Abstract: Consider the direct and inverse scattering problem of time harmonic acoustic waves by a two-dimensional elastic obstacle which contains an unknown impenetrable object inside. We apply the boundary integral equation method to solve the direct scattering problem. Since the obtained boundary integral system is a mixed form of scalar and vector equations, we consider the existence of the solution in the sense of a weak formulation. The factorization method is employed to solve the corresponding inverse scattering problem, that is, we try to recover the shape of the interaction surface from the knowledge of acoustic far-field patterns. With the help of a special data to pattern operator \( G \), we can reconstruct the shape of the interaction surface by using the classical factorization method. Furthermore, a new mixed reciprocity relation is proposed to show the relation between the interior elastic field and unitary matrix \( S \).

Keywords: Fluid-solid interaction problem, buried objects, boundary integral equation method, factorization method, mixed reciprocity relation

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1 Introduction

In this paper, we consider the fluid-solid interaction scattering problem with unknown buried objects. Mathematically, the fluid-solid interaction problem is modeled by the Helmholtz equation and the Navier equation which are coupled with the transmission conditions on the interface between the solid and the fluid medium. To give a precise description, let \( \Omega \) denote a bounded, simply connected domain in \( \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \), and its exterior complement \( \mathbb{R}^2 \setminus \overline{\Omega} \) is connected. The unknown buried object in \( \Omega \) is denoted by \( \Omega_0 \) with smooth boundary \( \partial \Omega_0 \). The domain \( \Omega_1 := \Omega \setminus \overline{\Omega_0} \) is occupied by a linear and isotropic elastic solid, and \( \mathbb{R}^2 \setminus \overline{\Omega} \) is filled with a compressible, inviscid fluid (Figure 1).

We denote by \( \omega > 0 \) the frequency, \( k = \frac{\omega}{c_0} > 0 \) the acoustic wave number, \( c_0 \) the speed of sound in the fluid, \( \rho \in \mathbb{R} \) the mass density of the solid, \( \rho_f \in \mathbb{R} \) the mass density of the fluid and \( \gamma := \rho_f \omega^2 \in \mathbb{R} \). The fluid-solid interaction scattering problem with buried objects we are dealing with is now modeled by

\[
\begin{align*}
\Delta p^I + k^2 p^I &= 0 \quad \text{in } R^2 \setminus \overline{\Omega}, \\
\Delta^* u + \rho \omega^2 u &= 0 \quad \text{in } \Omega_1, \\
yu \cdot n - \frac{\partial p^I}{\partial n} &= 0 \quad \text{on } \partial \Omega, \\
Tu + p^I n &= 0 \quad \text{on } \partial \Omega, \\
u &= 0 \quad \text{on } \partial \Omega_0.
\end{align*}
\]

(1.1)

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Here, \( p^i = p^i + p \) is the total field which is the superposition of the given incident plane wave \( p^i = e^{i k x \cdot d} \) (\( d \) denotes the unit vector on the incident direction) and the scattered wave \( p \). And \( p \) satisfies the Sommerfeld radiation condition, i.e.

\[
\lim_{{r \to \infty}} \sqrt{r} (\frac{\partial p}{\partial r} - i k p) = 0, \quad r = |x|.
\]  

(1.2)

The notation \( \Delta^\ast \) denotes the Lamé operator \( \mu \Delta + (\mu + \lambda)\nabla(\nabla \cdot) \), where \( \mu \) and \( \lambda \) are Lamé constants satisfying \( \mu > 0, 2\mu + \lambda > 0 \). The vector \( n \) denotes the exterior unit normal vector on \( \partial \Omega \) and \( \frac{\partial}{\partial n} \) is the normal derivative. Furthermore, the stress vector \( Tu \) is defined by

\[
Tu := 2\mu \frac{\partial u}{\partial n} + \lambda n \nabla \cdot u - \mu n^\perp \nabla^\perp \cdot u,
\]

where \( n^\perp \) is obtained by rotating \( n \) anticlockwise by \( \frac{\pi}{2} \) and

\[
\nabla \cdot u := \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \nabla^\perp \cdot u := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad u = [u_1, u_2]^T.
\]

**The direct scattering problem** is to show that there is a unique solution to problem (1.1) and (1.2). **The inverse scattering problem** is to recover the shape of the interaction surface from the knowledge of acoustic far-field patterns.

