An uncountable family of finitely generated residually finite groups

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Abstract. We study a family of finitely generated residually finite groups. These groups are doubles $F_2 *_H F_2$ of a rank-2 free group F_2 along an infinitely generated subgroup H . Varying H yields uncountably many groups up to isomorphism.

1 Introduction

A group G is *residually finite* if, for each $g \neq 1$, there is a finite quotient $G \rightarrow \bar{G}$ with $\bar{g} \neq 1$. Residually finite groups form a privileged and arguably rare class of groups. It is a routine exercise in small cancellation theory to provide uncountably many isomorphism classes of 2-generated groups. For instance, consider the following presentation, where (p_i) is a sequence of distinct primes ≥ 7 :

$$\langle a, b \mid (abab^2 \dots ab^{100i})^{p_i} : i \in \mathbb{N} \rangle.$$

Since the degrees of the torsion elements of the group are precisely the elements of $\{p_i\}$, these uncountably many groups are pairwise non-isomorphic.

This paper emerges from our curiosity to produce uncountably many finitely generated residually finite groups. The family of examples we construct have a very simple structure: they are all doubles $F *_H F$ of a free group F . Hence, they are particularly familiar examples in combinatorial group theory satisfying many routine properties (e.g. torsion-free, cohomological dimension 2, fundamental groups of non-positively curved spaces). Though our groups have a very simple structure, and form a very flexible family, we are not the first to wander in this direction.

Grigorchuk's family of *Grigorchuk groups* [5] provide uncountably many isomorphism classes of residually finite torsion groups. This has been revisited in [1] where uncountably many just-infinite branch pro-2 groups are provided. There are even earlier uncountable classes, again having torsion, which we discuss below.

There is an interesting interplay between residual finiteness and the word problem. Dyson observed that every finitely presented residually finite group has solvable word problem [3]. On the other hand, Higman showed that every finitely generated recursively presented group embeds as a subgroup of a finitely presented group [7]. Higman's embedding theorem demonstrates that there are finitely presented groups with unsolvable word problem, as there are finitely generated recursively presented groups having unsolvable word problem.

It is natural to attempt to relax the finite presentability hypothesis in Dyson's result. Following Higman's construction, Meskin gave an example of a finitely generated residually finite group with unsolvable word problem [10]. Around the same time, Dyson constructed a concrete family of examples of finitely generated residually finite groups with unsolvable word problem [4]. Rauzy used Dyson's examples to prove that Higman's embedding theorem cannot be extended to the category of finitely generated residually finite groups with solvable word problem [13]. The examples we construct can be recursively presented but still have unsolvable word problem. We suspect that Dyson's uncountable family actually provides uncountably many isomorphism classes. Dyson's examples are also doubles $L *_H L$, but of a lamplighter group instead of a free group. Hence, they are rich in torsion like Grigorchuk's examples.

By controlling subgroup growth, Pyber produced uncountably many pairwise non-isomorphic 4-generator residually finite groups with isomorphic profinite completions [12]. By counting irreducible representations, Kassabov–Nikolov also produced an uncountable family of pairwise non-isomorphic residually finite groups [8]. More recently, Segal provides a center-by-metabelian family, and Nikolov–Segal provide uncountably many non-isomorphic length-4 soluble groups that have the same profinite completion [11, 15].

Section 2 describes the groups $G_{\mathfrak{s}}$. Section 3 proves they are residually finite. Section 4 computes homologies of certain quotients of $G_{\mathfrak{s}}$. Section 5 uses this to show that these groups are pairwise non-isomorphic. Section 6 notes the simple relationship to unsolvable word and membership problems.

2 The groups $G_{\mathfrak{s}}$

Here we describe our family of finitely generated groups $G_{\mathfrak{s}}$.

Definition 2.1. Let F be a group with an isomorphic copy $\bar{F}$. Let $H \leq F$ and $\bar{H} \leq \bar{F}$ be corresponding subgroups. The *double of F along H* is the amalgamated product $G = F *_H \bar{F}$.

