Finite non-solvable groups whose real degrees are prime-powers

Lorenzo Bonazzi*

Communicated by Britta Späth

Abstract. We present a description of non-solvable groups in which all real irreducible character degrees are prime-power numbers.

1 Introduction

Let $G$ be a finite group. It is well known that $\text{cd}(G)$, the set of the degrees of all irreducible characters, has great impact on the structure of $G$. Manz in [11, 12] described the solvable and non-solvable groups in which all the real irreducible characters have prime-power degrees. In this paper, we give a structural description of non-solvable groups $G$ such that $\text{cd}_{rv}(G)$, the set of the degrees of all real irreducible characters, consists of prime-power numbers. In the following, $\text{Rad}(G)$ is the radical subgroup and $G^{(\infty)}$ is the last term of the derived series.

Theorem A. Let $G$ be a finite non-solvable group, and suppose that $\text{cd}_{rv}(G)$ consists of prime-power numbers. Then $\text{Rad}(G) = H \times O$ for a group $O$ of odd order and a 2-group $H$ of Chillag–Mann type. Furthermore, if $K = G^{(\infty)}$, then one of the following holds:

(i) $G = K \times \text{Rad}(G)$, and $K$ is isomorphic to $A_5$ or $L_2(8)$;

(ii) $G = (KH) \times O$, where $K \simeq \text{SL}_2(5)$, $HK = H \cdot K$, $K \cap H = Z(K) < H$.

About (ii), we remark that if $G$ is the SmallGroup(240,93), then $K \simeq \text{SL}_2(5)$, $|H| = 4$ and $K \cap H = Z(K)$.

As a corollary, we get control on the set of real character degrees. We recall that $\text{cd}_{rv,2^r}(G)$ is the set of odd real character degrees of a finite group $G$.

Theorem B. Let $G$ a non-solvable group such that $\text{cd}_{rv}(G)$ consists of prime-power numbers. Then either

(i) $\text{cd}_{rv}(G) = \text{cd}_{rv}(L_2(8))$ or

(ii) $\text{cd}_{rv,2^r}(G) = \text{cd}_{rv,2^r}(A_5)$.
2 Preliminary results and lemmas

Chillag and Mann are among the first authors that studied $\text{cd}_{rv}(G)$. They characterized the groups $G$ such that $\text{cd}_{rv}(G) = \{1\}$, namely where all real irreducible characters are linear. Now these groups are commonly known as groups of Chillag–Mann type.

**Theorem 2.1** ([2, Theorem 1.1]). Let $G$ be a finite group of Chillag–Mann type. Then $G = O \times T$, where $O$ is a group of odd order and $T$ is a 2-group of Chillag–Mann type.

One other important contribution was given by Dolfi, Navarro and Tiep in [3]. In that paper, there appears a version for real characters of the celebrated Ito–Michler Theorem for the prime $p = 2$. Recall that $\text{Irr}_{rv}(G)$ denotes the set of irreducible real-valued character of $G$.

**Theorem 2.2** ([3, Theorem A]). Let $G$ be a finite group and $T \in \text{Syl}_2(G)$. Then $2 \nmid \chi(1)$ for every non-linear $\chi \in \text{Irr}_{rv}(G)$ if and only if $T \leq G$ and $T$ is of Chillag–Mann type.

The corresponding condition for an odd prime $p$ was studied by Tiep in [15] and Isaacs and Navarro in [9]. Though a partial result, the techniques involved are deep. This confirms the special role of the prime 2 in the study of real character degrees.

**Theorem 2.3** ([15, Theorem A]). Let $G$ be a finite group and $p$ a prime. Suppose that $p \nmid \chi(1)$ for every $\chi \in \text{Irr}_{rv}(G)$ with Schur–Frobenius indicator 1. Then $O^p(G)$ is solvable; in particular, $G$ is $p$-solvable.

Navarro, Sanus and Tiep gave a version for real characters of Thompson’s Theorem for the prime 2 in [13]. Their work includes also a characterization of groups in which the real character degrees are powers of 2.

**Theorem 2.4** ([13, Theorem A]). Let $G$ be a finite group, and suppose that 2 divides $\chi(1)$ for all every real non-linear irreducible character of $G$. Then $G$ has a normal 2-complement.