In order to show the uniqueness of the solution to the direct problem, the so-called Jones frequency associated with the original transmission problem is excluded. More related results about the Jones frequency can be found in [31, 34] for the case when there is no buried object inside, that is, \( \Omega_0 = 0 \), in [14, 21] for the case \( Tu = 0 \) on \( \partial \Omega_0 \), in [38] for the case \( \mathcal{B} u = 0 \) on \( \partial \Omega_0 \), where \( \mathcal{B} \) stands for the first or the second boundary condition.

For the existence of the solution to the direct problem, we apply the boundary integral equation method. This method for solving the transmission scattering problems has been widely investigated in many different wave propagation, for example the acoustic transmission problem [3, 19, 28], the electromagnetic transmission problem [8, 30], the elastic transmission problem [7, 27] and also the fluid-solid interaction problem [18, 31, 39].

Based on Green’s and Betti’s representation and potential theory, we derive a boundary integral system equivalent to the original scattering problem. In the process to show the uniqueness of the obtained system, we construct reversely two potential functions and gain two independent problems which are easy to analyze. This knack depends closely on our selection of the unknown variables and the way we bring them into the boundary conditions. And since the transmission conditions are mixed type, that is, a scalar condition and a vector condition, we consider the existence of the solution in the sense of a weak formulation (see [39]).

We solve the inverse scattering problem by using the factorization method which has been firstly introduced by Kirsch in 1998 [22] for the acoustic scattering by an impenetrable obstacle with Dirichlet or Neumann boundary conditions. This method belongs to the class of qualitative methods for inverse scattering problems. The object is to determine the shape of the scatterer. And one main character is that it provides
a criterion for sampling points $z$, which is both necessary and sufficient. Therefore, it is viewed as an elegant tool in inverse scattering problems, both in acoustic [5, 6], electromagnetic [23], elastic [10] and fluid-solid interaction problems [25, 40].

Fluid-solid interaction problems have been studied for a long time, and many inversion algorithms have been proposed to reconstruct the shape of the elastic obstacle numerically when there are no embedded objects inside. For example, an iterative optimization algorithm [15, 16], the finite element method [12, 13, 32, 41, 42], the linear sampling method [34–36] and the factorization method [25, 40]. However, few results are found in the literature to the case that there is a buried object. In this paper, we aim at extending the factorization method to solve this case. Firstly, motivated by the work [25] in which a novel way to define the data to pattern operator $G$ was introduced, we establish a satisfactory factorization and also find that the type of the boundary condition on $\partial \Omega_0$ has little influence on our factorization, that is, the buried object does not affect the reconstruction of interaction interface. Furthermore, this is consistent with the uniqueness theorem of the inverse problem proven in [38] to determine the bounded elastic obstacle by using the fixed frequency acoustic far-field data.

Secondly, removing the buried object, we establish a novel mixed reciprocity relation (4.2) between the Green function and the total wave (refer to [6, 29, 37] for acoustic scattering in a layered medium). Since the fundamental solution $\Gamma(x, y)$ of the Navier equation is a symmetric matrix [1, 2], while the solution to the Navier equation is a vector function, we choose a fixed direction $\alpha$ to define the corresponding Green function $(G_k(x, z), G_\omega(x, z))$, and Theorems 4.1 and 4.3 hold for any fixed direction. In the future, we want to use these two relations to recover the buried object by the factorization method.

The remaining part of the paper is organized as follows. In Section 2, we will reduce the original problem (1.1) to a mixed system by the boundary integral equation method and then establish the well-posedness (uniqueness, existence and stability) of the direct scattering problem. Section 3 is devoted to the reconstruction of the interaction boundary $\partial \Omega$, where a special operator $G$ is defined to obtain a simple factorization, disregarding the content inside the elastic body. In Section 4, we consider a scattering problem with no buried object and give a novel mixed reciprocity relation to show the relation between the interior elastic field and unitary matrix $S$.

2 The direct scattering problem

In this section, we establish the well-posedness of scattering problem (1.1) by the boundary integral equation method. Since the incident field $p^i$ satisfies the Helmholtz equation $\Delta p^i + k^2 p^i = 0$ in all of $R^2$, it is easy to show that the scattering field $p$ and the total displacement field $u$ satisfy the following problem:

$$
\begin{align*}
\Delta p + k^2 p &= 0 \quad \text{in } R^2 \setminus \Omega, \\
\Delta^* u + \rho \omega^2 u &= 0 \quad \text{in } \Omega_1, \\
y u \cdot n - \frac{\partial p}{\partial n} &= f_1 \quad \text{on } \partial \Omega, \\
Tu + p n &= f_2 \quad \text{on } \partial \Omega, \\
u &= 0 \quad \text{on } \partial \Omega_0, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial p}{\partial r} - ikp \right) &= 0,
\end{align*}
$$

(2.1)

where $f_1 = \frac{\partial p^i}{\partial n} \in H^{-\frac{1}{2}}(\partial \Omega), f_2 = -p^i n \in [H^{\frac{1}{2}}(\partial \Omega)]^2$.