Definition 2.2. A sequence $\mathfrak{s} = (s_n)_{n=0}^\infty$ is *multiplying* if s_n and s_{n+1}/s_n are positive integers for all n , and $\lim s_n = \infty$.

Let $\mathfrak{s}$ be multiplying. Let $F = \langle a, b \rangle$ be a rank-2 free group. Let $H_{\mathfrak{s}} \leq F$ be the subgroup $H_{\mathfrak{s}} = \langle b^n a^{s_n} b^{-n} : n \geq 0 \rangle$. We define $G_{\mathfrak{s}}$ as the double of F along $H_{\mathfrak{s}}$,

$$G_{\mathfrak{s}} = F *_{H_{\mathfrak{s}} = \bar{H}_{\mathfrak{s}}} \bar{F}.$$

3 Residual Finiteness of $G_{\mathfrak{s}}$

In this section, we prove residual finiteness of $G_{\mathfrak{s}}$ by generalizing [17, Lemma 2].

A subgroup $H \leq G$ is *closed* (in the profinite topology) if H is the intersection of finite index subgroups of G . For instance, $\{1\} \subseteq G$ is closed if and only if G is residually finite. We use below the fact that every finitely generated subgroup of a free group F is closed [6, Theorem 5.1]. In particular, F is residually finite. Lemma 3.1 is used to prove that $G_{\mathfrak{s}}$ is residually finite. The proof of Lemma 3.1 is a special case of a result of G. Baumslag; see for instance [17].

Lemma 3.1. *Let H be a closed subgroup of a residually finite group F . Then the double $F *_{H = \bar{H}} \bar{F}$ is also residually finite.*

If $H_{\mathfrak{s}}$ is a closed subgroup of a free group F , then since F is residually finite, $G_{\mathfrak{s}}$ is also residually finite by Lemma 3.1, since $G_{\mathfrak{s}}$ is the double of F along $H_{\mathfrak{s}}$. Hence, to prove Lemma 3.3, it suffices to show that $G_{\mathfrak{s}}$ is residually finite. The proof uses the graphical viewpoint on subgroups of free groups as popularized in combinatorial group theory by Stallings [16] and [14, Remark 3.6].

Definition 3.2. The *based core* at x of a connected graph B is the smallest connected subgraph containing x and all closed cycles of B .

Lemma 3.3. *If $\mathfrak{s}$ is multiplying, then $H_{\mathfrak{s}}$ is a closed subgroup of F .*

Proof. As above, every finitely generated subgroup of a free group is closed. Hence, it suffices to show that $H_{\mathfrak{s}}$ is the intersection of a sequence of finitely generated subgroups of F .

For each $k \geq 0$, define $S_k \leq F$ as follows:

$$S_k = \langle b^k, b^j a^{s_j} b^{-j} : 0 \leq j < k \rangle.$$

To show $H_{\mathfrak{s}} \subseteq \bigcap S_k$, it suffices to show that $b^m a^{s_m} b^{-m} \in S_k$ for $m, k \geq 0$. Fix $m, k \geq 0$. If $m < k$, then $b^m a^{s_m} b^{-m} \in S_k$ by definition. If $m \geq k$, then let $m = qk + r$ for integers $q, r \geq 0$ with $0 \leq r < k$. Let $f = s_m/s_r$. Since $\mathfrak{s}$ is mul-

Figure 1. The core C of the based covering space associated with S_k .

Figure 2. The based graph C' after removing the (red) loops $a^{s_{k-m}}$ for $1 \leq m \leq |w|$ and the bottom $b^{|w|}$ path in Figure 1. The base point is bold.

tipling, f is a positive integer. Thus,

$$\begin{aligned} b^m a^{s_m} b^{-m} &= b^{qk+r} a^{s_r f} b^{-qk-r} \\ &= (b^k)^q (b^r a^{s_r} b^{-r})^f (b^{-k})^q \in S_k. \end{aligned}$$

It remains to show that $\bigcap S_k \subseteq H_{\mathfrak{s}}$. Consider $w \in \bigcap S_k$. Since $\lim s_n = \infty$, we may choose k with $k > 2|w|$ and $s_{k-|w|} > |w|$. The based core C of the covering space associated with S_k is shown in Figure 1. The graph C is the union of a based length k cycle of b -edges, and a length s_i cycle of a -edges attached at its i -th vertex for $0 \leq i < k$.