The next two lemmas appear in [4].

**Lemma 2.5.** Let $N$ be a normal subgroup of $G$ and $\chi \in \text{Irr}_{rv}(G)$. The following hold:

(i) if $\chi(1)$ is odd, then $N \leq \ker(\chi)$;

(ii) if $|N|$ is odd and $N$ centralizes a Sylow 2-subgroup of $G$, then $N \leq \ker(\chi)$. 
Proof. Point (ii) is [4, Lemma 1.4]. Point (i) follows from the discussion before [4, Lemma 1.4], keeping in mind that a group of odd order does not have any real non-trivial character.

Let $N$ be a normal subgroup of $G$ and $\theta \in \text{Irr}_{rv}(N)$. The next lemmas provide some sufficient conditions for the existence of a real character of $G$ above $\theta$.

**Lemma 2.6** ([14, Lemma 2.1 and Corollary 2.2]). Let $N$ be a normal subgroup of a group $G$ and $\theta \in \text{Irr}_{rv}(N)$. If $[G : N]$ is odd, then $\theta$ allows a unique real-valued extension to $I_G(\theta)$. Furthermore, there exists a unique real-valued character $\chi \in \text{Irr}_{rv}(G \mid \theta)$.

**Lemma 2.7** ([14, Theorem 2.3]). Let $G$ be a finite group and $N \trianglelefteq G$. Suppose that there is some $\theta \in \text{Irr}_{rv}(G)$ such that $\theta(1)$ is odd and $o(\theta) = 1$. Then $\theta$ extends to a character $\phi \in \text{Irr}_{rv}(I_G(\theta))$ and $\chi = \phi^G \in \text{Irr}_{rv}(G \mid \theta)$.

**Lemma 2.8.** Suppose that $N$ is a minimal normal subgroup of a group $G$ with $N = S_1 \times \cdots \times S_n$, where $S \cong S$ is a non-abelian simple group. Let $\sigma \in \text{Irr}_{rv}(S)$, and suppose that $\sigma$ extends to a real character of $\text{Aut}(S)$. Then $\sigma \times \cdots \times \sigma$ extends to a real character of $G$.

Proof. The extension $\chi$ is constructed in [1, Lemma 5]. We see that if $\sigma$ takes real values, then so does $\chi$.

The technique used in the proof of Lemma 2.8 is known as tensor induction; for further details, see [7, Section 4].

**Lemma 2.9** ([4, Lemma 1.6]). Let $G$ be a finite group that acts by automorphisms on the group $M$. For every involution $x \in C_G(M) \subseteq G/C_G(M)$, there exists a non-trivial character $\mu \in \text{Irr}(M)$ such that $\mu^x = \tilde{\mu}$.

3 Proofs

In the following, we call an integer a composite number if it is divisible by more than one prime. If $p$ is a prime, we denote by $p^*$ a general positive integer that is a power of $p$. Moreover, $\text{Rad}(G)$ is the solvable radical of $G$, namely the largest solvable normal subgroup of $G$.

**Theorem 3.1.** Let $G$ be a finite non-solvable group such that $\text{cd}_{rv}(G)$ consists of prime-power numbers. If $\text{Rad}(G) = 1$, then $G$ is isomorphic to $A_5$ or $\text{PSL}_2(8)$.
Proof. Let \( M \) be a minimal normal subgroup of \( G \). Then \( M = S_1 \times \cdots \times S_n \) is the product of simple groups, which are all isomorphic to a simple group \( S \). Since \( \text{Rad}(G) = 1 \), the group \( S \) is non-abelian.

**Step 1:** \( S \) is isomorphic to one of the following groups:

\[
A_5, A_6, \text{PSL}_2(8), \text{PSL}_3(3), \text{PSp}_4(3), \text{PSL}_2(7), \text{PSU}_3(3), \text{PSL}_2(17).
\]