Before we show the uniqueness of the solution to problem (2.1), we assume that $\omega$ is not a Jones frequency, that is, the system

$$
\begin{align*}
\Delta^* u + \rho \omega^2 u &= 0 \quad \text{in } \Omega_1, \\
\cdot u &= 0 \quad \text{on } \partial \Omega, \\
Tu &= 0 \quad \text{on } \partial \Omega, \\
u &= 0 \quad \text{on } \partial \Omega_0,
\end{align*}
$$

(2.2)
admits only the trivial solution $u = 0$ (refer to [31, 34] for the case when there is no buried object inside, that is, $\Omega_0 = \emptyset$, to [14, 21] for the case $Tu = 0$ on $\partial \Omega_0$, to [38] for the case $\mathcal{B}u = 0$ on $\partial \Omega_0$, where $\mathcal{B}$ stands for the first or the second boundary condition).

**Theorem 2.1.** Assume $\omega$ is not a Jones frequency of system (2.2). Then, for any $f_1 \in H^{-\frac{1}{2}}(\Omega)$, $f_2 \in [H^2(\Omega)]^2$, there exists at most one solution $(p, u) \in \mathcal{H}^1_0((\Omega) \times \mathcal{H}^1(\Omega_1))^2$ of problem (2.1).

**Proof.** With little confusion, we still use the notation $p$ and $u$ to denote the solution of the corresponding homogeneous equation (2.1), that is, $f_1 = 0, f_2 = 0$. So it is sufficient to show that $p = 0$ in $\Omega \setminus \Omega_0$ and $u = 0$ in $\Omega_1$. Choose a disk $B_R$ centered at the origin with radius $R$ large enough to contain $\Omega$, i.e. $\Omega \subset B_R$.

From Green’s formula, we obtain
\[
\int_{\partial B_R} p \frac{\partial \bar{p}}{\partial n} \, ds = \int_{\partial \Omega} p \frac{\partial \bar{p}}{\partial n} \, ds + \int_{\Omega \setminus \Omega_0} (|\nabla p|^2 - k^2 |p|^2) \, dx.
\]
Here and below, $n$ denotes the exterior unit normal vector to the corresponding boundary. Using the homogeneous boundary conditions $\gamma u \cdot n - \frac{\partial u}{\partial n} = 0$, $Tu + pn = 0$ on $\partial \Omega$ and $u = 0$ on $\partial \Omega_0$, and the Betti formula (refer to [4]), we have
\[
\int_{\partial \Omega} p \frac{\partial \bar{p}}{\partial n} \, ds = y \int_{\partial \Omega} p (\bar{u} \cdot n) \, ds = -y \int_{\partial \Omega} \bar{u} \cdot Tu \, ds = -y \left( \int_{\Omega_1} (\bar{u} \cdot \Lambda^* u + E(u, \bar{u})) \, dx \right)
\]
\[
= -y \left( E(u, \bar{u}) - \rho \omega^2 |u|^2 \right) \, dx.
\]
Here, the general form of $E(v, u)$ is given by
\[
E(v, u) = (2\mu + \lambda) \left( \frac{\partial v_1}{\partial x_1} \frac{\partial u_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \frac{\partial u_2}{\partial x_2} \right) + \mu \left( \frac{\partial v_1}{\partial x_2} \frac{\partial u_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \frac{\partial u_2}{\partial x_1} \right)
\]
\[
+ \lambda \left( \frac{\partial v_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} + \frac{\partial v_2}{\partial x_1} \frac{\partial u_1}{\partial x_2} \right) + \mu \left( \frac{\partial v_1}{\partial x_2} \frac{\partial u_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \frac{\partial u_2}{\partial x_1} \right).
\]
It is easy to see that $\text{Im} E(u, \bar{u}) = 0$; we obtain
\[
\text{Im} \left( \int_{\partial B_R} p \frac{\partial \bar{p}}{\partial n} \, ds \right) = 0.
\]
According to the Rellich’s lemma (see [9]), we have $p = 0$ in $\mathcal{H}^2 \setminus \overline{B_R}$, and it follows by the unique continuation principle (see [11]) that $p = 0$ in $\mathcal{H}^2 \setminus \Omega$.