Let ω be the immersed loop in C labelled by w . Then ω is shorter than the $a^{s_{k-m}}$ (red) loops for $1 \leq m \leq |w|$, because $s_{k-|w|} > |w|$. Thus, ω cannot traverse an edge of those (red) loops. Therefore, ω is loop immersed in the subgraph of C obtained by removing those (red) loops.

Moreover, ω cannot traverse an edge of $b^{|w|}$ at the bottom of the graph since ω cannot backtrack.

Therefore, ω is immersed in the subgraph C' obtained by discarding edges which ω cannot traverse as explained above. See Figure 2. Since C' is a based subgraph of the based core of the covering space associated with $H_{\mathfrak{s}}$, we see that $w \in H_{\mathfrak{s}}$. $\square$

Remark. It is not difficult to show that $G_{\mathfrak{s}}$ is residually a finite p -group if and only if all elements of $\mathfrak{s}$ are powers of p . Furthermore, $G_{\mathfrak{s}}$ fails to be residually finite without the requirement that $\mathfrak{s}$ is nonbounded.

4 The homologies of quotients of $G_{\mathfrak{s}}$

We first prove a simple homological lemma. Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

Lemma 4.1. *Let $\mathfrak{s}$ be multiplying. Let m be a positive integer. Let*

$$\hat{G}_{\mathfrak{s}} = G_{\mathfrak{s}} / \langle\langle a^m, \bar{a}^m \rangle\rangle.$$

Then

$$\begin{aligned} H_1(\hat{G}_{\mathfrak{s}}) &\cong \mathbb{Z}^2 \times \mathbb{Z}_m \times \mathbb{Z}_{[m, s_0]}, \\ H_2(\hat{G}_{\mathfrak{s}}) &\cong \bigoplus_{j=1}^{\infty} \mathbb{Z}_{m/[m, s_j]}, \end{aligned}$$

where $[a, b] = \gcd(a, b)$.

Proof. Recall that $G_{\mathfrak{s}} = F *_{H_{\mathfrak{s}} = \bar{H}_{\mathfrak{s}}} \bar{F}$. Images of subgroups in $G_{\mathfrak{s}} \rightarrow \hat{G}_{\mathfrak{s}}$ are hatted,

$$\begin{aligned} H_1(\hat{G}_{\mathfrak{s}}) &\cong (\hat{G}_{\mathfrak{s}})_{\text{ab}} = \langle a, b, \bar{a}, \bar{b} \mid H_{\mathfrak{s}} = \bar{H}_{\mathfrak{s}}, a^m, \bar{a}^m \rangle_{\text{ab}} \\ &= \langle a, b, \bar{a}, \bar{b} \mid a^{s_0} = \bar{a}^{s_0}, a^m, \bar{a}^m \rangle_{\text{ab}} \\ &= \langle a, b, \bar{a}, \bar{b} \mid a^{[m, s_0]} = \bar{a}^{[m, s_0]}, a^m, \bar{a}^m \rangle_{\text{ab}} \\ &\cong \mathbb{Z}^2 \times \mathbb{Z}_m \times \mathbb{Z}_{[m, s_0]}, \\ H_1(\hat{F}) &= (\hat{F})_{\text{ab}} = \langle a, b \mid a^m \rangle_{\text{ab}} \cong \mathbb{Z} \times \mathbb{Z}_m. \end{aligned} \tag{4.1}$$

We shall compute $H_2(\hat{G}_{\mathfrak{s}})$ via the Mayer–Vietoris sequence for amalgamated products. Firstly, we claim $\hat{G}_{\mathfrak{s}}$ decomposes as a double as follows:

- $\hat{G}_{\mathfrak{s}} \cong Q *_{K=\bar{K}} \bar{Q}$, where $Q = F / \langle\langle a^m \rangle\rangle = \hat{F}$ and $\bar{Q} = \bar{F} / \langle\langle \bar{a}^m \rangle\rangle = \bar{\hat{F}}$
- and where $K = \text{im}(H_{\mathfrak{s}})$ under $F \rightarrow Q$, and $\bar{K} = \text{im}(\bar{H}_{\mathfrak{s}})$ under $\bar{F} \rightarrow \bar{Q}$.

And hence $K \rightarrow \hat{H}_{\mathfrak{s}}$ and $\bar{K} \rightarrow \bar{\hat{H}}_{\mathfrak{s}}$ are isomorphisms.

The map $\hat{G}_{\mathfrak{s}} \rightarrow Q *_{K=\bar{K}} \bar{Q}$ is induced by $F \rightarrow Q$ and $\bar{F} \rightarrow \bar{Q}$. The retraction maps $G_{\mathfrak{s}} \rightarrow F$ and $G_{\mathfrak{s}} \rightarrow \bar{F}$ project to retractions $\hat{G}_{\mathfrak{s}} \rightarrow Q$ and $\hat{G}_{\mathfrak{s}} \rightarrow \bar{Q}$. Moreover, the retractions $G_{\mathfrak{s}} \rightarrow F$ and $G_{\mathfrak{s}} \rightarrow \bar{F}$ project to $\hat{G}_{\mathfrak{s}} \rightarrow \hat{F}$ and $\hat{G}_{\mathfrak{s}} \rightarrow \bar{\hat{F}}$, respectively. The inclusions $Q \hookrightarrow \hat{G}_{\mathfrak{s}}$ and $\bar{Q} \hookrightarrow \hat{G}_{\mathfrak{s}}$ induce $Q *_{K=\bar{K}} \bar{Q} \rightarrow \hat{G}_{\mathfrak{s}}$, which provides the inverse. Thus, $\hat{G}_{\mathfrak{s}} \cong Q *_{K=\bar{K}} \bar{Q}$.

To calculate $H_1(\hat{H}_{\mathfrak{s}})$, we first determine $\hat{H}_{\mathfrak{s}}$ through the following sequence of isomorphisms. The first holds by declaring $\hat{a}, \hat{b}$ to be the images of a, b . The second holds by the Normal Form Theorem for free products [9, Theorem IV.1.2]. The third holds for each factor as s_j has order $m/\lfloor s_j, m \rfloor$ in $\mathbb{Z}/m\mathbb{Z}$.

$$\hat{H}_{\mathfrak{s}} = \langle \hat{b}^j \hat{a}^{s_j} \hat{b}^{-j} : j \geq 0 \rangle \cong \bigstar_{j=0}^{\infty} \langle \hat{b}^j \hat{a}^{s_j} \hat{b}^{-j} \rangle \cong \bigstar_{j=0}^{\infty} \mathbb{Z}_{m/\lfloor s_j, m \rfloor}. \quad (4.2)$$

Therefore, as homology is summable over free products, we have

$$H_1(\hat{H}_{\mathfrak{s}}) \cong \bigoplus_{j=0}^{\infty} \mathbb{Z}_{m/\lfloor m, s_j \rfloor}. \quad (4.3)$$

To compute $H_2(\hat{F})$, we construct an aspherical complex X with $\pi_1 X \cong \hat{F}$, so $H_2(\hat{F}) = H_2(X)$ (see [2, Proposition 2.4.1]). Let X^2 be the standard 2-complex of $\langle a, b \mid b^m \rangle$. The aspherical complex X is formed by adding appropriate n -cells to X^2 for $n \geq 3$.

