Projective representations of Heisenberg groups over the rings of order $p^2$

Sumana Hatui, E. K. Narayanan and Pooja Singla*

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Abstract. We describe the 2-cocycles, Schur multiplier and representation group of discrete Heisenberg groups over the unital rings of order $p^2$. We also describe all projective representations of Heisenberg groups with entries from the rings $\mathbb{Z}/p^2\mathbb{Z}$ and $\mathbb{F}_p[t]/(t^2)$ for odd primes $p$ and obtain a classification of their degenerate and non-degenerate 2-cocycles.

1 Introduction

The theory of projective representations of finite groups was first studied by Schur in a series of papers [16–18]. A projective representation of a group $G$ is a homomorphism from $G$ to the projective general linear group $\text{PGL}(V)$, where $V$ is a complex vector space. A projective representation is thus a map $\rho: G \to \text{GL}(V)$ such that $\rho(1) = \text{Id}_V$ and there is a 2-cocycle $\alpha: G \times G \to \mathbb{C}^\times$ satisfying

$$\rho(xy) = \alpha(x,y)\rho(x)\rho(y), \quad x, y \in G.$$ 

In this case, we say $\rho$ is an $\alpha$-representation. An $\alpha$-representation is also frequently called a projective representation with factor set $\alpha$.

For cyclic groups, the irreducible projective representations are the same as the ordinary representations (up to equivalence) and hence one-dimensional. However, this is not true in general for abelian groups. Several authors have investigated the case of abelian groups; see [6, 14, 15].

Recently, the first- and third-named authors studied the projective representations of the discrete Heisenberg group over cyclic rings in [5]. This paper may be considered as a continuation of [5], where we study the projective representations of the discrete Heisenberg groups over unital rings $R$ of order $p^2$, where $p$...
is an odd prime. The discrete Heisenberg group of rank one over $R$, denoted by $H_{2n+1}(R)$, is the set $R \times R^n \times R^n$ with the multiplication given by

$$(c_1, b_1, \ldots, b_n, a_1, \ldots, a_n)(c_1', b_1', b_2', \ldots, b_n', a_1', a_2', \ldots, a_n') = (c_1 + c_1' + \sum_{i=1}^{n} a_i \ast b_i', a_1 + b_1', \ldots, b_n + b_n', b_1 + a_1', \ldots, a_n + a_n'),$$

where $a \ast b$ denotes the multiplication in $R$.

From now onwards, we consider $R$ to be a ring of order $p^2$ with unity. It is easy to prove that any such ring $R$ is commutative and is isomorphic to one of the following:

- $\mathbb{Z}/p^2\mathbb{Z}$,
- $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$,
- $\mathbb{F}_p[t]/(t^2)$,
- $\mathbb{F}_{p^2}$.

We note that the results in [5] account for the case $H_{2n+1}(R)$ when $n > 1$. Indeed, for $n > 1$, every irreducible projective representation of $H_{2n+1}(R)$ (where $R$ is any commutative ring) is obtained via inflation from an irreducible projective representation of the abelian group $R^{2n}$. As a result, we will focus on the case $n = 1$, i.e. on $H_3(R)$, in this article.

In the study of projective representations of a finite group $G$, the main ingredients are to describe the Schur multiplier of $G$, to determine a representation group $G^*$ of $G$ and then describe the ordinary representations of $G^*$. We refer the reader to [11, 12] for any unexplained terms or notation in this article. Following [2], we say a 2-cocycle $\alpha \in Z^2(G, \mathbb{C}^\times)$ is non-degenerate if the twisted group algebra $\mathbb{C}^\alpha[G]$ is a simple algebra. If a non-degenerate 2-cocycle exists, the group $G$ is said to be of central type. These groups play an important role in classifying the semisimple triangular complex Hopf algebras; see [4]. Now let $R$ be one of the aforementioned rings of order $p^2$. The main results obtained in this article can be summarized as follows:

(a) description of the Schur multiplier $H^2(H_3(R), \mathbb{C}^\times)$,

(b) explicit description of the 2-cocycles of $H_3(R)$,

(c) construction of a representation group $G^*$ for $H_3(R)$ and its irreducible representations,

(d) description of the non-degenerate 2-cocycles of $H_3(\mathbb{F}_p[t]/(t^2)), H_3(\mathbb{Z}/p^2\mathbb{Z})$. 

We now provide statements of our results. Our first result gives

$$H^2(\mathbb{H}_3(\mathbb{F}_p[t]/(t^r)), \mathbb{C}^\times)$$

for an odd prime $p$ and $r \geq 2$. For $r = 1$, this result is already known in the literature; see [12, Theorem 3.3.6].

**Theorem 1.1.** Assume that $p$ is an odd prime and $r \in \mathbb{N}$. Then

$$H^2(\mathbb{H}_3(\mathbb{F}_p[t]/(t^r)), \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^2r^2.$$  

We next describe $H^2(\mathbb{H}_3(R), \mathbb{C}^\times)$, where $R$ is a unital ring of order $p^2$ and $p$ is an odd prime. For $R = \mathbb{Z}/p^2\mathbb{Z}$, $H^2(\mathbb{H}_3(R), \mathbb{C}^\times) \cong (\mathbb{Z}/p^2\mathbb{Z})^2$ by [10, Theorem 1.1]. For $R = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, we use $H^2(\mathbb{H}_3(\mathbb{Z}/p\mathbb{Z}), \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^2$ (see [12, Theorem 3.3.6]) and the following result due to Schur (see [12, Theorem 2.2.10]) to describe $H^2(\mathbb{H}_3(R), \mathbb{C}^\times)$.

For finite groups $G_1$ and $G_2$,

$$H^2(G_1 \times G_2, \mathbb{C}^\times) \cong H^2(G_1, \mathbb{C}^\times) \times H^2(G_2, \mathbb{C}^\times) \times \text{Hom}\left(\frac{G_1}{G'_1} \otimes \frac{G_2}{G'_2}, \mathbb{C}^\times\right),$$

where $G'_1$ and $G'_2$ denote the commutator subgroups of $G_1$ and $G_2$ respectively. Therefore, $H^2(\mathbb{H}_3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}), \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^8$. The following result describes $H^2(\mathbb{H}_3(R), \mathbb{C}^\times)$ for the remaining cases of $R$.

**Theorem 1.2.** Suppose $p$ is an odd prime. Then

$$H^2(\mathbb{H}_3(\mathbb{F}_p[t]/(t^2)), \mathbb{C}^\times) \cong H^2(\mathbb{H}_3(\mathbb{F}_p^{p^2}), \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^8.$$  

The proofs of the above results are given in Section 3 and mainly use the results of Blackburn and Evens [3]. Since $\mathbb{H}_3(\mathbb{F}_p[t]/(t^r))$ and $\mathbb{H}_3(\mathbb{F}_p^{p^2})$ are $p$-groups of nilpotency class 2 with the property that their abelianization is elementary abelian, we can apply the main results from [3]. The hypothesis of $p \neq 2$ is used in our proofs because (a) this gives $\mathbb{F}_p^{p^2} \cong \mathbb{F}_p[t]/(t^2 - k)$ for some $k \in \mathbb{F}_p^{p^2} \setminus \mathbb{F}_p$ and (b) the groups $\mathbb{H}_3(R)$ are of exponent $p$, so we have easy presentations to work with. As a different treatment is required for the $p = 2$ case, we assume $p \neq 2$ throughout this article. From now onwards, we will fix $k$ such that $\mathbb{F}_p^{p^2} \cong \mathbb{F}_p[t]/(t^2 - k)$ and use this wherever required.

Our next result is the description of the 2-cocycles of the group $\mathbb{H}_3(R)$. We prove the following for $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2))$. 

Theorem 1.3. Every 2-cocycle of $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2))$ is cohomologous to the following cocycles:

$$\alpha(x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}, x_1^{a'_1}x_2^{a'_2}y_1^{b'_1}y_2^{b'_2}) = \mu_1^{a_2a'_1} \mu_2^{b_2a'_1} \mu_3^{b_2b'_1} \mu_4^{b_2b'_1} \mu_5^{b_1(b'_2)} + a'_1c_1$$

$$\mu_6^{b_2(a'_1)} + a'_1c_2 + a'_2c_1 \mu_7^{a'_1(b_1)} - b_1c'_1 \mu_8^{a'_2(b_1)} - b_2c'_1 - b_1c'_2,$$

where $\mu_i \in \mathbb{C}^\times$ such that $\mu_i^p = 1$. Furthermore, any two distinct cocycles of the above form are not cohomologous to each other.

Here

$$x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}, z_1^{z_1}z_2^{z_2}$$

denotes a general element in the group $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2))$. See Section 2.2 for more details. A description of the 2-cocycles of $\mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z})$ already appeared in [5]. For other groups, a description of the 2-cocycles is given in Section 5.

As is well known, the projective representations of a group are obtained from ordinary representations of its representation group. To study the projective representations of $\mathbb{H}_3(R)$, we construct a representation group of $\mathbb{H}_3(R)$. In this direction, we have the following result for $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2))$. For a group $G$ and $x, y \in G$, the commutator $x^{-1}y^{-1}xy$ is denoted by $[x, y]$. Whenever we write a presentation of a group, we assume all the commutators $[x, y]$ for generators $x, y$, which are not explicitly stated in the presentation, are trivial.

Theorem 1.4. A representation group of $G = \mathbb{H}_3(\mathbb{F}_p[t]/(t^2))$ is given by

$$G^* = \langle x_i, y_i, i = 1, 2 \mid [y_1, x_1] = z_1, [y_1, x_2] = z_2, [x_2, x_1] = w_1, [y_2, x_1]z_2^{-1} = w_2, [y_2, x_2] = w_3, [y_2, y_1] = w_4, [z_1, x_i] = u_i, [z_1, y_i] = u_i, [z_2, x_1] = [z_1, x_2], [z_2, y_1] = [z_1, y_2], x_i^p = y_i^p = z_i^p = w_i^p = 1, 1 \leq j \leq 4 \rangle.$$

The parallel results for $\mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z})$ have already appeared in [5]. For other groups, a description of their representation group is included in Section 4. In Section 6, we give a construction of all projective representations of $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2))$ and $\mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z})$. We use our construction of projective representations to classify the degenerate and non-degenerate cocycles of $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2))$ and $\mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z})$. More specifically, we prove the following result.
Theorem 1.5. For a finite group \( G \), let
\[
X(G) = \{ [\alpha] \in \mathsf{H}^2(G, \mathbb{C}^\times) \mid [\alpha] \text{ is non-degenerate} \}.
\]

Then
\[
|X(G)| = \begin{cases} 
p^2(p-1)^2, & G = \mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z}), 
p^5(p-1)^2(p+1), & G = \mathbb{H}_3(\mathbb{F}_p[t]/(t^2)). \end{cases}
\]

For a proof of the above, see Section 6. We obtain the above from a construction of the irreducible representations of \( G^* \); see Sections 6.2 and 6.4.