Consequently, the displacement field $u$ satisfies the homogeneous system (2.2). Under the assumption that $\omega$ is not a Jones frequency of the system, we conclude that $u = 0$ in $\Omega_1$. The proof is completed. \hfill $\Box$

Next, we will transform problem (2.1) into a boundary integral system to show the existence of its solution. At first, we introduce the fundamental solution $\Phi(x, y)$ of the Helmholtz equation as well as the fundamental solution $\Gamma(x, y)$ of the Navier equation (see [1, 2, 26]) in $\mathcal{H}^2$, which are given by
\[
\Phi(x, y) = \frac{i}{4} H^{(1)}_0(k|x - y|), \quad x \neq y,
\]
\[
\Gamma(x, y) = \frac{i}{4\mu} H^{(1)}_0(k_s|x - y|)I + \frac{i}{4\omega^2} \nabla^2 \nabla_x (H^{(1)}_0(k_s|x - y|) - H^{(1)}_0(k_p|x - y|)), \quad x \neq y.
\]
Here, $H^{(1)}_0(\cdot)$ is the Hankel function of the first kind of order zero, and $I$ is the identity matrix. And $k_p$ is the compressional wave number, $k_s$ is the shear wave number. They are given by the following forms respectively:
\[
k_p = \omega \sqrt{\frac{\rho}{2\mu + \lambda}}, \quad k_s = \omega \sqrt{\frac{\rho}{\mu}}.
\]
In addition, the subscript $x$ is used to denote differentiation with respect to the corresponding variable, and the superscript $\tau$ denotes transpose.

The following boundary integral operators in terms of the fundamental solutions will be used in the sequel ($\Sigma \in \{\partial \Omega_0, \emptyset\}$):

\[
(K'\Phi)(x) = \int_{\partial \Omega} \frac{\partial \Phi(x, y)}{\partial n(x)} \psi(y) \, ds(y), \quad x \in \partial \Omega, \quad \psi \in H^{-\frac{1}{2}}(\partial \Omega),
\]

\[
(N\Phi)(x) = \frac{\partial}{\partial n(x)} \int_{\partial \Omega} \frac{\partial \Phi(x, y)}{\partial n(y)} \psi(y) \, ds(y), \quad x \in \partial \Omega, \quad \psi \in H^\frac{1}{2}(\partial \Omega),
\]

\[
(S_{\Sigma\Omega_0}\varphi)(x) = \int_{\Sigma} \Gamma(x, y) \varphi(y) \, ds(y), \quad x \in \partial \Omega_0, \quad \varphi \in \mathbb{H}^{-\frac{1}{2}}(\Sigma)^2,
\]

\[
(K_{\Sigma\Omega_0}\varphi)(x) = \int_{\Omega} [T_y \Gamma(x, y)]^\tau \varphi(y) \, ds(y), \quad x \in \partial \Omega_0, \quad \varphi \in \mathbb{H}^\frac{1}{2}(\partial \Omega),
\]

\[
(N_{\Sigma\Omega_0}\varphi)(x) = T_x \int_{\partial \Omega} [T_y \Gamma(x, y)]^\tau \varphi(y) \, ds(y), \quad x \in \partial \Omega, \quad \varphi \in \mathbb{H}^\frac{1}{2}(\partial \Omega)^2.
\]

Referring to the book [33], they have following mapping properties:

\[
K': H^{-\frac{1}{2}}(\partial \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega), \quad N: H^\frac{1}{2}(\partial \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega),
\]

\[
S_{\Sigma\Omega_0}: [H^{-\frac{1}{2}}(\Sigma)^2]^2 \rightarrow [H^{\frac{1}{2}}(\partial \Omega_0)^2]^2, \quad K_{\Sigma\Omega_0}: [H^{\frac{1}{2}}(\partial \Omega)]^2 \rightarrow [H^{\frac{1}{2}}(\partial \Omega_0)]^2,
\]

\[
K_{\Sigma\Omega_0}': [H^{-\frac{1}{2}}(\Sigma)^2]^2 \rightarrow [H^{-\frac{1}{2}}(\partial \Omega)]^2, \quad N_{\Sigma\Omega_0}: [H^{\frac{1}{2}}(\partial \Omega)]^2 \rightarrow [H^{-\frac{1}{2}}(\partial \Omega_0)]^2.
\]