$\partial_2: C_2(X) \rightarrow C_1(X)$ is injective since the attaching map of the 2-cell has degree m . Thus,

$$H_2(\hat{F}) = H_2(X) = \frac{\ker \partial_2}{\text{im } \partial_3} = 0. \quad (4.4)$$

The following exact sequence is deduced from the Mayer–Vietoris sequence for amalgamated products [2, Proposition 2.7.7]:

$$H_2(\hat{F}) \oplus H_2(\hat{F}) \rightarrow H_2(\hat{G}_{\mathfrak{s}}) \rightarrow H_1(\hat{H}_{\mathfrak{s}}) \rightarrow H_1(\hat{F}) \oplus H_1(\hat{F}).$$

Applying equations (4.1), (4.3) and (4.4), we obtain the following exact sequence:

$$0 \rightarrow H_2(\hat{G}_{\mathfrak{s}}) \rightarrow \bigoplus_{j=0}^{\infty} \mathbb{Z}_{m/\lfloor m, s_j \rfloor} \xrightarrow{\phi} (\mathbb{Z} \times \mathbb{Z}_m)^2.$$

The map $\phi: H_1(\hat{H}_{\mathfrak{s}}) \rightarrow H_1(\hat{F}) \oplus H_1(\hat{F})$ is induced by $\hat{H}_{\mathfrak{s}} \hookrightarrow \hat{F}$ and $\hat{H}_{\mathfrak{s}} \hookrightarrow \hat{F}$. By (4.2), $\phi([\hat{b}^j \hat{a}^{s_j} \hat{b}^{-j}]) = ([\hat{a}^{s_j}], [\hat{a}^{s_j}])$. Since $\{\hat{b}^j \hat{a}^{s_j} \hat{b}^{-j} : j \geq 0\}$ generates $H_1(\hat{H}_{\mathfrak{s}})$, we have $\text{im}(\phi) = \lfloor m, s_0 \rfloor \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}_{m/\lfloor m, s_0 \rfloor}$ as $s_0 \mid s_j$ for $j \geq 0$. Therefore, by the first isomorphism theorem, we reach the conclusion

$$H_2(\hat{G}_{\mathfrak{s}}) = \ker \phi \cong \frac{\bigoplus_{j=0}^{\infty} \mathbb{Z}_{m/\lfloor m, s_j \rfloor}}{\mathbb{Z}_{m/\lfloor m, s_0 \rfloor}} = \bigoplus_{j=1}^{\infty} \mathbb{Z}_{m/\lfloor m, s_j \rfloor}. \quad \square$$

5 Canonical subgroups and pairwise non-isomorphism

We shall first identify certain canonical subgroups of $G_{\mathfrak{s}}$. We then use this to show that $G_{\mathfrak{s}} \not\cong G_{\mathfrak{t}}$ for multiplying sequences $\mathfrak{s} \neq \mathfrak{t}$.

Definition 5.1. An element $g \in G$ is *loose* if $C = \langle g \rangle$ is a maximal cyclic subgroup and there exists a maximal cyclic subgroup C' such that $C \cap C'$ is a proper finite index subgroup of C .

Proposition 5.2. *If $\mathfrak{s} \neq (1)$, then $g \in G_{\mathfrak{s}}$ is loose if and only if g is conjugate to $a^{\pm 1}$ or $\bar{a}^{\pm 1}$.*

Proof. Choose p with $s_p > 1$. Let $C = \langle b^p a b^{-p} \rangle$ and $C' = \langle \bar{b}^p \bar{a} \bar{b}^{-p} \rangle$. Then $C \cap C' = \langle b^p a^{s_p} b^{-p} \rangle = \langle \bar{b}^p \bar{a}^{s_p} \bar{b}^{-p} \rangle$ is a proper finite index subgroup of C and C' . Hence, $b^p a b^{-p}$ and $\bar{b}^p \bar{a} \bar{b}^{-p}$ are loose and thus a and $\bar{a}$ are loose, since looseness is preserved by conjugation.