As an application of this result, we compute certain \( 1 \)-cocycles of \( \mathbb{H}_3(\mathbb{F}_p) \). Consider \( G = \mathbb{H}_3(\mathbb{F}_p[t]/(t^2)) \) and the following exact sequence obtained via the natural projection map from \( G \) onto \( \mathbb{H}_3(\mathbb{F}_p) \):
\[
1 \to \langle x_2, y_2, z_2 \rangle \cong (\mathbb{F}_p)^3 \to G \to \mathbb{H}_3(\mathbb{F}_p) \to 1.
\]

Let \( Q = \mathbb{H}_3(\mathbb{F}_p) \) and \( A = \langle x_2, y_2, z_2 \rangle \). Then \( G \) is a semidirect product of \( Q \) by \( A \). The following result describes \( \mathsf{H}^1(Q, \hat{A}) \) and enumerates its bijective classes. The elements of \( \mathsf{H}^1(Q, \hat{A}) \) can also be described explicitly; see Remark 6.2.

Corollary 1.6. With the above notation, the following hold.

1. \( \mathsf{H}^1(Q, \hat{A}) \cong (\mathbb{Z}/p\mathbb{Z})^5 \).

2. The number of bijective classes in \( \mathsf{H}^1(Q, \hat{A}) \) is \( p^2(p-1)^2(p+1) \).

Combining the results of [2] with a description of the elements of
\[
\mathsf{H}^2(\mathbb{H}_3(\mathbb{F}_p[t]/(t^2)), \mathbb{C}^\times)
\]
yields a proof of the above; see Section 6.

2 Preliminaries

In this section, we fix the notation and recall a few results which will be used in the upcoming sections. The center and commutator subgroup of \( G \) are denoted by \( Z(G) \) and \( G' \) respectively.

A central extension
\[
1 \to A \to G \to G/A \to 1 \tag{2.1}
\]
is called a stem extension if \( A \subseteq Z(G) \cap G' \). For a central extension (2.1), the map \( \operatorname{tra}: \text{Hom}(A, \mathbb{C}^\times) \to \mathsf{H}^2(G/A, \mathbb{C}^\times) \) given by \( f \mapsto [\operatorname{tra}(f)] \),
\[
\operatorname{tra}(f)(\tilde{x}, \tilde{y}) = f(\mu(\tilde{x})\mu(\tilde{y})\mu(\tilde{x}\tilde{y})^{-1}), \quad \tilde{x}, \tilde{y} \in G/A,
\]
for a section $\mu: G/A \to G$, is a group homomorphism and is called the transgression homomorphism. The map

$$\inf: H^2(G/A, \mathbb{C}^\times) \to H^2(G, \mathbb{C}^\times)$$

given by $[\alpha] \mapsto [\inf(\alpha)]$, where $\inf(\alpha)(x, y) = \alpha(xA, yA)$, is a group homomorphism and is called the inflation homomorphism. By [7, Theorem 2, p. 129] and [9, Proposition 1.1], the spectral sequence for cohomology of groups yields the following exact sequence:

$$1 \longrightarrow \text{Hom}(A, \mathbb{C}^\times) \xrightarrow{\text{tra}} H^2(G/A, \mathbb{C}^\times) \xrightarrow{\inf} H^2(G, \mathbb{C}^\times) \xrightarrow{(\text{res}, \chi)} H^2(A, \mathbb{C}^\times) \oplus \text{Hom}(G/G' \otimes A, \mathbb{C}^\times),$$

where the map $\chi$, defined by Iwahori and Matsumoto [9], is given by

$$\chi([\alpha])(gG' \otimes a) = \alpha(g, a)\alpha(a, g)^{-1} \quad \text{for } g \in G, a \in A.$$

**Lemma 2.1** (Hall–Witt identity). Let $G$ be a finite group of nilpotency class 3. For $x, y, z \in G$, we have $[x, y^{-1}, z][y, z^{-1}, x][z, x^{-1}, y] = 1$.

### 2.1 Projective representations of a finite group

In this section, we include the results that we require regarding the projective representations of a finite group. We will use these in Section 6.

Let $G$ be a finite group. We use $Z^2(G, \mathbb{C}^\times)$ to denote the set of all 2-cocycles of $G$. For $\alpha \in Z^2(G, \mathbb{C}^\times)$, we use $\text{Irr}^\alpha(G)$ to denote the set of equivalence classes of irreducible $\alpha$-representations of $G$. For $\alpha = 1$, we use $\text{Irr}(G)$ instead of $\text{Irr}^\alpha(G)$ and call this the set of ordinary irreducible characters of $G$. Let $G^*$ be a representation group of $G$ with $A \cong H^2(G, \mathbb{C}^\times)$ such that

$$1 \to A \to G^* \to G \to 1$$

is a stem extension. The existence of such $G^*$ follows from [12, Theorem 2.1.4]. Let $\text{Irr}(G|\chi)$ denote the set of inequivalent irreducible representations of $G$ lying above $\chi$, that is $\rho \in \text{Irr}(G|\chi)$ if and only if $\text{Hom}_N(\rho|N, \chi)$ is non-trivial. The following well-known result relates the projective representations of $G$ and the ordinary ones of $G^*$.

**Theorem 2.2.** Let $\alpha$ be a 2-cocycle of $G$. Suppose that $\chi \in \text{Hom}(A, \mathbb{C}^\times)$ is such that $\text{tra}(\chi) = [\alpha]$. Then there is a bijective correspondence between

$$\text{Irr}^\alpha(G) \leftrightarrow \text{Irr}(G^*|\chi)$$
obtained via lifting a projective representation of $G$ to an ordinary representation of $G^\ast$. In particular, we obtain the following:

$$\bigcup_{[\alpha] \in H^2(G, \mathbb{C}^\times)} \text{Irr}^\alpha(G) \leftrightarrow \text{Irr}(G^\ast).$$

A proof of this follows from the proof of [11, Chapter 3, Section 3]; see also [5, Theorem 3.2]. To determine the projective representations of $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2))$ and $\mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z})$, it therefore suffices to determine the ordinary representations of their representation groups. We now discuss a method that will work in our situation.

In this direction, we first recall the results of Clifford theory regarding the ordinary characters of a finite group. For proofs, see [8, Theorem 6.11, Corollary 6.17].

**Theorem 2.3.** Let $G$ be a finite group and $N$ a normal subgroup. For any irreducible representation $\rho$ of $N$, let $I_G(\rho) = \{g \in G \mid \rho^g \cong \rho\}$ denote the stabilizer of $\rho$ in $G$. Then the following hold.

(a) The map

$$\theta \mapsto \text{Ind}^G_{I_G(\rho)}(\theta)$$

is a bijection of $\text{Irr}(I_G(\rho) \mid \rho)$ onto $\text{Irr}(G \mid \rho)$.

(b) Let $H$ be a subgroup of $G$ containing $N$, and suppose that $\rho$ is an irreducible representation of $N$ which has an extension $\tilde{\rho}$ to $H$ (i.e. $\tilde{\rho}|_N = \rho$). Then the representations $\delta \otimes \tilde{\rho}$ for $\delta \in \text{Irr}(H/N)$ are irreducible, distinct for distinct $\delta$ and

$$\text{Ind}^H_N(\rho) = \bigoplus_{\chi \in \text{Irr}(H/N)} \chi \otimes \tilde{\rho}.$$

Let $G$ be a finite group with an abelian normal subgroup $N$ such that $G/N$ is abelian. Let $\chi : N \to \mathbb{C}^\times$ be a one-dimensional representation of $N$ and let

$$I_G(\chi) = \{g \in G \mid \chi^g = \chi\}$$

be the inertia group of $\chi$ in $G$. By Theorem 2.3, the problem of determining $\text{Irr}(G \mid \chi)$ reduces to that of determining $\text{Irr}(I_G(\chi) \mid \chi)$. We now mention a method that helps us to determine $\text{Irr}(I_G(\chi) \mid \chi)$ for every $\chi$.

Let $K$ be a finite group with an abelian normal subgroup $N$ such that $K/N$ is abelian. Let $\chi : N \to \mathbb{C}^\times$ be a one-dimensional representation of $N$ such that $\chi^k = \chi$ for all $k \in K$, that is $I_K(\chi) = K$. Let $T$ be a fixed set of left coset representatives of $N$ in $K$. Define a map $\chi' : K \to \mathbb{C}^\times$ by $\chi'(kn) = \chi(n)$ for all $k \in T$ and $n \in N$. Following [8, Chapter 11], let $\alpha \in Z^2(K, \mathbb{C}^\times)$ be a 2-cocycle of $K$ associated to $\chi'$ and let $\beta \in Z^2(K/N, \mathbb{C}^\times)$ be defined by $\beta(gN, hN) = \alpha(g, h)$ for $g, h \in K$. 
Lemma 2.4. The following are equivalent.

(1) The character \( \chi \) extends to \( K \).
(2) \([K, K] \subseteq \text{Ker}(\chi)\).
(3) \([\beta] = 1\).

Proof. The equivalence of (1) and (2) follows from the fact that \([K, K] \subseteq N\) and \(N/[K, K]\) is an abelian group. The equivalence of (1) and (3) follows from [8, Theorem 11.7].

With the above notation, the following result can be obtained from [11, Theorem 4.2, Chapter 6].

Lemma 2.5. There is a dimension-preserving bijection between the sets \( \text{Irr}(K|\chi) \) and \( \text{Irr}^{\beta^{-1}}(K/N) \).

Hence it boils down to understanding the projective representations of the quotient group \( K/N \). For our case, this quotient group will turn out to be an abelian group. The projective representations of abelian groups are well studied, and we use some of these results. In particular, we will use the following lemma without further reference.

Lemma 2.6. Let \( G \) be a finite abelian group. Then the following hold.

(1) For any fixed \( \alpha \in Z^2(G, \mathbb{C}^\times) \), all irreducible representations in \( \text{Irr}^\alpha(G) \) have equal dimension.
(2) For \( |G| \in \{p^2, p^3\} \) and \( 1 \neq [\alpha] \in H^2(G, \mathbb{C}^\times) \), every \( \rho \in \text{Irr}^\alpha(G) \) has dimension \( p \).
(3) For \( |G| = p^4 \) and \( 1 \neq [\alpha] \in H^2(G, \mathbb{C}^\times) \), every \( \rho \in \text{Irr}^\alpha(G) \) has dimension either \( p \) or \( p^2 \).

Proof. Here (1) follows from [1, Theorem 1]. Assertions (2) and (3) follow from the fact that any \( \rho \in \text{Irr}^\beta(G) \) for \([\beta] \neq 1\) satisfies

\[
1 < \text{dim}(\rho) \leq \sqrt{|G|}
\]

and \( \text{dim}(\rho) \) divides the order of \( G \).