Let \( p \in \pi(M) \). Since \( M \) is minimal normal in \( G \), we have \( M \leq O^p(G) \), so \( O^p(G) \) is non-solvable. By Theorem 2.3, there is a real irreducible character \( \chi \) of \( G \) such that \( p \mid \chi(1) \). By the hypothesis, \( \chi(1) = p^* > 1 \). This means that, for every prime \( p \in \pi(M) \), there is \( \chi \in \text{Irr}_{rv}(G) \) such that \( \chi(1) = p^* > 1 \). By [3, Theorem B], if \( \Delta_{rv}(G) \) is the prime graph on real character degrees of \( G \), then the number of connected components of \( \Delta_{rv}(G) \) is at most three. In our hypotheses, \( \Delta_{rv}(G) \) consists of isolated vertices, and hence the number of primes that appear as divisors of the degree of some real irreducible character is at most 3. It follows that \( M \), and hence \( S \), is divisible by exactly 3 primes. Now, by [16, Lemma 2.1], the simple groups having order divided by exactly 3 distinct primes are those stated.

**Step 2:** \( S \) is isomorphic to one of the following groups: \( A_5, \text{PSL}_2(8), A_6 \).

If \( S \in \{\text{PSp}_4(3), \text{PSL}_3(3), \text{PSU}_3(3)\} \), there is a non-linear character \( \sigma \in \text{Irr}_{rv}(S) \) such that \( \sigma(1) \) is an odd composite number. Let \( \theta = \sigma \times \cdots \times \sigma \in \text{Irr}_{rv}(M) \). Then we have \( 2 \mid \theta(1) \) and \( o(\theta) = 1 \) since \( M \) is perfect. So, by Lemma 2.7, there is \( \chi \in \text{Irr}_{rv}(G \mid \theta) \). As \( \theta(1) \) divides \( \chi(1) \), the degree of \( \chi \) is a composite number, contrary to the hypothesis. Suppose that \( S \in \{\text{PSL}_2(7), \text{PSL}_2(17)\} \). There is a real character \( \sigma \in \text{Irr}_{rv}(S) \) such that \( \sigma(1) \) is a composite number and \( \sigma \) extends to a real character of \( A = \text{Aut}(S) \). By tensor induction (Lemma 2.8), the character \( \theta = \sigma \times \cdots \times \sigma \) extends to a real character \( \chi \in \text{Irr}_{rv}(G) \). Again, we get that \( \chi(1) = \theta(1) = \sigma(1)^n \) is a composite number.

**Step 3:** \( n = 1 \) and \( M \) is a simple group.

The only remaining possibilities are \( S \in \{A_5, \text{PSL}_2(8), A_6\} \). Checking the character table of these groups, there are two non-linear characters \( \sigma, \rho \in \text{Irr}_{rv}(S) \) such that \( \sigma(1) = p^* > 1 \) and \( \rho(1) = q^* > 1 \) for \( p, q \) odd distinct primes. Let \( \theta = \sigma \times 1 \times \cdots \times 1 \in \text{Irr}_{rv}(M) \). Since \( o(\theta) = 1 \) and \( \theta(1) \) is odd, the character \( \theta \) extends to a character \( \phi \in \text{Irr}_{rv}(I_G(\theta)) \) by Lemma 2.7, and \( \chi = \phi^G \) has degree \( p^* \); hence \( [G : I_G(\theta)] = p^* > 1 \). Since \( I_G(\theta) \leq N_G(S_1) \), we have that

\[
n = [G : N_G(S_1)] \text{ divides } [G : I_G(\theta)] = p^* > 1,
\]

so \( n = p^* \). By the same argument with \( \rho \) in place of \( \sigma \), we get that \( n = q^* \) and \( n \mid (p^*, q^*) = 1 \).
Step 4: $C_G(M) = 1$.

Suppose, by contradiction, that $C_G(M) > 1$, and take $N$ to be a minimal normal subgroup of $G$ contained in $C_G(M)$. By the same arguments as used on $M$, we have that $N$ is isomorphic to a simple group of the following: $A_5$, PSL$_2(8)$, $A_6$. As before, take $\sigma \in \text{Irr}_{rv}(M)$ with $\sigma(1) = p^*$ and $\rho \in \text{Irr}_{rv}(N)$ with $\rho(1) = q^*$ for $p, q$ odd distinct primes. Note that $[M, N] \leq M \cap N \leq M \cap C_G(M) = 1$ since $M$ is simple and non-abelian. So $MN = M \times N$ is perfect normal in $G$, and $G = M \times N$. Note that $o(\theta) = 1$ and $2 \nmid \theta(1)$. By Lemma 2.7, there is $\chi \in \text{Irr}_{rv}(G | \theta)$, and this is impossible since $\chi(1)$ is not a composite number.