For the other direction, consider the action of $G_{\mathfrak{s}}$ on the Bass–Serre tree. Suppose $\langle g \rangle \neq \langle k \rangle$ are maximal cyclic subgroups with $\langle g \rangle \cap \langle k \rangle$ infinite. Note that either both are hyperbolic or both are elliptic. Indeed, if g is hyperbolic, then k is hyperbolic since $k^n \in \langle g \rangle$ for some n .

Suppose g and k are hyperbolic, then since $\langle g \rangle$ and $\langle k \rangle$ are commensurable, they stabilize the same axis in the tree and hence act on the axis in the same way. By maximality of $\langle g \rangle$ and $\langle k \rangle$, we have $\langle g \rangle = \langle k \rangle$, which contradicts our assumption.

Suppose g and k are elliptic. Let u and v be the vertices stabilized by g and k , and let A be the geodesic from u to v . Note that $u \neq v$ since free groups have no loose subgroups. Note that the pointwise-stabilizer of A is $\langle g \rangle \cap \langle k \rangle$.

Let $A = e_1 e_2 \cdots e_m$, with vertices $u = u_0, u_1, \dots, u_m = v$. Since g stabilizes u_0 , there exists ℓ such that g stabilizes u_ℓ but not $e_{\ell+1}$. Thus, g is in the vertex group associated to u_ℓ but not in the edge group associated to e_ℓ . However, g^n is in the edge group associated to e_ℓ for some $n > 0$.

It thus suffices to show that $H < F$ has the property that if $g^r \in H$ for some r but $g \notin H$, then g is conjugate into $\langle a \rangle$. Equivalently, we show that if a path S in the graph associated to H has a power S^m which is closed, then either S is closed or S is a conjugate of a power of a .

Observe that the b -exponent sum of S equals 0, for otherwise, no S^r is closed. Assume without loss of generality that S is cyclically reduced, and moreover, if S is not a power of a , then S starts with b . We then deduce that S starts on the b ray, for otherwise, S^r embeds in a tree attached to the interior of an a -loop. If S ends on the b ray, then S is closed since the b -exponent sum of S is 0. Finally, if S ends

in the interior of an a -loop, then as above, S^{r-1} embeds in a tree attached to the interior of an a -loop, so S^r is not closed. $\square$

Corollary 5.3. *Each $G_{\mathfrak{s}}$ is finitely generated and residually finite. Moreover, we have $G_{\mathfrak{s}} \not\cong G_{\mathfrak{t}}$ when $\mathfrak{s} \neq \mathfrak{t}$.*

Proof. Let $\mathfrak{s}$ and $\mathfrak{t}$ be multiplying sequences. Suppose $\phi: G_{\mathfrak{s}} \rightarrow G_{\mathfrak{t}}$ is an isomorphism. Denote generators for $G_{\mathfrak{s}}$ as $a, b, \bar{a}, \bar{b}$ and that for $G_{\mathfrak{t}}$ as $a', b', \bar{a}', \bar{b}'$. Denote $A_m = \langle\langle a^m, \bar{a}^m \rangle\rangle \subseteq G_{\mathfrak{s}}$ and $A'_m = \langle\langle a'^m, \bar{a}'^m \rangle\rangle \subseteq G_{\mathfrak{t}}$.

By Proposition 5.2, loose elements $a^{\pm 1}, \bar{a}^{\pm 1}$ can be mapped to conjugates of $(a')^{\pm 1}$ or $(\bar{a}')^{\pm 1}$. For $s_0 = 1$, since $a = \bar{a}$, so $\phi(A_m) = A'_m$ for all m . For $s_0 > 1$, by abelianization of $G_{\mathfrak{s}}$ and $G_{\mathfrak{t}}$, $\phi(a)$ is conjugate to $(a')^{\pm 1}$ and $\phi(\bar{a})$ is conjugate to $(\bar{a}')^{\pm 1}$, or $\phi(a)$ is conjugate to $(\bar{a}')^{\pm 1}$ and $\phi(\bar{a})$ is conjugate to $(a')^{\pm 1}$. Therefore, $\phi(A_m) = A'_m$ for all m .