At this point, we mention the ways to identify the non-degenerate cocycles of a group \( G \) by using the ordinary representations of its representation group \( G^* \). Consider the stem extension (2.3). Let \( \alpha \) be a 2-cocycle of \( G \) and \( \chi \in \text{Hom}(A, \mathbb{C}^\times) \) such that \( \text{tra}(\chi) = [\alpha] \).
Lemma 2.7. The following conditions are equivalent.

(1) The 2-cocycle $\alpha$ is non-degenerate.

(2) $|\text{Irr}^\alpha(G)| = 1$.

(3) There exists $\rho \in \text{Irr}^\alpha(G)$ such that $\dim(\rho) = \sqrt{|G|}$.

(4) There exists $\rho \in \text{Irr}(G^*|\chi)$ such that $\dim(\rho) = \sqrt{|G|}$.

(5) $|\text{Irr}(G^*|\chi)| = 1$.

Proof. By definition, a 2-cocycle $\alpha$ is non-degenerate if and only if $G$ has a unique irreducible $\alpha$-character if and only if the group algebra $\mathbb{C}^\alpha[G]$ is simple. The equivalence of (1), (2) and (3) follows from this. Equivalence of (2) and (5) as well as of (3) and (4) follows from Theorem 2.2. □

We use this result to classify the non-degenerate cocycles of $H_3(\mathbb{F}_p[t]/(t^2))$ and $H_3(\mathbb{Z}/p^2\mathbb{Z})$ in Section 6.

2.2 Presentation and matrix form of groups

In this section, we give a presentation of $H_3(\mathbb{F}_p[t]/(t^r))$ and $H_3(\mathbb{F}_p[t]/(t^2 - k))$ that we use throughout this article.

The groups $H_3(\mathbb{F}_p[t]/(t^r))$ for $r \geq 1$ are of nilpotency class 2 and have the following presentation:

$$H_3\left(\mathbb{F}_p[t]\big/(t^r)\right) = \langle x_m, y_m, 1 \leq m \leq r \mid [y_j, x_{i-j+1}] = z_i, x^p_m = y^p_m = 1, 1 \leq i \leq r, 1 \leq j \leq i \rangle.$$

As mentioned earlier, $\mathbb{F}_{p^2} \cong \mathbb{F}_p[t]/(t^2 - k)$ for $p \neq 2$ and $k \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$. Therefore,

$$H_3(\mathbb{F}_{p^2}) \cong H_3\left(\mathbb{F}_p[t]\big/(t^2 - k)\right) \text{ for } p \neq 2.$$

We have the following presentation of $H_3(\mathbb{F}_p[t]/(t^2 - k))$:

$$H_3\left(\mathbb{F}_p[t]\big/(t^2 - k)\right) = \langle x_1, x_2, y_1, y_2 \mid [y_1, x_1] = z_1, [y_2, x_2] = z_1^k, [y_1, x_2] = [y_2, x_1] = z_2, x^p_i = y^p_i = 1, i = 1, 2 \rangle.$$

It is helpful to think of $H_3(R)$ in its matrix form, that is as a group of $3 \times 3$ matrices with entries from the ring $R$. In the above two presentations, the element

$$x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2} z_1^{c_1} z_2^{c_2}$$
corresponds to
\[
\begin{pmatrix}
1 & (b_1, b_2) & (c_1, c_2) \\
0 & 1 & (a_1, a_2) \\
0 & 0 & 1
\end{pmatrix}
\]
in the matrix form of \(H_3(R)\).

### 2.3 Schur multiplier of certain \(p\)-groups

Let \(G\) be a \(p\)-group of nilpotency class 2 with \(G/G'\) elementary abelian. In this section, we recall the theory given in [3, Section 3] to compute the Schur multiplier of \(G\).

Consider \(G/G'\) and \(G'\) as vector spaces over \(\mathbb{F}_p\), denoted by \(V\) and \(W\) respectively. For \(g_1, g_2, g_3 \in G\), let \(U_1\) be the subspace of \(V \otimes W\) spanned by the elements of the form
\[\bar{g}_i \otimes [g_2, g_3] + \bar{g}_2 \otimes [g_3, g_1] + \bar{g}_3 \otimes [g_1, g_2],\]
where \(\bar{g}_i = g_i G' \in V\) for \(i = 1, 2, 3\). Let \(U_2\) be the subspace of \(V \otimes W\) spanned by all \(\bar{g}_1 \otimes g_1^p\). Now consider \(U = U_1 + U_2\).

We have the following result from [3, Theorem 3.1].

**Proposition 2.8.** Let \(G\) be a \(p\)-group of nilpotency class 2 such that \(G/G'\) is elementary abelian. Then
\[|H^2(G, \mathbb{C}^\times)| = \left| \frac{V \wedge V}{W} \right| \left| \frac{V \otimes W}{U} \right|.\]

We will continue to use this result in the next section.

**Lemma 2.9.** Let \(p\) be an odd prime and \(G\) a \(p\)-group of exponent \(p\), of nilpotency class 2. Then \(H^2(G, \mathbb{C}^\times)\) is an elementary abelian \(p\)-group.

**Proof.** If \(G\) has a free presentation \(F/\mathcal{R}\), then
\[H^2(G, \mathbb{C}^\times) \cong H_2(G, \mathbb{Z}) \cong \frac{F' \cap \mathcal{R}}{[F, \mathcal{R}]},\]
from [12, Theorem 2.4.6]. For \(x, y \in F\) and for odd \(p\),
\[1 \equiv [x^p, y] \equiv [x, y]^p ([x, y], x)^{\binom{p}{2}} \equiv [x, y]^p \pmod{[F, \mathcal{R}]}\]
implies that \((F')^p \subset [F, \mathcal{R}]\). Therefore, for any \(x \in F' \cap \mathcal{R}\), \(x^p \in [F, \mathcal{R}]\). Since
\[H^2(G, \mathbb{C}^\times) \cong \frac{(F' \cap \mathcal{R})}{[F, \mathcal{R}]},\]
\(H^2(G, \mathbb{C}^\times)\) is an elementary abelian \(p\)-group. \(\square\)
The following result will help us to describe the 2-cocycles of \( \mathbb{H}_3(\mathbb{F}_p[t]/(t^2)) \).

**Proposition 2.10.** Let \( p \) be an odd prime, \( G \) a \( p \)-group of exponent \( p \) and of nilpotency class 2 such that there is a central subgroup \( Z \subseteq Z(G) \cap G' \) with the property

\[
|H^2(G, \mathbb{C}^\times)| = \left| \frac{H^2(G/Z, \mathbb{C}^\times)}{\text{Hom}(Z, \mathbb{C}^\times)} \right| |\text{Hom}(G/G' \otimes Z, \mathbb{C}^\times)|.
\]

Then

\[
H^2(G, \mathbb{C}^\times) \cong \frac{H^2(G/Z, \mathbb{C}^\times)}{\text{Hom}(Z, \mathbb{C}^\times)} \times \text{Hom}(G/G' \otimes Z, \mathbb{C}^\times).
\]

**Proof.** By (2.2), we have the following exact sequence:

\[
1 \rightarrow \frac{H^2(G/Z, \mathbb{C}^\times)}{\text{Hom}(Z, \mathbb{C}^\times)} \rightarrow H^2(G, \mathbb{C}^\times) \rightarrow \text{Hom}(G/G' \otimes Z, \mathbb{C}^\times) \rightarrow 1.
\]

The group \( H^2(G, \mathbb{C}^\times) \) is an elementary abelian \( p \)-group by Lemma 2.9. Therefore, the result follows. \( \square \)

### 3 Schur multiplier of \( \mathbb{H}_3(\mathbb{F}_p[t]/(t^r)) \) and \( \mathbb{H}_3(\mathbb{R}) \)

In this section, we prove Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** By Lemma 2.9, \( H^2(G, \mathbb{C}^\times) \) is an elementary abelian \( p \)-group for an odd prime \( p \). Let \( N = \langle z_r \rangle \). Then

\[
\frac{G}{N} \cong \mathbb{H}_3\left(\frac{\mathbb{F}_p[t]}{(t^r-1)}\right) \times (\mathbb{Z}/p\mathbb{Z})^2.
\]

For \( r = 1 \), we have \( H^2(\mathbb{H}_3(\mathbb{F}_p), \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^2 \) from [12, Theorem 3.3.6]. Now we use induction on \( r \) to prove our result. Let \( G \) have a free presentation \( F/\mathcal{R} \), where \( F \) is the free group generated by the symbols \( \langle x_m, y_m, 1 \leq m \leq r \rangle \). Viewing \( H^2(G, \mathbb{C}^\times) \) as \( (F' \cap \mathcal{R})/[F, \mathcal{R}] \), consider the exact sequence from [12, Theorem 2.5.6],

\[
\frac{G}{G'} \otimes N \xrightarrow{\lambda} H^2(G, \mathbb{C}^\times) \rightarrow H^2(\frac{G}{N}, \mathbb{C}^\times) \rightarrow N \rightarrow 1,
\]

where \( \lambda \) is the Ganea map defined by

\[
\lambda(x_m F' R \otimes x_s R) = [x_m, x_s][F, R].
\]

Our claim is that \( \text{Im}(\lambda) \cong (\mathbb{Z}/p\mathbb{Z})^2 \).
From [3, Remark, p. 110], it follows that $\text{Ker} \lambda = (G/G' \otimes N) \cap U$. Observe that here $U_2 = 1$ so that $U = U_1$. For $2 \leq m \leq r$, by the Hall–Witt identity in $F$ (mod $[F, R]$), we have

$$[x_m, z_r] = [x_m, [y_r, x_1]] = [[x_1, x_m], y_r] [[x_m, y_r], x_1] = 1,$$

$$[y_m, z_r] = [y_m, [y_1, x_r]] = [[x_r, y_m], y_1] [[y_m, y_1], x_r] = 1,$$

which says that

$$\tilde{x}_m \otimes [y_r, x_1] = \tilde{x}_m \otimes z_r \in \text{Ker} \lambda, \quad \tilde{y}_m \otimes [y_1, x_r] = \tilde{y}_m \otimes z_r \in \text{Ker} \lambda.$$

