. Since Y satisfies the minimal condition, Y contains an infinite G -invariant subgroup D_1 which is G -quasifinite. If $D_1^p \neq D_1$, then D_1^p is finite since D_1 is quasifinite, and D_1/D_1^p is finite since it is an elementary abelian p -group with the minimal condition; hence D_1 is finite, a contradiction. Therefore, $D_1^p = D_1$, and D_1 is divisible. Thus $V = D_1 R$ for some subgroup R such that R is G -invariant, the intersection $D_1 \cap R$ is finite and $(D_1 \cap R)^{|C|} = \langle 1 \rangle$ (see, for example, [13, Corollary 5.11]). Put $|C| = p^n$; then $D_1 \cap R \leq \Omega_n(V)$. It is not hard to prove that the subgroup $[D_1, C]$ is G -invariant. If we suppose that $[D_1, C]$ is a proper subgroup of D_1 , then the fact that D_1 is G -quasifinite implies that $[D_1, C]$ must be finite. Then $D_1 C$ is a finite-by-abelian p -group so that $D_1 C$ is nilpotent. Being Chernikov, $D_1 C$ is central-by-finite (see, for example, [6, Corollary 3.2.10]). It follows that $D_1 \leq \zeta(D_1 C)$. Consider the factor group G/R . We have $V/R = D_1 R/R \simeq D_1/(D_1 \cap R)$. The equality $[D_1, C] = \langle 1 \rangle$ implies that $[V/R, CR/R] = [D_1 R/R, CR/R] = [D_1, C]R/R = \langle 1 \rangle$. It follows that $V/R \leq \zeta(G/R)$. But, in this case, $(CR/R)^{G/R} = CR/R$, and we obtain a contradiction with Lemma 2.1. This contradiction shows that $[D_1, C] = D_1$. Choose in the subgroup R a Prüfer p -subgroup Q_2 . Again, Q_2 has only finitely many conjugates so that Q_2^G is a divisible Chernikov subgroup. As above, Q_2^G contains an infinite G -invariant subgroup D_2 , which is G -quasifinite. Arguing as before, it is possible to prove that D_2 is divisible. Then, by [13, Corollary 5.11], $R = D_2 R_1$ for some subgroup R_1 such that R_1 is G -invariant and the intersection $D_2 \cap R_1$ is finite; moreover, $D_2 \cap R_1 \leq \Omega_n(V)$. Using the above arguments,

we obtain that $[D_2, C] = D_2$. Put $L_1 = \Omega_n(D_1)$; then $D_1/L_1 \cap RL_1/L_1 = \langle 1 \rangle$ and $L_1 \leq \Omega_n(V)$. Similarly, put $L_2 = \Omega_n(D_2)$; then $D_2/L_2 \cap R_1L_2/L_2 = \langle 1 \rangle$ and $L_2 \leq \Omega_n(V)$. Repeating these arguments and using transfinite induction, we obtain that the subgroup V has a family of G -invariant G -quasifinite subgroups $\{D_\mu \mid \mu \in M\}$ such that $V = \langle D_\mu \mid \mu \in M \rangle$, $[D_\mu, C] = D_\mu$ for all $\mu \in M$, as required. Moreover, we have $V/\Omega_n(V) = \times_{\mu \in M} D_\mu \Omega_n(V)/\Omega_n(V)$. $\square$

Now we can prove the following corollary.

Corollary 4.2. *Let G be a periodic locally nilpotent abelian-by-finite group. If G contains a finite contranormal subgroup, then G satisfies the following conditions:*

- (i) $G = VC$, where V is a normal divisible abelian subgroup and C is a finite contranormal subgroup of G ;
- (ii) $\Pi(G) = \Pi(C)$; in particular, the set $\Pi(G)$ is finite;
- (iii) V has a family of G -invariant G -quasifinite subgroups $\{D_\mu \mid \mu \in M\}$ such that $V = \langle D_\mu \mid \mu \in M \rangle$;
- (iv) $[D_\mu, C] = D_\mu$ for all $\mu \in M$; in particular, $[V, C] = V$.