**Conclusion:** We have proved, so far, that $S \leq G \leq \text{Aut}(S)$ and that $S$ is isomorphic to either $A_5, A_6$ or PSL$_2(8)$. Now, $S$ cannot be the alternating group $A_6$ because each of the 5 subgroups between $S$ and $\text{Aut}(S)$ has a rational irreducible character of degree 10 (it is possible to check this with the software GAP), so $S \not\in \{A_5, \text{PSL}_2(8)\}$. In any of these cases, $[\text{Aut}(S) : S]$ is a prime number and there is only one subgroup strictly above $S$, namely $\text{Aut}(S)$ itself. But both $\text{Aut}(A_5)$ and $\text{Aut}(\text{PSL}_2(8))$ have a real irreducible character with composite degree. Hence $G = A_5$ or $G = \text{PSL}_2(8)$.

**Proposition 3.2.** Let $G$ be a finite non-solvable group such that $\text{cd}_{rv}(G)$ consists of prime-power numbers. Then $G = KR$ with $R = \text{Rad}(G)$ and $K = G^{(\infty)}$. Moreover, $K \cap R = L$ is a 2-group, and $K/L$ is isomorphic to $A_5$ or PSL$_2(8)$.

**Proof.** Let $K = G^{(\infty)}$ be the last term of the derived series of $G$, and define $\tilde{G} = G/K \cap R$. Observe that quotients preserve the hypotheses. Hence, by Theorem 3.1, $G/R$ is a simple group. Since $1 < KR/R \leq G/R$, we have that $G = KR$ and $\tilde{K} \simeq G/R$ is isomorphic to $A_5$ or PSL$_2(8)$. Moreover, $\tilde{G} = \tilde{K} \times \tilde{R}$ because $[K, R] \leq L$.

Suppose by contradiction that there is some $\theta \in \text{Irr}_{rv}(\tilde{R})$ of non-trivial degree. By Theorems 2.4 and 2.2, there are two non-linear characters $\phi_1, \phi_2 \in \text{Irr}_{rv}(\tilde{K})$ such that $\phi_1(1)$ is even and $\phi_2(1)$ is odd. If $\theta(1)$ is odd, consider $\chi = \theta \phi_1$, and if $\theta(1)$ is even, consider $\chi = \theta \phi_2$. In any case, $\chi$ is a composite number, but this is impossible. It follows that every real character of $R/L$ is linear, and by Theorem 2.1, $\tilde{R} = \tilde{O} \times \tilde{H}$, where $O \in \text{Hall}_2(R)$ and $H \in \text{Syl}_2(R)$. Write $G_0$ for the preimage in $G$ of $\tilde{K} \tilde{H}$; note that $G_0$ is a normal subgroup of odd index in $G$. Note that $G_0 = LKH = KH$. By Lemma 2.6, $\text{cd}_{rv}(G_0)$ consists of prime-power numbers. Moreover, $K = G_0^{(\infty)}$ and $\text{Rad}(G_0) \cap K = L$. Hence we can assume that $G = G_0$. This implies that $O \leq L$.

Suppose, working by contradiction, that $O > 1$, namely $L$ is not a 2-group. Consider $M/M_0$, the first term (from above) of a principal series of $G$ such that $M, M_0 \leq L$ and $M/M_0$ is not a 2-group. Hence $M/M_0$ is an elementary abelian
\[ p \text{-group for } p \text{ odd, and } L/M \text{ is a } 2\text{-group. Possibly replacing } G \text{ with } G/M_0, \text{ we can assume that } M_0 = 1 \text{ and } M \text{ is a minimal normal subgroup of } G. \]

Since \( K/L \) is simple, \( C_K(M) = K \) or \( C_K(M) \leq L \). If \( C_K(M) = K \), then \( M \) has a direct complement \( N \) in \( L \), and we consider \( \bar{K} = K/N \). Note that

\[
1 < \bar{M} \leq Z(\bar{K}) \cap \bar{K}'
\]

since \( K = K' \) is perfect, and hence \(|M| \text{ divides } |M(G)|\) by [6, Corollary 11.20], where \( M(G) \) denotes the Schur multiplier of \( G \). But this is impossible since \(|M(A_5)| = 2 \) and \( M(\text{PSL}_2(8)) = 1 \).