Now, for $1 < j < r$, by the Hall–Witt identity in $F$ (mod $[F, R]$), we have the following:

$$[x_1, z_r] = [x_1, [y_j, x_{r-j+1}]]$$

$$= [[x_{r-j+1}, x_1], y_j] [[x_1, y_j], x_{r-j+1}] = [z_j^{-1}, x_{r-j+1}],$$

$$[y_1, z_r] = [y_1, [y_j, x_{r-j+1}]]$$

$$= [[x_{r-j+1}, y_1], y_j] [[y_1, y_j], x_{r-j+1}] = [z_j^{-1}, y_j],$$

which says that

$$\tilde{x}_1 \otimes [y_j, x_{r-j+1}] + \tilde{x}_{r-j+1} \otimes [x_1, y_j]$$

$$= \tilde{x}_1 \otimes z_r + \tilde{x}_{r-j+1} \otimes z_j^{-1} \in U \setminus \text{Ker} \lambda,$$

$$\tilde{y}_1 \otimes [y_j, x_{r-j+1}] + \tilde{y}_j \otimes [x_{r-j+1}, y_1]$$

$$= \tilde{y}_1 \otimes z_r + \tilde{y}_j \otimes z_j^{-1} \otimes \in U \setminus \text{Ker} \lambda.$$
We now proceed to prove the result for $G = \mathbb{H}_3(\mathbb{F}_p^2)$. By Lemma 2.9 and Proposition 2.8, $H^2(G, \mathbb{C}^\times)$ is an elementary abelian group and

$$|H^2(G, \mathbb{C}^\times)| = \frac{p^{12}}{|U|},$$

where

$$U = ((\tilde{y}_1 \otimes z_1^k + \tilde{y}_2 \otimes z_2^{-1}), (\tilde{y}_2 \otimes z_1 + \tilde{y}_1 \otimes z_2^{-1}),
(\tilde{x}_1 \otimes z_1^k + \tilde{x}_2 \otimes z_2^{-1}), (\tilde{x}_2 \otimes z_1 + \tilde{x}_1 \otimes z_2^{-1})).$$

Therefore, $|U| = p^4$ and we have

$$H^2\left(\mathbb{H}_3\left(\frac{\mathbb{F}_p[t]}{(t^2 - k)}\right), \mathbb{C}^\times\right) \cong (\mathbb{Z}/p\mathbb{Z})^8.$$

## 4 Representation group of $\mathbb{H}_3(R)$

### 4.1 $G = \mathbb{H}_3(\mathbb{F}_p[t]/(t^2))$

In this section, we prove Theorem 1.4.

**Proof of Theorem 1.4.** Recall that $G^*$ is given by the following:

$$G^* = \langle x_i, y_i, i = 1, 2 \mid [y_1, x_1] = z_1, [y_1, x_2] = z_2, [x_2, x_1] = w_1, [y_2, x_1]z_2^{-1} = w_2, [y_2, x_2] = w_3, [y_2, y_1] = w_4, [z_1, x_i] = v_i, [z_1, y_i] = u_i, [z_2, x_1] = [z_1, x_2], [z_2, y_1] = [z_1, y_2], x_i^p = y_i^p = z_i^p = w_j^p = 1, 1 \leq j \leq 4 \rangle.$$

If $G^*$ is a group of order $p^{14}$, then we have the stem extension

$$1 \rightarrow Z \rightarrow G^* \rightarrow G \rightarrow 1$$

for

$$Z = \langle u_i, v_i, w_j, 1 \leq i \leq 2, 1 \leq j \leq 4 \rangle \cong (\mathbb{Z}/p\mathbb{Z})^8.$$
the following notation:

\[
[y_1, x_1] = z_1, \quad [y_1, x_2] = z_2, \quad [y_2, x_1]z_2^{-1} = w_2,
\]

\[
x_2, x_1 = w_1, \quad [y_2, x_2] = w_3, \quad [y_2, y_1] = w_4,
\]

and

\[
[z_1, x_i] = v_i, \quad [z_1, y_i] = u_i, \quad i = 1, 2.
\]

To prove our result, we show that \( G^* \) is isomorphic to a certain quotient group of \( F \).

Let \( H_1 = F/\langle y_3(F), F^p, w_1, w_2, w_3, w_4 \rangle \). Then

\[
H_1 \cong \langle y_1, y_2, x_1, x_2 \mid [y_1, x_1] = z_1, [y_1, x_2] = [y_2, x_1] = z_2, x_i^p = y_i^p = 1 \rangle
\]

\[
\cong \mathbb{H}_3(\mathbb{F}_p[t]/(t^2))
\]

is a group of order \( p^6 \).

In \( F \) modulo \( \langle y_4(F), [w_j, x_i], [w_j, y_i], i = 1, 2, 1 \leq j \leq 4 \rangle \), we have the following identities:

\[
[x_1^{-1}, y_1^{-1}, x_2] = [z_1, x_2]^{-1}, \quad [x_1^{-1}, y_1^{-1}, y_2] = [z_1, y_2]^{-1},
\]

\[
x_1^{-1}, y_2^{-1}, x_2 = [z_2, x_2]^{-1}, \quad [x_2^{-1}, y_1^{-1}, y_2] = [z_2, y_2]^{-1},
\]

\[
[y_1, x_2^{-1}, x_1^{-1}] = [z_2, x_1].
\]

Observe that, using the Hall–Witt identity, we have the following relations in \( F \) modulo \( \langle y_4(F), [w_j, x_i], [w_j, y_i], i = 1, 2, 1 \leq j \leq 4 \rangle \):

\[
[z_2, x_1] = [z_1, x_2], \quad [z_2, y_1] = [z_1, y_2], \quad [z_2, y_2] = [z_2, x_2] = 1. \quad (4.1)
\]

Now consider the group

\[
H_2 = F/\langle y_4(F), F^p, w_1, w_2, w_3, w_4, [z_1, x_2], [z_1, y_i], i = 1, 2 \rangle.
\]

By (4.1), we have

\[
H_2 \cong \langle x_1, x_2, y_1, y_2 \mid [y_1, x_1] = z_1, [y_1, x_2] = [y_2, x_1] = z_2,
\]

\[
[z_1, x_1] = v_1, x_i^p = 1 \rangle
\]

and \( H_2/\langle v_1 \rangle \cong H_1 \). Hence \( H_2 \) is of order \( p^7 \). We proceed further step by step considering

\[
H_3 = F/\langle y_4(F), F^p, w_1, w_2, w_3, w_4, [z_1, y_i], i = 1, 2 \rangle,
\]

\[
H_4 = F/\langle y_4(F), F^p, w_1, w_2, w_3, w_4, [z_1, y_2] \rangle,
\]

\[
H_5 = F/\langle y_4(F), F^p, w_1, w_2, w_3, w_4 \rangle.
\]
Using the identities in (4.1), we see that

\[ H_5 \cong \langle x_1, x_2, y_1, y_2 \mid [y_1, x_1] = z_1, [y_1, x_2] = [y_2, x_1] = z_2, \]
\[ [z_1, x_1] = v_1, [z_1, y_1] = u_1, [z_2, x_1] = [z_1, x_2] = v_2, \]
\[ [z_2, y_1] = [z_1, y_2] = u_2, x_i^p = 1 \rangle. \]

and \( H_5/\langle u_2 \rangle \cong H_4, H_4/\langle u_1 \rangle \cong H_3, H_3/\langle v_2 \rangle \cong H_2 \). Hence \( H_5 \) is of order \( p^{10} \).

Now consider the group

\[ K_1 = F/\langle y_4(F), F^p, w_1, w_3, w_4, [w_2, x_i], [w_2, y_i], i = 1, 2 \rangle. \]

By (4.1), we have

\[ [z_2, x_1] = [z_1, x_2] = v_2, [z_2, y_1] = [z_1, y_2] = u_2, [z_2, y_2] = [z_2, x_2] = 1. \]

So

\[ K_1 \cong \langle y_1, y_2, x_1, x_2 \mid [y_1, x_1] = z_1, [y_1, x_2] = z_2, [y_2, x_1]z_2^{-1} = w_2, \]
\[ [z_1, x_1] = v_1, [z_1, y_1] = u_1, [z_2, x_1] = [z_1, x_2] = v_2, \]
\[ [z_2, y_1] = [z_1, y_2] = u_2, x_i^p = 1 \rangle. \]

Therefore, \( K_1/\langle w_2 \rangle \cong H_5 \). Hence \( K_1 \) is of order \( p^{11} \). In a similar way, step by step, we consider the groups

\[ K_2 = F/\langle y_4(F), F^p, w_3, w_4, [w_j, x_i], [w_j, y_i], i = 1, 2, j = 1, 2 \rangle, \]
\[ K_3 = F/\langle y_4(F), F^p, w_4, [w_j, x_i], [w_j, y_i], i = 1, 2, j = 1, 2, 3 \rangle, \]
\[ K_4 = F/\langle y_4(F), F^p, [w_j, x_i], [w_j, y_i], i = 1, 2, j = 1, 2, 3, 4 \rangle, \]

and we see that \( K_2/\langle w_1 \rangle \cong K_1, K_3/\langle w_3 \rangle \cong K_2, K_4/\langle w_4 \rangle \cong K_3 \). Finally, we see that \( K_4 \cong G^* \). Hence \( G^* \) is of order \( p^{14} \).

In the following corollary, we point out the cardinality of a specific group that appeared in the above proof. We require it for later.

**Corollary 4.1.** Consider the following nilpotent class three group:

\[ \tilde{G} = \langle x_i, y_i, i = 1, 2 \mid [y_1, x_1] = z_1, [y_1, x_2] = [y_2, x_1] = z_2, \]
\[ [z_1, x_1] = v_1, [z_1, y_1] = u_1, [z_2, x_1] = [z_1, x_2], \]
\[ [z_2, y_1] = [z_1, y_2], x_i^p = y_i^p = z_i^p = 1 \rangle. \]

(4.2)

The group \( \tilde{G} \) is of order \( p^{10} \).

**Proof.** Let \( H_5 \) be the group that appeared in the proof of Theorem 1.4, where we also proved that \( |H_5| = p^{10} \). The result follows because \( \tilde{G} \cong H_5 \).
4.2 $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2 - k))$

In this section, we give a representation group of $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2 - k))$.

**Theorem 4.2.** For an odd prime $p$, a representation group of

$$G = \mathbb{H}_3\left(\frac{\mathbb{F}_p[t]}{(t^2 - k)}\right)$$

is the following group of nilpotency class three:

$$G^* = \langle x_i, y_i, i = 1, 2 \mid [y_1, x_1] = z_1, [y_1, x_2] = z_2, [y_2, x_1]z_2^{-1} = w_2, [y_2, x_2]z_1^{-k} = w_3, [z_1, y_i] = u_i, [z_1, x_i] = v_i, [z_2, x_1] = [z_1, x_2], [z_2, y_1] = [z_1, y_2], [z_2, y_2] = [z_1, y_1]^k, [x_2, x_1] = w_1, [y_2, y_1] = w_4, x_i^p = y_i^p = w_i^p = 1 \rangle.$$

**Proof.** Consider the group

$$H = \langle x_i, y_i, i = 1, 2 \mid [y_1, x_1] = z_1, [y_2, x_2] = z_1^k, [y_1, x_2] = [y_2, x_1] = z_2, [z_1, y_1] = u_i, [z_1, x_i] = v_i, [z_2, x_1] = [z_1, x_2], [z_2, y_1] = [z_1, y_2], [z_2, y_2] = [z_1, y_1]^k, [z_2, x_2] = [z_1, x_1]^k, x_i^p = y_i^p = w_i^p = 1 \rangle.$$

By using a method similar to the proof of Theorem 1.4, we obtain that $H$ is of order $p^{10}$ and $G^*$ is of order $p^{14}$. Hence it follows that $G^*$ is a representation group of $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2 - k))$. 

\[\square\]

4.3 $\mathbb{H}_3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$

**Theorem 4.3.** A representation group of $G = \mathbb{H}_3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$ is the following nilpotency class three group:

$$G^* = \langle x_1, x_2, y_1, y_2 \mid [x_1, x_2] = z_1, [y_1, y_2] = z_2, [z_1, x_1] = t_1, [z_1, x_2] = t_2, [z_2, y_1] = t_3, [z_2, x_2] = t_4, [x_1, y_1] = t_5, [x_1, y_2] = t_6, [x_2, y_1] = t_7, [x_2, y_2] = t_8, x_i^p = y_i^p = z_i^p = z_2^p = 1, i = 1, 2 \rangle.$$. 