Suppose sequences $\mathfrak{s} \neq \mathfrak{t}$ first differ at index k , i.e. $s_j = t_j$ for $j < k$ and $s_k \neq t_k$. Without loss of generality, let $s_k > t_k$. Using $s_j \mid s_k$ for $j \leq k$, by Lemma 4.1,

$$\begin{aligned} H_1(G_{\mathfrak{s}}/A_{s_k}) &\cong \mathbb{Z}^2 \times \mathbb{Z}_{s_k} \times \mathbb{Z}_{s_0}, \\ H_2(G_{\mathfrak{s}}/A_{s_k}) &\cong \bigoplus_{j=1}^{k-1} \mathbb{Z}_{s_k/s_j}, \\ H_1(G_{\mathfrak{t}}/A'_{s_k}) &\cong \mathbb{Z}^2 \times \mathbb{Z}_{s_k} \times \mathbb{Z}_{\lfloor s_0, t_0 \rfloor}, \\ H_2(G_{\mathfrak{t}}/A'_{s_k}) &\cong \left[\bigoplus_{j=1}^{k-1} \mathbb{Z}_{s_k/s_j} \right] \oplus \left[\bigoplus_{j=k}^{\infty} \mathbb{Z}_{s_k/\lfloor s_k, t_j \rfloor} \right]. \end{aligned}$$

If $k = 0$, then $\mathbb{Z}_{\lfloor s_0, t_0 \rfloor} = \mathbb{Z}_{t_0}$. Thus, $H_1(G_{\mathfrak{s}}/A_{s_k}) \not\cong H_1(G_{\mathfrak{t}}/A'_{s_k})$. This is impossible.

If $k \geq 1$, then $\lfloor s_k, t_k \rfloor < s_k$. Thus, $\bigoplus_{j=k}^{\infty} \mathbb{Z}_{s_k/\lfloor s_k, t_j \rfloor} \not\cong 1$, so

$$H_2(G_{\mathfrak{s}}/A_{s_k}) \not\cong H_2(G_{\mathfrak{t}}/A'_{s_k}).$$

This is impossible. $\square$

6 Most $G_{\mathfrak{s}}$ have unsolvable word problem

Definition 6.1. A group G generated by a finite set S has *solvable word problem* if there is a computer program that can determine whether a word in S represents 1_G . A subgroup H of G has *solvable membership problem* if there is a computer program that determines whether or not a word in S represents an element in H .

A sequence $\mathfrak{s}$ is *computable* if there is a computer program that outputs s_n for input n . *Unsolvable* means not solvable and *uncomputable* means not computable.

Corollary 6.2. *There are uncountably many finitely generated residually finite groups with unsolvable word problem.*

Proof. Consider groups $G_{\mathfrak{s}}$. We claim that if $\mathfrak{s}$ is an uncomputable sequence, then $H_{\mathfrak{s}}$ has unsolvable membership problem. Assume there is a program that determines whether $(b^n a^p b^{-n}) \in H_{\mathfrak{s}}$. Then enumerating p gives a program that outputs s_n on input n .

We now show $G_{\mathfrak{s}}$ has unsolvable word problem if $H_{\mathfrak{s}}$ has unsolvable membership problem. Assume there is a program that determines whether a word is trivial in $G_{\mathfrak{s}}$. Consider the involution $\phi: G_{\mathfrak{s}} \rightarrow G_{\mathfrak{s}}$ with $\phi(a) = \bar{a}$ and $\phi(b) = \bar{b}$. Then determining whether the word $w\phi(w)^{-1}$ represents 1_G for any word w in $\{a, b\}$ solves the membership problem for H .

It thus suffices to prove that there are uncountably many uncomputable multiplying sequences $\mathfrak{s}$. There are countably many computable sequences since there are countably many (finite) computer programs. However, there are uncountably many multiplying sequences, e.g. let $s_{n+1}/s_n \in \{2, 4\}$. $\square$

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