Hence \( C_K(M) \leq L \), and the action of \( K \) on \( M \) is non-trivial. Moreover, \( K/L \) has even order, so by Lemma 2.9, there are an element \( \lambda \in \bar{M} \) and \( x \in K \) such that \( \lambda^x = \bar{\lambda} \). Let \( I = I_G(\lambda) \), and note that \( x \in N_G(I) \setminus I \), so \( 2 \) divides \([G : I]\).

Let \( \bar{I} = I/\ker(\lambda) \) (we remark that the “bar” notation here is not the same as in first part of the proof), and observe that \( \bar{M} \leq Z(\bar{I}) \). Take \( P \in \text{Syl}_p(I) \); since the index of \( K \) in \( G \) is a 2-power, every subgroup of \( G \) with odd order is contained in \( K \); hence \( P \leq K \). Moreover, \( \bar{M} \leq Z(\bar{P}) \), \( \bar{P} \in \text{Syl}_p(\bar{I}) \), and \( PL/L \) is a \( p \)-subgroup of the simple group \( K/L \) that is isomorphic to \( A_5 \) or \( \text{PSL}_2(8) \). Now, if \( p \) is an odd prime, every Sylow \( p \)-subgroup of \( A_5 \) or \( \text{PSL}_2(8) \) is cyclic (see Tables 1 and 2). Hence \( P/M \simeq \bar{P}/\bar{M} \simeq PL/L \) is cyclic and \( \bar{P} \) is abelian.

Since \( \bar{M} \leq Z(\bar{I}) \), we have that \( \bar{M} \neq \bar{I} \) by [8, Theorem 5.3]. In addition, we have \( \bar{M} \cap \bar{I}' = 1 \) because \( \bar{M} \) has order \( p \). Write

\[
\bar{I}/\bar{I}' = Q \times B, \quad \text{where } B \in \text{Hall}_p'(\bar{I}/\bar{I}') \text{ and } Q \in \text{Syl}_p(\bar{I}/\bar{I}').
\]

Note that \( Q \) and \( B \) are \( x \)-invariant as \( x \) normalizes \( I \). By abuse of notation, we write \( M \leq Q \) in the place of \( \bar{M} \bar{I}'/\bar{I}' \leq Q \). In this notation, \( M \) is a group of order \( p \), and \( \lambda \) is a faithful character of \( \bar{M} \). The 2-group \( \langle x \rangle \) acts on the abelian group \( Q \); hence by Maschke’s Theorem [10, Theorem 8.4.6], there is an \( \langle x \rangle \)-invariant complement \( T \) for the \( \langle x \rangle \)-invariant subgroup \( M \), so \( Q = M \times T \). Letting \( \hat{\lambda} = \lambda \times 1_T \in \text{Irr}(Q) \) and \( \delta = \hat{\lambda} \times 1_B \in \text{Irr}(\bar{I}/\bar{I}') \), we have that

\[
\delta^x = \hat{\lambda}^x \times 1_B = (\lambda^x \times 1_{T^x}) \times 1_B = (\hat{\lambda} \times 1_T) \times 1_B = \bar{\delta}.
\]

We return to the previous notation, so \( \delta \) lifts to a character of \( I \) that we also call \( \delta \). Note that \( I < G \) as \( 2 \) divides \([G : I]\).

If \( IH < G \), then \( IH/H \) is a proper subgroup of \( G/H \) that is a simple group isomorphic to \( A_5 \) or \( \text{PSL}_2(8) \). The maximal subgroups of these two groups are known as well as their indexes; see Tables 1 and 2. In particular, there always is an odd prime \( q \) such that \( q \) divides \([G : IH]\), and hence \( 2q \) divides \([G : I]\). Note that \( \delta \in \text{Irr}(I \mid \lambda) \), so \( \chi = \delta^G \in \text{Irr}(G) \). Moreover,

\[
\bar{\chi} = (\delta^G)^G = (\delta^x)^G = \delta^G = \chi.
\]
Hence \( \chi \) is a real character of \( G \), and \( 2q | \chi(1) \) since \( 2q | [G : I] \), and this is impossible.