Proof. We have \( H^2(G, \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^8 \) and
\[
G^* \cong \langle (x_1, x_2 \mid [x_1, x_2] = z_1, [z_1, x_1] = t_1, [z_1, x_2] = t_2) \times \langle t_5, t_6, t_7, t_8 \rangle \rangle 
\times \langle y_1, y_2 \mid [y_1, y_2] = z_2, [z_2, y_1] = t_3, [z_2, y_2] = t_4 \rangle 
\cong (H' \times (\mathbb{Z}/p\mathbb{Z})^4) \rtimes K',
\]
where \( H' \cong K' \) is of order \( p^5 \). Thus \( G^* \) is a semidirect product of a normal subgroup of order \( p^9 \) and a subgroup of order \( p^5 \). So \( G^* \) of order \( p^{14} \) such that the sequence
\[
1 \rightarrow \langle t_i \rangle_{i=1}^8 \cong H^2(G, \mathbb{C}^\times) \rightarrow G^* \rightarrow G \rightarrow 1
\]
is exact. Hence the result follows.

\section{Description of 2-cocycles of \( \mathbb{H}_3(R) \)}

In this section, we describe the 2-cocycles of \( \mathbb{H}_3(R) \).

\subsection{\( \mathbb{H}_3(\mathbb{F}_p[t]/(t^2)) \)}

We start with a proof of Theorem 1.3.

\textit{Proof of Theorem 1.3.} Let \( G = \mathbb{H}_3(\mathbb{F}_p[t]/(t^2)) \). We show that \( H^2(G, \mathbb{C}^\times) \) is the direct product of two groups which we compute explicitly. In order to do this, consider the central subgroup \( Z = \langle z_1 \rangle \) of \( G \). Then
\[
H = G/Z \cong \langle y_1, y_2, x_1, x_2 \mid [y_1, x_2] = z_2, x_i^p = y_i^p = 1 \rangle
\]
is an extra-special \( p \)-group of order \( p^5 \). From (2.2), we obtain the exact sequence
\[
1 \rightarrow \text{Hom}(Z, \mathbb{C}^\times) \xrightarrow{\text{tra}} H^2(H, \mathbb{C}^\times) \xrightarrow{\text{inf}} H^2(G, \mathbb{C}^\times) \xrightarrow{\chi} \text{Hom}(G/G' \otimes Z, \mathbb{C}^\times). \]
Since \( H^2(G, \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^8 \), \( H^2(H, \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^5 \),
\[
\text{Im} \text{(inf)} \cong \frac{H^2(H, \mathbb{C}^\times)}{\text{Im} \text{(tra)}} \quad \text{and} \quad \text{Hom}(G/G' \otimes Z, \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^4,
\]
we have, by Proposition 2.10,
\[
H^2(G, \mathbb{C}^\times) \cong \text{Im} \text{(inf)} \times \mu(\text{Hom}(G/G' \otimes Z, \mathbb{C}^\times)) \quad (5.1)
\]
for a section \( \mu: \text{Hom}(G/G' \otimes Z, \mathbb{C}^\times) \rightarrow H^2(G, \mathbb{C}^\times) \) which is an injective homomorphism (\( \mu \) will be defined in Step 2 below). We describe
\[
\text{Im} \text{(inf)} \quad \text{and} \quad \mu(\text{Hom}(G/G' \otimes Z, \mathbb{C}^\times))
\]
in the following two steps. For simplification of notation, let

\[ g = x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2} z_1^{c_1} z_2^{c_2}, \quad g' = x_1^{a'_1} x_2^{a'_2} y_1^{b'_1} y_2^{b'_2} z_1^{c'_1} z_2^{c'_2} \]

such that \( g \) and \( g' \) are arbitrary elements of \( G \) and let \( N = \langle Z, z_2 \rangle \) be the subgroup generated by \( Z \) and \( z_2 \).

**Step 1:** To describe \( \text{Im}(\inf) \), we proceed as follows. Consider the exact sequence

\[ 1 \rightarrow \text{Hom}(\langle z_2 \rangle, \mathbb{C}^\times) \xrightarrow{\text{tra}} H^2(H/\langle z_2 \rangle, \mathbb{C}^\times) \xrightarrow{\inf} H^2(H, \mathbb{C}^\times). \]

Since \( H^2(H, \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^5 \) and \( H/\langle z_2 \rangle \cong (\mathbb{Z}/p\mathbb{Z})^4 \), the map

\[ \inf: H^2(H/\langle z_2 \rangle, \mathbb{C}^\times) \rightarrow H^2(H, \mathbb{C}^\times) \]

is surjective. Hence every \( 2 \)-cocycle \( \beta \) of \( H \) is of the form

\[ [\beta] = \inf([\delta]), \delta \in H^2(H/\langle z_2 \rangle, \mathbb{C}^\times). \]

The \( 2 \)-cocycles of \( H/\langle z_2 \rangle \), being an elementary abelian group, are well known. Therefore,

\[ \beta(gZ, g'Z) = \delta(gN, g'N) = \mu_1^{a_2a'_1} \lambda_1^{b_1a'_1} \lambda_2^{b_2a'_1} \lambda_3^{b_3a'_1} \mu_3^{b_4a'_1}, \]

where \( \mu_j, j \in \{1, 3, 4\} \), and \( \lambda_\ell, \ell \in \{1, 2, 3\} \), are scalars whose \( p \)-th power is one.

Now recall (5.1) and define \([\alpha_1] = \inf([\beta]) \in H^2(G, \mathbb{C}^\times) \) for \( \beta \in H^2(H, \mathbb{C}^\times) \). Then

\[ \alpha_1(g, g') = \beta(gZ, g'Z) = \mu_1^{a_2a'_1} \lambda_1^{b_1a'_1} \lambda_2^{b_2a'_1} \lambda_3^{b_3a'_1} \mu_3^{b_4a'_1}. \]

For \( i \in \{1, 2\} \), define maps \( f_i: G \rightarrow \mathbb{C}^\times \) by

\[ f_1(g) = \lambda_1^{c_1}, \quad f_2(g) = \lambda_3^{c_2} \quad \text{for} \quad g = x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2} z_1^{c_1} z_2^{c_2} \in G. \]

This shows that \( \lambda_1^{b_1a'_1}, \lambda_3^{b_2a'_1 + b_1a'_1} \) are coboundaries in \( G \). Hence \( \alpha_1 \) is cohomologous to the following cocycle (again denoted by \( \alpha_1 \)):

\[ \alpha_1(g, g') = \mu_1^{a_2a'_1} (\lambda_2^{b_2a'_1} + \lambda_3^{b_2a'_1}) \mu_3^{b_4a'_1} = \mu_1^{a_2a'_1} \mu_2^{b_2a'_1} \mu_3^{b_2a'_1} \mu_4^{b_2a'_1}. \]

Therefore, in (5.1), \( \text{Im}(\inf) \) consists of \([\alpha_1] \) such that \( \alpha_1 \) is a \( 2 \)-cocycle of the above form.
**Step 2:** Consider the group $\tilde{G}$ from (4.2). We have the central exact sequence

$$1 \rightarrow \langle u_1, u_2, v_1, v_2 \rangle \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \rightarrow 1.$$ 

Observe that, in $\tilde{G}$, we have

$$[y^i_1, x^j_1] = z^i_j v^j_1 u^i_1, \quad [y^i_2, x^j_2] = z^i_j u^j_2, \quad [y^i_2, x^j_1] = z^i_j v^j_2.$$ 

For $g = x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2} z_1^{c_1} z_2^{c_2}$, $g' = x_1^{a'_1} x_2^{a'_2} y_1^{b'_1} y_2^{b'_2} z_1^{c'_1} z_2^{c'_2}$ in $G$,

$$gg' = x_1^{a_1 + a'_1} x_2^{a_2 + a'_2} y_1^{b_1 + b'_1} y_2^{b_2 + b'_2} z_1^{c_1 + c'_1} z_2^{c_2 + c'_2} + b_1 a_2 + b_2 a'_1.$$ 

We denote the numbers

$$b_1 \left( \frac{a'_1}{2} \right) + a'_1 c_1, \quad b_2 \left( \frac{a'_1}{2} \right) + a'_1 c_2 + a' c_1,$$

$$a_1 \left( \frac{b_1}{2} \right) - b_1 c_1, \quad a'_1 \left( \frac{b_1}{2} \right) - b_2 c_1 - b_1 c'_1$$

by $k_5, k_6, k_7, k_8$ respectively. Define a section $s: G \rightarrow \tilde{G}$ by

$$s(x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2} z_1^{c_1} z_2^{c_2}) = x_1^{a_1} x_2^{a_2} z_1^{c_1} z_2^{c_2} y_1^{b_1} y_2^{b_2} \in \tilde{G}.$$ 

Then

$$s(g)s(g') = (x_1^{a_1} x_2^{a_2} z_1^{c_1} z_2^{c_2} y_1^{b_1} y_2^{b_2})(x_1^{a'_1} x_2^{a'_2} z_1^{c'_1} z_2^{c'_2} y_1^{b'_1} y_2^{b'_2})$$

$$= x_1^{a_1 + a'_1} x_2^{a_2 + a'_2} z_1^{c_1 + c'_1 + c_2'} + b_1 a'_1 c_1 + b_2 a'_1 c_2 + b_1 a'_2 + b_2 a'_1$$

$$y_1^{b_1 + b'_1} y_2^{b_2 + b'_2} v_1^{k_5} v_2^{k_6} u_2^{k_7} u_1^{k_8} = u_1^{k_7 - k_2 a'_1} u_2^{k_8 - b_1 (b_1 a_2' + b_2 a_1')}.$$ 

Consider the exact sequence

$$1 \rightarrow \text{Hom}(\langle u_1, u_2, v_1, v_2 \rangle, \mathbb{C}^\times) \xrightarrow{\text{tra}} \text{H}^2(G, \mathbb{C}^\times) \xrightarrow{\text{inf}} \text{H}^2(\tilde{G}, \mathbb{C}^\times).$$