Proof. Let A be a normal abelian subgroup of G having finite index, and let C be a finite contranormal subgroup of G . Then, arguing as above, we have $G = CA$. Suppose that $\Pi(G) \neq \Pi(C)$, and choose a prime $q \in \Pi(G) \setminus \Pi(C)$. The equality $G = AC$ implies that A contains a Sylow q -subgroup Q of G . We have $A = Q \times R$, where R is a Sylow q' -subgroup of A . Then

$$G/R = QR/R \times CR/R,$$

which shows that CR/R cannot be a contranormal subgroup of G/R . Thus we obtain a contradiction with Lemma 2.1. This contradiction proves $\Pi(G) = \Pi(C)$. We have $G = \times_{p \in \Pi(G)} S_p$, where S_p is a Sylow p -subgroup of G . The isomorphism $S_p \simeq G/(\times_{q \in \Pi(G), q \neq p} S_q)$ and an application of Proposition 4.1 prove the result. $\square$

Recall that a group G is called $\mathcal{F}$ -perfect if G does not contain a proper subgroup of finite index. In every group, the subgroup $\mathcal{F}(G)$, generated by all $\mathcal{F}$ -perfect subgroups, is $\mathcal{F}$ -perfect. It is the largest $\mathcal{F}$ -perfect subgroup of G . Clearly, $\mathcal{F}(G)$ is a characteristic subgroup of G , and the factor group $G/\mathcal{F}(G)$ does not contain $\mathcal{F}$ -perfect subgroups. The subgroup $\mathcal{F}(G)$ is called the $\mathcal{F}$ -perfect part of G . Let $\mathcal{X}$ be a class of groups. If G is a group, then we denote by $G_{\mathcal{X}}$ the intersection of all normal subgroups H of G such that $G/H \in \mathcal{X}$. The subgroup $G_{\mathcal{X}}$ is called the $\mathcal{X}$ -residual of the group G . If $\mathcal{X} = \mathcal{F}$ is the class of all finite groups, then $G_{\mathcal{F}}$ is called the finite residual of G .

Lemma 4.3. *Let G be a locally nilpotent periodic group. If G contains a finite contranormal subgroup, then the $\mathcal{F}$ -perfect part of G has finite index.*

Proof. If G does not contain proper subgroups of finite index, then G is $\mathcal{F}$ -perfect and the result is proved. Therefore, we suppose that G contains proper subgroups of finite index. Let S be a finite contranormal subgroup of G . Then S is nilpotent. Let k be the nilpotence class of S . If H is a normal subgroup of G such that G/H is finite, then Lemma 2.1 shows that SH/H is a contranormal subgroup of G/H . On the other hand, G/H is nilpotent, and a nilpotent group does not contain proper contranormal subgroups. It follows that $SH/H = G/H$. In particular, G/H has nilpotence class at most k . Let $\mathcal{S}$ be the family of all normal subgroups of G having finite index, and let $L = \bigcap_{H \in \mathcal{S}} H$. By Remak's theorem, there is an embedding $G/L \lesssim \text{Cr}_{H \in \mathcal{S}} G/H$. Since G/H has nilpotence class at most k for every $H \in \mathcal{S}$, this implies that G/L is a nilpotent group. It follows that G/L does not contain proper contranormal subgroups, and we obtain the equality $G/L = SL/L$. This means that G/L is finite. If we suppose that L contains a proper subgroup K having finite index in L , then K has finite index in G . Then $D = \text{Core}_G(K)$ is normal in G and has finite index in G . Thus $D \in \mathcal{S}$, and therefore $L \leq D$, a contradiction. This contradiction proves that L is $\mathcal{F}$ -perfect and L coincides with the $\mathcal{F}$ -perfect part of G . $\square$

Corollary 4.4. *Let G be a hypercentral periodic group. If G contains a finite contranormal subgroup, then G is abelian-by-finite.*

Proof. Let L be the $\mathcal{F}$ -perfect part of G . Lemma 4.3 implies that L has finite index in G . The result follows since a periodic hypercentral $\mathcal{F}$ -perfect group is abelian (see [4, Chapter 2, n. 2, Theorem 2.2]). $\square$

Lemma 4.5. *Let G be a locally nilpotent group. If G is not periodic, then G does not contain finite contranormal subgroups.*

Proof. Suppose the contrary, and let S be a finite contranormal subgroup of G . Since G is locally nilpotent, the set $\text{Tor}(G)$ of all elements of G having finite order is a characteristic subgroup of G . Since G is not periodic, $G \neq \text{Tor}(G)$. Then the inclusion $S \leq \text{Tor}(G)$ implies that $S^G \neq G$, and we obtain a contradiction which proves the result. $\square$

Now we can prove Theorem D.

Proof of Theorem D. Lemma 4.5 implies that the group G must be periodic. By Corollary 4.4, G is abelian-by-finite, and the result follows from Corollary 4.2. $\square$

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