Suppose now \( IH = G \). In this case, \( I/I \cap H \cong G/H \), which is isomorphic to \( A_5 \) or \( \text{PSL}_2(8) \). These groups have a unique rational character \( \phi \) of odd degree. The element \( x \) stabilizes the section \( I/I \cap H \); hence by uniqueness, \( \phi^x = \phi \).

By Gallagher’s Theorem [6, Theorem 6.17], \( \phi \delta \in \text{Irr}(I | \lambda) \), and by the Clifford correspondence, \( \chi = (\phi \delta)^G \in \text{Irr}(G) \). Since \( \phi \) is a real \( x \)-invariant character and \( \delta^x = \delta \), we have that \( (\phi \delta)^x = \overline{\phi \delta} \). Hence, as before, \( \chi \) is a real irreducible character. Now \( \theta(1) | \chi(1) \), and there is an odd prime \( q \) such that \( q | \chi(1) \).

Moreover, \( 2 | \chi(1) \) since \( 2 | [G : I] \). So \( \chi(1) \) is a composite number, and this is impossible.

We give the list of maximal subgroups of \( A_5 \) and \( \text{PSL}_2(8) \) and their indices.

<table>
<thead>
<tr>
<th>( A_4 )</th>
<th>( D_{10} )</th>
<th>( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( F_{56} )</th>
<th>( D_{18} )</th>
<th>( D_{14} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>56</td>
<td>18</td>
<td>14</td>
</tr>
<tr>
<td>9</td>
<td>28</td>
<td>72</td>
</tr>
</tbody>
</table>

Table 1. Maximal subgroups of \( A_5 \). Table 2. Maximal subgroups of \( \text{PSL}_2(8) \).

**Lemma 3.3.** Let \( K \) be a perfect group and \( M \) a minimal normal subgroup of \( K \) that is an elementary abelian 2-group. Suppose that \( M \) is non-central in \( K \) and \( K/M \) is isomorphic to \( L_2(8) \) or \( A_5 \). Then \( K \) has an irreducible non-linear real character with odd composite degree.

**Proof.** Since \( G/M \) is simple, we have that \( C_G(M) = M \). Suppose that \( K/M \) is isomorphic to \( A_5 \). There are two non-isomorphic irreducible \( A_5 \)-modules \( W_1, W_2 \) of \( A_5 \) over \( \text{GF}(2) \). Both have dimension 4, and \( H^2(A_5, W_1) = H^2(A_5, W_2) = 0 \).

Hence \( M \) has a complement \( S \) in \( K \). It is easy to construct these groups, and we see that \( K = M \rtimes S = W_i \rtimes A_5 \) has a real irreducible character of degree 15. Suppose now that \( K/M \cong L_2(8) \). Let \( W_1, W_2, W_3 \) be the non-isomorphic irreducible \( L_2(8) \)-modules over \( \text{GF}(2) \), where \( \dim(W_1) = 6, \dim(W_2) = 8, \dim(W_3) = 12 \).

If \( M \cong M_i \) with \( i = 2, 3 \), then \( H^2(L_2(8), W_i) = 0 \), and hence \( M_i \) has a complement \( S \) in \( K \). Then, as before, we conclude by observing that \( W_i \rtimes L_2(8) \) has a real irreducible character of degree 63. Suppose that \( M \cong W_1 \). Then we have \( \dim H^2(L_2(8), W_1) = 3 \). Nevertheless, there are just two perfect groups of order \( 2^6 \cdot |L_2(8)| \). Both these groups have an irreducible real character of degree 63. \( \Box \)

In the previous lemma, dimensions of cohomology groups and all the perfect groups of a given order is information that is accessible with the \text{GAP} functions \text{cohomolo} and \text{PerfectGroup}.
Proposition 3.4. Let $G$ be a finite non-solvable group, and suppose that $\text{cd}_{tv}(G)$ consists of prime-power numbers. Let $K = G^{(\infty)}$ and $R = \text{Rad}(G)$. Then we have $|K \cap R| \leq 2$, and if equality holds, then $K \cong \text{SL}_2(5)$.