Let $f \in \text{Hom}(\langle u_1, u_2, v_1, v_2 \rangle, \mathbb{C}^\times)$ and $f(v_1), f(v_2), f(u_1), f(u_2)$ be denoted by $\mu_5, \mu_6, \mu_7, \mu_8 \in \mathbb{C}^\times$ respectively. Then $[\alpha_2] = \text{tra}(f) \in \text{H}^2(G, \mathbb{C}^\times)$ is defined by

$$\alpha_2(g, g') = f(s(g)s(g')s(g'))^{-1}$$

$$= f(v_1^{k_5} v_2^{k_6} + b_1 a'_1 a_2' u_1^{k_7 - k_2 a'_1} u_2^{k_8 - b_1 (b_1 a_2' + b_2 a_1')})$$

$$= \mu_5^{k_5} \mu_6^{k_6} + b_1 a'_1 a_2' \mu_7^{k_7 - k_2 a'_1} \mu_8^{k_8 - b_1 (b_1 a_2' + b_2 a_1')}.$$
For $i \in \{1, 2, 3\}$, define the maps $f_i : G \to \mathbb{C}^\times$ by
\[
f_1(g) = \mu_6^{c_1}, \quad f_2(g) = \mu_7^{c_1}, \quad f_3(g) = \mu_8^{c_2}
\]
for $g = x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}z_1^{c_1}z_2^{c_2} \in G$.
This gives that $\mu_6^{b_1a'_1a'_2}, \mu_7^{-b_2a'_1}, \mu_8^{(b_1a'_2+b_2a'_1)}$ are coboundaries in $G$. Therefore, $\alpha_2$ is cohomologous to the following cocycle:
\[
\alpha_2(g, g') = \mu_5^{k_5} \mu_6^{k_6} \mu_7^{k_7} \mu_8^{k_8}.
\] (5.3)

Consider the subgroup $S$ of $H^2(G, \mathbb{C}^\times)$ consisting of $[\alpha_2]$ such that $\alpha_2$ is of the form (5.3). Now
\[
\chi([\alpha_2])(x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}z_1^{c_1}z_2^{c_2}) = \alpha_2(x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}z_1^{c_1}z_2^{c_2})\alpha_2(z_1^{c_1}x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2})^{-1} = \mu_5^{-a_1c_1} \mu_6^{-a_2c_1} \mu_7^{-b_1c_1} \mu_8^{-b_2c_1}.
\]

Define a section $\mu : \text{Hom}(G/G' \otimes Z, \mathbb{C}^\times) \to S$ by
\[
\mu(f)(g, g') = f(x_1 \otimes z_1)^{-k_5} f(x_2 \otimes z_1)^{-k_6} f(y_1 \otimes z_1)^{-k_7} f(y_2 \otimes z_1)^{-k_8}.
\]
It is easy to check that $\mu$ is a homomorphism and $\chi|_S \circ \mu = \text{id}|_{\text{Hom}(G/G' \otimes Z, \mathbb{C}^\times)}$ and $\mu \circ \chi|_S = \text{id}|_S$. In particular,
\[
\mu(\text{Hom}(G/G' \otimes Z, \mathbb{C}^\times)) \cong S.
\]

We are now in a position to complete the proof. Using (5.2) and (5.3), we define $\alpha = \alpha_1\alpha_2$, and so we have
\[
\alpha(x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}z_1^{c_1}z_2^{c_2}, x_1^{a_1'}x_2^{a_2'}y_1^{b_1'}y_2^{b_2'}z_1^{c_1'}z_2^{c_2'}) = \mu_1^{a_2a'_1} \mu_2^{b_2a'_1} \mu_3^{b_2a'_2} \mu_4^{b_2b'_1} \mu_5^{k_5} \mu_6^{k_6} \mu_7^{k_7} \mu_8^{k_8}.
\]
This gives that every cocycle of $H^3(\mathbb{F}_p[t]/(t^2))$ is cohomologous to one of the above form. By the above description, there are at most $p^8$ elements in $H^2(H^3(\mathbb{F}_p[t]/(t^2)), \mathbb{C}^\times)$.

By Theorem 1.2, we have $H^2(H^3(\mathbb{F}_p[t]/(t^2)), \mathbb{C}^\times) \cong (\mathbb{Z}/p\mathbb{Z})^8$. Therefore, all of the above cocycles must be not cohomologous to each other. This completes the proof of our result. \qed
5.2 $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2 - k))$

We denote the numbers

\[ b_1\left(\frac{a'_1}{2}\right) + kb_1\left(\frac{a'_2}{2}\right) + kb_2a'_1a'_2 + a'_1c_1 + ka'_2c_2, \]
\[ b_2\left(\frac{a'_1}{2}\right) + kb_2\left(\frac{a'_2}{2}\right) + b_1a'_1a'_2 + a'_1c_2 + a'_2c_1, \]
\[ a'_1\left(\frac{b_1}{2}\right) + ka'_1\left(\frac{b_2}{2}\right) - b'_1a'_1 - b'_1c'_1 - kb'_2c'_2, \]
\[ a'_2\left(\frac{b_1}{2}\right) + ka'_2\left(\frac{b_2}{2}\right) - kb'_2a'_2 - b'_2c'_1 - b'_1c'_2 \]

by $q_5$, $q_6$, $q_7$ and $q_8$ respectively.

**Theorem 5.1.** Every 2-cocycle of $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2 - k))$ is cohomologous to a cocycle of the following form:

\[ \alpha(x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}z_1^{c_1}z_2^{c_2}, x_1^{a'_1}x_2^{a'_2}y_1^{b'_1}y_2^{b'_2}z_1^{c'_1}z_2^{c'_2}) = \mu_1^{a_2a'_1}\mu_2^{b_2a'_1}\mu_3^{b_2a'_2}\mu_4^{b_2b'_1}\mu_5^{q_5}\mu_6^{q_6}\mu_7^{q_7}\mu_8^{q_8}. \]

where $\mu_i \in \mathbb{C}^\times$ such that $\mu_i^p = 1$.

**Proof.** The proof of this result goes along the same lines as the proof of Theorem 1.3, using the representation group of $\mathbb{H}_3(\mathbb{F}_p[t]/(t^2 - k))$ that appeared in Section 4.2. \(\square\)

5.3 $\mathbb{H}_3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$

In the following result, the element $x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}z_1^{c_1}z_2^{c_2}$ denotes the element

\[ \begin{pmatrix} 1 & (b_1, b_2) & (c_1, c_2) \\ 0 & 1 & (a_1, a_2) \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}_3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}). \]

We use

$\mathbb{H}_3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{H}_3(\mathbb{Z}/p\mathbb{Z}) \times \mathbb{H}_3(\mathbb{Z}/p\mathbb{Z})$

and [13, Theorem 9.6] to describe elements of $\mathbb{H}_3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$. More precisely, every 2-cocycle $\alpha$ of $\mathbb{H}_3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$ is cohomologous to a cocycle of
the following form:

\[
\alpha(x_1 a_1 y_1 b_1, x_2 a_2 y_2 b_2) = \lambda_1 c_1 + a_1\lambda_2 c_1 b_1 + a'_1(b_2)\lambda_3 c_2 + a_2 b_2 a'
\]

where \(\lambda_i \in \mathbb{C}^*\) such that \(\lambda_i^p = 1\).

5.4 \(\mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z})\)

In the following result, the element \(x^a y^b z^c\) denotes the element

\[
\begin{pmatrix}
1 & b & c \\
0 & 1 & a \\
0 & 0 & 1
\end{pmatrix} \in \mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z}).
\]

From [5, Lemma 2.2], every 2-cocycle of \(\mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z})\) is cohomologous to a cocycle of the following form:

\[
\alpha(x^a y^b z^c, x'^a y'^b z'^c) = \lambda_1^{c'a + b(b') + aba' + a'(b')} \lambda_2^{c'b + a'(b')} \lambda_3^{p^2} = \lambda_2^{p^2} = 1.
\]

6 Projective representations of \(\mathbb{H}_3(\mathbb{R})\)

In this section, we give a construction of all projective irreducible representations of \(\mathbb{H}_3(\mathbb{F}_p[t]/(t^2))\) and \(\mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z})\). We also classify their non-degenerate cocycles. In particular, Theorem 1.5 and Corollary 1.6 follow from this section.

6.1 Projective representations of \(\mathbb{H}_3(\mathbb{F}_p[t]/(t^2))\)

By Theorem 1.4, a representation group of \(G = \mathbb{H}_3(\mathbb{F}_p[t]/(t^2))\) is given by

\[
G^* = \langle x_i, y_i, i = 1, 2 | [y_1, x_1] = z_1, [y_1, x_2] = z_2, [x_2, x_1] = w_1, [y_2, x_1]z_2^{-1} = w_2, [y_2, x_2] = w_3, [y_2, y_1] = w_4, \\
[z_1, x_i] = u_i, [z_1, y_i] = u_i, [z_2, x_1] = z_1, [z_2, x_2] = z_2, \\
[z_2, y_1] = [z_1, y_2], x_i^p = y_i^p = z_i^p = w_i^p = 1, \\
1 \leq j \leq 4 \rangle.
\]

Consider the normal subgroup

\[
N = \langle z_i, u_i, v_i, w_j, i = 1, 2, 1 \leq j \leq 4 \rangle \cong (\mathbb{Z}/p\mathbb{Z})^{10}
\]
of $G^*$. Any one-dimensional ordinary representation, say $\chi$, of $N$ is given by

$$\chi(w_j) = \mu_j, \quad 1 \leq j \leq 4,$$
$$\chi(u_1) = \mu_7, \quad \chi(u_2) = \mu_6, \quad \chi(v_1) = \mu_5,$$
$$\chi(v_2) = \mu_8, \quad \chi(z_1) = \mu_9, \quad \chi(z_2) = \mu_{10},$$

where $\mu_i \in \mathbb{C}^\times$ such that $\mu_i^p = 1$ for all $i$. If $\mu_6 \neq 1$, assume that $\mu_i = \mu_6^{r_i}$ for $1 \leq i \leq 8$ with $r_6 = 1$, and if $\mu_8 \neq 1$, assume that $\mu_i = \mu_8^{t_i}$ for $1 \leq i \leq 7$ with $t_8 = 1$.

Our first step is to determine the stabilizer $I_{G^*}(\chi)$ of the character $\chi$. An element $g = x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}n$ for $n \in N$ satisfies $g \in I_{G^*}(\chi)$ if and only if

$$\chi([x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}, z_1]) = 1 \quad \text{and} \quad \chi([x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}, z_2]) = 1.$$  

From the definition of $G^*$, we observe that

$$\begin{bmatrix}
x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}z_1^{c_1}z_2^{c_2}
\end{bmatrix} = \begin{bmatrix}
x_1^{a_1'}x_2^{a_2'}y_1^{b_1'}y_2^{b_2'}z_1^{c_1'}z_2^{c_2'}
\end{bmatrix}^{a_1^{b_1}-a_1b_1'+a_1b_2'+a_1b_2-2a_1b_2'+a_1'a_2-b_1b_2'
\end{bmatrix} + b_1b_1'+a_1'a_2-b_1b_2'
\begin{bmatrix}
w_1^{a_1'}/2 - a_1(b_1'/2) + a_1b_1' + (b_1'c_2 - b_1c_1')
+ a_1'b_2(b_1 + b_1') + (b_1'c_2 - b_1c_1') + (b_2'c_2 - b_2c_1')
\end{bmatrix} \quad \text{(6.1)}$$

Therefore, $g \in I_{G^*}(\chi)$ if and only if

$$\chi(v_1)^{a_1}\chi(v_2)^{a_2}\chi(u_1)^{b_1}\chi(u_2)^{b_2} = 1 \quad \text{and} \quad \chi(v_2)^{a_1}\chi(u_2)^{b_1} = 1. \quad \text{(6.2)}$$

Our next goal is to describe $\text{Irr}(I_{G^*}(\chi)|\chi)$. We note that $I_{G^*}(\chi)/N$ is abelian and $|I_{G^*}(\chi)/N| \leq p^4$. Either $\chi$ extends to $I_{G^*}(\chi)$ or there exists a normal subgroup $N'$ of $I_{G^*}(\chi)$ such that $N'/N$ is cyclic, $|I_{G^*}(\chi)/N'| \leq p^3$ and $\chi$ extends to $N'$ by Lemma 2.4. In both cases, by Theorem 2.3, Lemma 2.4 and Lemma 2.6, all representations in $\text{Irr}(I_{G^*}(\chi)|\chi)$ for a fixed $\chi$ will have the same dimension. We now consider various cases.

(i) Assume $\mu_6 = \mu_8 = 1$. Then, by (6.2), we have $g \in I_{G^*}(\chi)$ if and only if $\chi(v_1)^{a_1}\chi(u_1)^{b_1} = 1$. Hence $|I_{G^*}(\chi)/N| \in \{p^3, p^4\}$. We consider these cases separately.
Suppose \(|I_G(\chi)/N| = p^4\). For this case, \(I_G(\chi) = G\). By Lemma 2.5, representations are determined by certain projective representations of the abelian group \(G^*/N\). Since \(|G^*/N| = p^4\), any irreducible representation of \(\text{Irr}(G^*/\chi)\) will be of dimension either \(p\) or \(p^2\).

Next suppose \(|I_G(\chi)/N| = p^3\). Again by Lemma 2.5, representations are determined by certain projective representative representations of the abelian group \(I_G(\chi)/N\). Since \(|G^*/N| = p^4\), any irreducible representation of \(\text{Irr}(G^*/\chi)\) in this case will also be of dimension either \(p\) or \(p^2\).

(ii) Assume that \(\mu_6 = 1\), \(\mu_8 \neq 1\). Then, by (6.2),

\[ g = x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2} n \in I_G(\chi) \]

if and only if \(a_1 = 0\) and \(a_2 = -t_7 b_1\). Therefore, we have \(|I_G(\chi)/N| = p^2\) in this case. By Lemma 2.6 and the discussion before it, an irreducible representation of \(\text{Irr}(I_G(\chi)/\chi)\) will be of dimension one or \(p\). Further it will be one-dimensional if and only if \(\chi\) extends to \(I_G(\chi)\). By Lemma 2.4, this holds if and only if \([I_G(\chi), I_G(\chi)] \subseteq \text{Ker}(\chi), \) that is

\[ \chi([x_2^{-t_7 b_1} y_1^{b_1} y_2^{b_2} z_1^{c_1} z_2^{c_2}, x_2^{-t_7 b'_1} y_1^{b'_1} y_2^{b'_2} z_1^{c'_1} z_2^{c'_2}]) = 1. \]

By using (6.1), this is equivalent to

\[ \chi(v_2)^{(t_3 t_7 - t_4)(b_1 b'_2 - b'_1 b_2)} = 1, \]

where \(b_1, b_2, b'_1, b'_2\) are arbitrary. So we must have \(t_4 = t_3 t_7\).

This discussion altogether implies that, for \(t_4 = t_3 t_7\), \(\text{Irr}(G^*/\chi)\) consists of \(p^2\)-dimensional irreducible representations, and for \(t_4 \neq t_3 t_7\), all representations of \(\text{Irr}(G^*/\chi)\) are of dimension \(p^3\).

(iii) Similar to (ii), in this case, any representation of \(\text{Irr}(G^*/\chi)\) is of dimension \(p^2\) if \(r_1 = -r_3 r_5\) and is of dimension \(p^3\) if \(r_1 \neq -r_3 r_5\).

(iv) Assume that \(\mu_6 \neq 1\), \(\mu_8 \neq 1\). Then, by (6.2),

\[ g = x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2} n \in I_G(\chi) \]

if and only if \(b_1 = -r_8 a_1\) and \(b_2 = -r_8 a_2 + (r_7 r_8 - r_5) a_1\). Therefore, we have \(|I_G(\chi)/N| = p^2\) in this case. As earlier, we only need to answer whether \(\chi\) extends to \(I_G(\chi)\) or not. Now \(\chi\) extends to \(I_G(\chi)\) if and only if

\[ \chi([x_2^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2} z_1^{c_1} z_2^{c_2}, x_2^{a'_1} x_2^{a'_2} y_1^{b'_1} y_2^{b'_2} z_1^{c'_1} z_2^{c'_2}]) = 1. \]

By using (6.1) and substituting the values \(\chi(w_i), \chi(v_j)\) for \(1 \leq i \leq 4, 1 \leq j \leq 2\) in terms of \(\chi(u_2)\), we get

\[ \chi(u_2)^{(a'_1 a_2 - a_1 a'_2)(r_1 - r_2 r_8 - r_3 (r_7 r_8 - r_5) + r_4 r_8^2 - r_8^2 + r_7^2)} = 1. \]
Since \(a_1, a'_1, a_2, a'_2\) are arbitrary, we must have
\[
 r_1 - r_2 r_8 - r_3 (r_7 r_8 - r_5) + r_4 r_8^2 + \binom{r_8}{2} = 0. \tag{6.3}
\]
As earlier, this implies that \(\text{Irr}(G^* | \chi)\) consists of \(p^2\)-dimensional irreducible representations if (6.3) holds and of dimension \(p^3\) otherwise. This completes our discussion regarding the irreducible representations of \(G^*\).

### 6.2 Non-degenerate 2-cocycles of \(\mathbb{H}_3(\mathbb{F}_p[t] / (t^2))\)

In this section, we describe the degenerate and non-degenerate cocycles of
\[
\mathbb{H}_3 \left( \frac{\mathbb{F}_p[t]}{(t^2)} \right).
\]
Recall that every 2-cocycle \(\alpha\) of \(\mathbb{H}_3(\mathbb{F}_p[t] / (t^2))\) is cohomologous to a cocycle of the following form:
\[
\alpha(x_1, x_2, y_1, y_2, z_1, z_2, a_1, a'_1, b_1, b'_1, c_1, c'_1, a_2, a'_2, b_2, b'_2, c_2, c'_2)
\]
\[
= \mu_1^{a_2 a'_1} \mu_2^{b_2 a'_1} \mu_3^{b_2 a'_1} \mu_4^{b_2 b'_1} \mu_5^{b_1(a'_1)} + a'_1 c_1
\]
\[
\mu_6^{b_2(a'_1)} + a'_1 c_2 + a'_2 c_1 + a_1^{b_1} c_1 + a'_1 c_2 + a_2 c_1 - b_1 c_1 + a'_1 (b_1) c_2 + a_1^{b_1} c_2 - b_2 c_1 - b_1 c_2, \tag{6.4}
\]
where \(\mu_i \in \mathbb{C}^\times\) such that \(\mu_i^p = 1\) for all \(i\). Following the notation of Section 6.1, if \(\mu_6 \neq 1\), we assume that \(\mu_i = \mu_6^{r_i}\) for \(1 \leq i \leq 8\) with \(r_6 = 1\), and if \(\mu_8 \neq 1\), we assume that \(\mu_i = \mu_8^{t_i}\) for \(1 \leq i \leq 7\) with \(t_8 = 1\). Let \(G^*, N, \chi\) be as given in Section 6.1.

**Theorem 6.1.** The cocycle \(\alpha\) as given in (6.4) is non-degenerate if and only if one of the following holds.

(i) \(\mu_6 = 1, \mu_8 \neq 1\) and \(t_4 \neq t_3 t_7\).

(ii) \(\mu_8 = 1, \mu_6 \neq 1\) and \(r_1 \neq -r_3 r_5\).

(iii) \(\mu_8 \neq 1, \mu_6 \neq 1\) and \(r_1 - r_2 r_8 - r_3 (r_7 r_8 - r_5) + r_4 r_8^2 + \binom{r_8}{2} \neq 0\).

In particular, the number of non-degenerate cocycles of the group \(\mathbb{H}_3(\mathbb{F}_p[t] / (t^2))\) is \(p^5(p - 1)^2(p + 1)\).

**Proof.** Let
\[
A = \langle u_i, v_i, w_j, i = 1, 2, 1 \leq j \leq 4 \rangle \cong (\mathbb{Z} / p\mathbb{Z})^8
\]
be a subgroup of \(G^*\). Then it follows that \(A\) is a central subgroup of \(G^*\) such that
\[
1 \to A \to G^* \to G \to 1
\]
is a stem extension. For this case, note that \(\text{tra}(\chi|_A) = [\alpha]\). This fact, Lemma 2.7 and Section 6.1 give the result.

**Proof of Corollary 1.6.** By [2, Section 3], the groups \(H^1(Q, \hat{A})\) and the quotient group

\[
\text{Ker(res}^G_A)/\text{Im}(\inf^G_Q)
\]

are isomorphic. Let

\[
X = x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}z_1^{c_1}z_2^{c_2} \quad \text{and} \quad Y = x_1^{a_1'}x_2^{a_2'}y_1^{b_1'}y_2^{b_2'}z_1^{c_1'}z_2^{c_2'}
\]

be two elements in \(G\). By Theorem 1.3, it follows that

\[
\text{Im}\,(\text{res}^G_A) = \{[\alpha] \mid \alpha(X, Y) = \mu_3^{b_2a_2'}, \mu_3^p = 1\} \cong \mathbb{Z}/p\mathbb{Z}.
\]

Hence,

\[
\text{Ker}(\text{res}^G_A) \cong (\mathbb{Z}/p\mathbb{Z})^7.
\]

By [5, Lemma 2.2 (ii)], it follows that every cocycle of \(H^3_3(\mathbb{Z}/p\mathbb{Z})\) is cohomologous to a cocycle of the form

\[
\sigma(x_1^{a_1}y_1^{b_1}z_1^{c_1}, x_1^{a_1'}y_1^{b_1'}z_1^{c_1'}) = \lambda c'_1b_1 + a'_1 b_1 \frac{(b_1-1)}{2} \mu a'_1c_1 + b_1 \frac{(a'_1-1)}{2},
\]

\[\lambda^p = \mu^p = 1.
\]

Therefore, it is easy to see that

\[
\text{Im}(\inf^G_Q) = \{[\alpha] \mid \alpha(X, Y) = \mu_5^{b_4} + a_1' c_1 \mu_7 b_1 c_1, \mu_5^p = \mu_7^p = 1\} \cong (\mathbb{Z}/p\mathbb{Z})^2.
\]

Thus, using Theorem 1.3, we have

\[
\text{Ker}(\text{res}^G_A)/\text{Im}(\inf^G_Q) = \{[\alpha] \mid \alpha(X, Y) = \mu_1^{a_2a_1'} \mu_2^{b_2a_1'} \mu_4^{b_2b_1'} \mu_6^{a_1' c_2 + a_2' c_1} \mu_8^{b_1' c_1 c_2} - b_1 c_1 c_2\} \cong (\mathbb{Z}/p\mathbb{Z})^5.
\]

This proves (1).