Proof. By Proposition 3.2, we have that $N = K \cap R$ is a 2-group. We prove that if $N > 1$, then $|N| = 2$ and $K$ is isomorphic to $\text{SL}_2(5)$. Let $V = N/\Phi(N)$; then $V$ is a normal elementary abelian 2-subgroup of $G/\Phi(N)$. Let $V > V_1 > \cdots > V_n$ be a $K$-principal series of $V$. Let $N > N_1 > \cdots > N_n$ be such that $N_i$ is the preimage in $N$ of $V_i$. Then $N/N_1$ is an irreducible $K/N$-module, and $K/N$ is isomorphic $A_5$ or $L_2(8)$ by Proposition 3.2. By Lemmas 3.3 and 2.7, $N/N_1$ is central in $K/N_1$. Since $K$ is perfect, we have that $N/N_1$ is isomorphic to a subgroup of the Schur multiplier $M(K/N)$. The only possibility is $|N/N_1| = 2$ and $K/N_1 \cong \text{SL}_2(5)$, the Schur covering of $A_5$. Suppose by contradiction that $N_1/N_2 > 1$; write $\bar{K} = K/N_2$. Since $M(\text{SL}_2(5)) = 1$, $\bar{N}_1$ cannot be central in $\bar{K}$. Let $t \in K$ be a 2-element such that $\langle tN_1 \rangle = Z(K/N_1)$, namely the unique central involution in $\text{SL}_2(5)$ and $\langle tN_1 \rangle = O_2(K/N_1)$. Since $N_1$ is an irreducible module over $\text{GF}(2)$, we have that $t$ acts trivially on $\bar{N}_1$. Suppose that $\bar{t}^2 \neq 1$; then $\langle \bar{t}^2 \rangle$ would be a proper, non-trivial submodule of $\bar{N}_1$, against irreducibility. This means that $\bar{t}^2 = 1$ and hence $\langle \bar{t} \rangle$, which centralizes $\bar{N}_2$, is a minimal normal subgroup of $\bar{N}_2$. Observe that $\bar{K}/\langle t \rangle$ is a quotient of $K$ that satisfies the hypotheses of Lemma 3.3. Hence, by Lemma 2.7, we derive a contradiction. □

We now prove Theorem A, which we restate for convenience of the reader.

Theorem 3.5. Let $G$ be a finite non-solvable group, and suppose that $\text{cd}_{tv}(G)$ consists of prime-power numbers. Then $\text{Rad}(G) = H \times O$ for a group $O$ of odd order and a 2-group $H$ of Chillag–Mann type. Furthermore, if $K = G^{(\infty)}$, then one of the following holds:

(i) $G = K \times R$, and $K$ is isomorphic to $A_5$ or $L_2(8)$;

(ii) $G = (KH) \times O$, where $K \cong \text{SL}_2(5)$, $HK = H \gamma K$, $K \cap H = Z(K) < H$.

Proof. By Proposition 3.4 and Proposition 3.2, if $K = G^{(\infty)}$ and $R = \text{Rad}(G)$, then $G = KR$, and either $K \cap R = 1$ and $K$ is simple isomorphic to $A_5$ or $L_2(8)$ or $K \cong \text{SL}_2(5)$ and $K \cap R = Z(K)$. In the first case, point (i) follows. Suppose $K = \text{SL}_2(5)$ and $K \cap R = Z(K) = Z$. Note that $Z$ is a normal subgroup of order 2, hence is central in $R$. Consider $\bar{G} = G/Z$. Then $\bar{G} = \bar{K} \times \bar{R}$, and hence $\bar{R}$ is a group of Chillag–Mann type since $\bar{K}$ is simple and has irreducible real non-linear characters of both odd and even degree. This means that $\bar{R} = \bar{O} \times \bar{H}$ for $O \in \text{Hall}_2(R)$ and $H \in \text{Syl}_2(R)$; note that $\bar{H}$ is of Chillag–Mann type. We have
that \( R \) is 2-closed, and hence \( R = H \rtimes O \). Clearly, \( O \) acts trivially on \( H/Z \), so

\[
H = C_H(O)Z \leq C_H(O)Z(R) \cap H.
\]