By [2, Theorem A], the bijective classes in \(H^1(Q, \hat{A})\) are in one to one correspondence with the non-degenerate cohomology classes in \(\text{Ker}(\text{res}^G_A)\) modulo \(\text{Im}(\inf^G_Q)\). By using the description of \(\text{Ker}(\text{res}^G_A)/\text{Im}(\inf^G_Q)\) given in (6.5) and Theorem 6.1, we obtain that the number of non-degenerate cocycles in \(\text{Ker}(\text{res}^G_A)\) modulo \(\text{Im}(\inf^G_Q)\) is exactly \(p^2(p-1)^2(p+1)\). This proves (2) \(\square\)
Remark 6.2. In [2, Section 3.2], an explicit isomorphism between
\[ \text{Ker}(\text{res}^G) / \text{Im}(\text{inf}^G) \quad \text{and} \quad H^1(Q, \hat{A}) \]
is given. Using the description of \( \text{Ker}(\text{res}^G) / \text{Im}(\text{inf}^G) \), as given in (6.5), all elements of \( H^1(Q, \hat{A}) \) can be explicitly described.

6.3 Projective representations of \( \mathbb{H}_3(Z/p^2Z) \)

In this section, we first give a construction of projective representations of \( G = \mathbb{H}_3(Z/p^2Z) \).

As mentioned earlier, it is enough to give a construction of all ordinary irreducible representations of a representation group of \( G \). From [5, Theorem 1.2],
\[ G^* = \langle x, y \mid [y, x] = z, [z, x] = z_1, [z, y] = z_2, x^{p^2} = y^{p^2} = z^{p^2} = 1 \]
is a representation group of \( G \). Note that \( x^ay^bz^c, x^{a'}y^{b'}z^{c'} \in G^* \) satisfy
\[ [x^ay^bz^c, x^{a'}y^{b'}z^{c'}] = z^{a'b-abb'z_1^{b-2}} + (a'c-ac')z_2^{b-2} - a(b') + (b'c-bc') \]

Consider \( N = \langle z, z_1, z_2 \rangle \cong (\mathbb{Z}/p^2\mathbb{Z})^3 \), a normal subgroup of \( G^* \). Any character \( \chi: N \to \mathbb{C}^\times \) is given by \( \chi(z_i) = \lambda_i \) for \( i = 1, 2 \) and \( \chi(z) = \lambda \), where \( \lambda_i^{p^2} = 1 \) for \( i = 1, 2 \) and \( \lambda^{p^2} = 1 \). The inertia group \( S_{\chi} = I_{G^*}(\chi) \) of \( \chi \) consists of the elements \( x^ay^bn, n \in N \), such that \( \chi([x^ay^bz]) = 1 \), that is \( \chi(z_1)^a\chi(z_2)^b = 1 \). We consider the various cases of \( \lambda_i \)’s.

(i) Assume \( \lambda_1 = \lambda_2 = 1 \). In this case, \( S_{\chi} = G^* \). We note that
\[ |S_{\chi}/N| = |G^*|/|N| = p^4, \]
\( S_{\chi}/N \) is abelian. Hence, by Theorem 2.3 and Lemma 2.6, depending on \( \chi(z) \), \( \text{Irr}(G^*|\chi) \) consists of representations of dimension either 1, \( p \) or \( p^2 \).

(ii) Assume \( \lambda_1 = 1, \lambda_2 \neq 1 \) such that \( \lambda_2^p = 1 \). Then \( S_{\chi} = \langle N, x, y^p \rangle \). Therefore, \( |S_{\chi}/N| = p^3 \) and \( S_{\chi}/N \) is abelian. Hence \( \text{Irr}(G^*|\chi) \) consists of \( p \)-dimensional representations if \( \chi \) extends to \( S_{\chi} \), and if \( \chi \) does not extend to \( S_{\chi} \), then \( \text{Irr}(G^*|\chi) \) consists of \( p^2 \)-dimensional representations. We use Lemma 2.4 and
\[ \chi(z_2)^{pt} = \chi(z_2)^{pt_1} = 1 \]
to observe that \( \chi \) extends to \( S_{\chi} \) if and only if
\[ \chi([x^ay^{pt}z^c, x^{a'}y^{pt_1}z^{c'}]) = \chi(z)^{p(ta'-t_1a)} = 1. \]
Therefore, \( \chi \) extends if and only if \( \lambda^p = 1 \).
(iii) Assume $\lambda_1 = 1$ and order of $\lambda_2$ is $p^2$. In this case, we obtain $S_\chi = \langle N, x \rangle$. Since $S_\chi/N$ is cyclic of order $p^2$, the character $\chi$ extends to $S_\chi$ by Lemma 2.4. Therefore, by Theorem 2.3, $\text{Irr}(G^* | \chi)$ consists of $p^2$-dimensional irreducible representations.

(iv) For $\lambda_2 = 1$ and $\lambda_1 \neq 1$, we proceed as above.

(v) Suppose $\lambda_1 \neq 1$, $\lambda_2 \neq 1$ and $\lambda_1^p = \lambda_2^p = 1$. Assume that $\chi(z_1) = \chi(z_2)$ for some $1 \leq r < p - 1$. For the inertia group computations, $a$ and $b$ are such that $\chi(z_1)^a \chi(z_2)^b = 1$, that is $\chi(z_2)^{b+ra} = 1$. This implies

$$b \in \{ -ra + pt \mid 0 \leq t \leq p - 1 \}.$$ 

Therefore, we get $S_\chi = \langle N, x^a y^{-ra+pt} \rangle$. So $|S_\chi/N| = p^3$ and $S_\chi/N$ is abelian. As in (ii), $\text{Irr}(G^* | \chi)$ consists of $p$-dimensional representations if $\chi$ extends to $S_\chi$, and if $\chi$ does not extend to $S_\chi$, then $\text{Irr}(G^* | \chi)$ consists of $p^2$-dimensional representations. So it remains to determine the conditions for which $\chi$ extend to $S_\chi$. By Lemma 2.4, $\chi$ extends to $S_\chi$ if and only if

$$\chi([x^a y^{-ra+pt} z^c, x^{a'} y^{-ra'+pt'} z^{c'}]) = 1.$$ 

This is equivalent to

$$\chi(z)^{p(a'+a' \cdot t' - a \cdot t)} \chi(z_2)^{r^2 (aa' + a-a')} = 1,$$

where $a, a', t, t'$ are arbitrary. Therefore, we must have $\lambda_2^{r^2} = 1$ and $\chi(z)^p = 1$. This implies that $r$ is a multiple of $p$. This in turn gives that $\lambda_1 = \lambda_2^p = 1$, a contradiction. Hence $\chi$ does not extend to $S_\chi$ in this case. Therefore, $\text{Irr}(G^* | \chi)$ consists of $p^2$-dimensional representations.

(vi) Assume that $\lambda_1 \neq 1$ and $\lambda_2$ is of order $p^2$. Assume that $\lambda_1 = (\lambda_2)^r$. It is easy to see that $S_\chi = \langle N, x^a y^{-ra} \rangle$, and therefore $S_\chi/N$ is an abelian group of order $p^2$. As in (v), we obtain that $\chi$ extends to $S_\chi$ if and only if $\lambda_1^p = 1$. Hence, in this case, $\text{Irr}(G^* | \chi)$ consists of $p^2$-dimensional representations. If both $\lambda_1$ and $\lambda_2$ are of order $p^2$, then $\text{Irr}(G^* | \chi)$ consists of $p^3$-dimensional representations.

(vii) Assume that the order of $\lambda_1$ is $p^2$ and that of $\lambda_2$ is $p$. This case can be done parallel to (v). Here we obtain that $\text{Irr}(G^* | \chi)$ consists of $p^2$-dimensional representations.

### 6.4 Non-degenerate 2-cocycles of $\mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z})$

In this section, we describe the non-degenerate cocycles of the group $\mathbb{H}_3(\mathbb{Z}/p^2\mathbb{Z})$.

**Theorem 6.3.** The cocycle $\alpha$ as given in (6.6) is non-degenerate if and only if $\lambda_1$ and $\lambda_2$ are of order $p^2$. In particular, the number of non-degenerate cocycles is $(p - 1)^2 p^2$. 
Proof. Recall that every 2-cocycle of $G = \mathbb{H}_3(Z/p^2\mathbb{Z})$ is cohomologous to a cocycle of the following form:

$$\alpha(x^{a_1}y^{b_1}z^{c_1}, x^{a'_1}y^{b'_1}z^{c'_1}) = \lambda_1^{c_1}a_1 + b_1\left(\frac{a_1}{2}\right) + a_1b_1\lambda_2^{c'_1}a_1' + a_1'\left(\frac{b_1}{2}\right),$$

(6.6)

Let $Z$ be a subgroup of $G^*$ generated by $z_1$ and $z_2$. Then $Z \subseteq N$ and

$$1 \to Z \to G^* \to G \to 1$$

is a stem extension. For any character $\chi$ of $N$ as given above, let $\chi|_Z$ denote its restriction to the group $Z$. We note that $\text{tra}(\chi|_Z) = [\alpha]$. The result now follows from the above description of ordinary irreducible representations of $G^*$, Theorem 2.2 and Lemma 2.7.

\[\square\]

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Bibliography


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**Author information**

Corresponding author:
Pooja Singla, Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208016, India. E-mail: psingla@iitk.ac.in

Sumana Hatui, School of Mathematical Sciences, National Institute of Science Education and Research, An OCC of Homi Bhabha National Institute, Bhubaneswar 752050, Odisha, India. E-mail: sumanahatui@niser.ac.in

E. K. Narayanan, Department of Mathematics, Indian Institute of Science, Bangalore 560012, India. E-mail: naru@iisc.ac.in