It follows that \( O \) centralizes \( H \) and \( R = H \rtimes O \). By the Dedekind modular law, \( HK \cap O \leq HK \cap R \leq H(K \cap R) \leq H \), and hence \( HK \cap O \leq H \cap O = 1 \). This means that \( G = (KH) \times O \) and \( K \cap H = Z \), which has order 2. If \( H \) and \( K \) commute, then \( KH = K \gamma H \). Suppose by contradiction that \( [H, K] = Z \); hence there is \( aZ \in H/Z \) that acts non-trivially by conjugation on \( K/Z \). But this is impossible since \( KH/Z = \tilde{K} \times \tilde{H} \). We now prove that \( H \) is of Chillag–Mann type. Suppose that this is not the case, so there is \( \phi \in \text{Irr}_{rv}(H) \) such that \( \phi(1) > 1 \). Since \( \tilde{H} \) is of Chillag–Mann type, we have that \( Z \notin \ker \phi \), so \( \phi_Z = \phi(1)\lambda \), with \( \lambda \neq 1_Z \). On the other hand, if \( \theta \) is the unique character of \( K \) of degree 6, then \( Z \notin \ker \theta \) and \( \theta_Z = \theta(1)\lambda \). Now, \( KH = K \times H/N \), where \( N = \{(z,z) \mid z \in Z\} \) (see [5, I9.10]) and \( \psi = \theta \times \phi \in \text{Irr}_{rv}(K \times H) \). Moreover,

\[
\psi_N = \phi(1)\theta(1)\lambda^2 = \phi(1)\theta(1)1_N,
\]

so it follows that \( N \leq \ker \psi \) and \( \psi \in \text{Irr}_{rv}(KH) \). If \( \chi = \psi \times 1_O \), then \( \chi \in \text{Irr}(G) \) takes real values and has composite degree, which is impossible. Since \( \text{SL}_2(5) \) does not satisfy the hypotheses, we have that \( Z < H \). Point (ii) follows.

We remark that, in [6, Problem 4.4], we can find a stronger version of the argument used in the proof above.

As a consequence, we get Theorem B.

**Corollary 3.6.** Let \( G \) be a non-solvable group, and suppose that \( \text{cd}_{rv}(G) \) consists of prime-power numbers. Then either

\[
\text{cd}_{rv}(G) = \text{cd}_{rv}(L_2(8)) \quad \text{or} \quad \text{cd}_{rv,2'}(G) = \text{cd}_{rv,2'}(A_5).
\]

**Proof.** Apply Theorem 3.5. In case (i), there is nothing to prove. Suppose (ii). Then we have that \( G = (KH) \times O \) with \( O \) of odd order, \( K = G^{(\infty)} \) and \( H \) a normal 2-subgroup. Let \( S \) denote the simple section \( KH/H \); hence \( S \simeq A_5 \). Take \( \chi \in \text{Irr}_{rv}(G) \) to be a real non-linear character of odd degree. Hence \( \chi(1) = p^n \) with \( p \) odd and \( \chi \) is a character of \( HK \) since, by Lemma 2.5, \( O \leq \ker(\chi) \). The degree of every irreducible constituent of \( \chi_H \) divides \( |H|, \chi(1) = 1 \), and hence \( \chi_H = e \sum \lambda_i \) for \( \lambda_i \in \text{Lin}(H) \). By hypothesis, we have that \( \chi(1) = p^* > 1 \) for an odd prime \( p \), and by [6, Corollary 11.29], \( \chi(1)/\lambda(1) \) divides \( |HK:H| = |S| \), where \( S \simeq A_5 \). Hence \( p \leq \chi(1) \leq |S|p \), the \( p \)-part of the number \( |S| \), which is equal to \( p \) if \( p \) is an odd prime. It follows that \( \chi(1) = p \). We have proved that \( \text{cd}_{rv,2'}(G) \subseteq \{3,5\} = \text{cd}_{rv,2'}(A_5) \). The right-to-left inclusion follows by observing that \( A_5 \) is a quotient of \( G \).
Acknowledgments. The author would like to thank Professor Silvio Dolfi for his suggestions and comments to the first manuscript.

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Received July 21, 2021; revised December 9, 2021

Author information

Corresponding author:
Lorenzo Bonazzi, Department of Mathematics and Informatics, University of Florence,
Viale Giovanni Battista Morgagni, 67/a, 50134 Firenze, FI, Italy.
E-mail: lorenzo.bonazzi@unifi.it