Then we will use the boundary integral equation method to show the existence of the solution to problem (2.1). Based on Green’s and Betti’s representation theory, we denote the solution in the following form:

\[
p(x) = \int_{\partial \Omega} \left[ p(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \frac{\partial p}{\partial n}(y) \Phi(x, y) \right] \, ds(y), \quad x \in R^2 \setminus \overline{\Gamma},
\]

\[
u(x) = \int_{\partial \Omega} \left[ \Gamma(x, y) T_u(y) - (T_y \Gamma(x, y))^\tau u(y) \right] \, ds(y) - \int_{\partial \Omega_0} \Gamma(x, y) T_u(y) \, ds(y), \quad x \in \Omega_1.
\]

Using the jump relations of the single- and double-layer potentials (refer to [33]) and letting $x$ approach the boundary of $\partial \Omega$ and $\partial \Omega_0$,

\[
\frac{\partial p}{\partial n}|_{\partial \Omega} = Np - (K' \frac{\partial p}{\partial n} - \frac{1}{2} \frac{\partial p}{\partial n}),
\]

\[
T_u|_{\partial \Omega} = K'_{\Sigma\Omega_0}(Tu) + \frac{1}{2} Tu - N_{\Sigma\Omega_0} u - K_{\Sigma\Omega_0}(Tu),
\]

\[
u|_{\partial \Omega_0} = S_{\Sigma\Omega_0}(Tu) - K_{\Sigma\Omega_0} u - S_{\Sigma\Omega_0}(Tu).
\]

Let $a = p|_{\partial \Omega}$, $b = u|_{\partial \Omega}$, $c = Tu|_{\partial \Omega}$. Then $a \in H^\frac{1}{2}(\partial \Omega_0)$, $b \in [H^{\frac{1}{2}}(\partial \Omega_0)]^2$ and $c \in [H^{-\frac{1}{2}}(\partial \Omega_0)]^2$. Using the boundary conditions, we obtain a boundary integral system

\[
-Na + y (K' + \frac{1}{2} I)(b \cdot n) = (K' + \frac{1}{2} I)f_1, \quad (2.3)
\]

\[
\left( \frac{1}{2} I - K'_{\Sigma\Omega_0} \right) \left( an \right) - N_{\Sigma\Omega_0} b - K_{\Sigma\Omega_0} c = \left( \frac{1}{2} I - K'_{\Sigma\Omega_0} \right) f_2, \quad (2.4)
\]

\[
S_{\Sigma\Omega_0}(an) + K_{\Sigma\Omega_0} b + S_{\Sigma\Omega_0} c = S_{\Sigma\Omega_0} f_2. \quad (2.5)
\]

**Theorem 2.2.** Assume that $k^2$ is not a Neumann eigenvalue of $-\Delta$ in $\Omega$, $\omega$ is neither a Dirichlet eigenvalue of $-\Delta^\tau$ in $\Omega_0$ nor a Jones frequency. Then the above boundary integral system (2.3)–(2.5) has at most one solution.
Proof. It is sufficient to prove that the corresponding homogeneous system has only a trivial solution. Suppose that \((a_0, b_0, c_0) \in H^2(\partial \Omega) \times [H^2(\partial \Omega)]^2 \times [H^{-1}(\partial \Omega_0)]^2\) is a solution of the corresponding homogeneous system (2.3)–(2.5).

Firstly, we construct two potential functions \(\bar{p} \in H^1(\Omega), \bar{u} \in [H^1_{\text{loc}}(R^2\setminus \Omega)]^2\) which are converse to the original problem (2.1),

\[
\bar{p}(x) = \int_{\partial \Omega} \left[ a_0(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - y(b_0(y) \cdot n)\Phi(x, y) \right] ds(y), \quad x \in \Omega. \tag{2.6}
\]

\[
\bar{u}(x) = - \int_{\partial \Omega} \left[ \Gamma(x, y)(a_0(y)n) + (T, \Gamma(x, y))^T b_0(y) \right] ds(y) - \int_{\partial \Omega_0} \Gamma(x, y)c_0(y) ds(y), \quad x \in R^2\setminus \Omega. \tag{2.7}
\]

Then the jump relations of the single- and double-layer potentials and the homogeneous system (2.3), (2.4) imply that

\[
\begin{align*}
\frac{\partial \bar{p}_-}{\partial n} &= Na_0 - y(K' + \frac{1}{2} I)(b_0 \cdot n) = 0, \quad \text{on } \partial \Omega, \\
T\bar{u}_- &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

here and below, \((\cdot, \cdot)_\Omega\) denotes the limit of function or its normal derivative on the corresponding surface from exterior or interior of its domain.

Under the assumption that \(k^2\) is not a Neumann eigenvalue of \(\Delta\) in \(\Omega\) and the well-posedness of the exterior elastic Neumann problem,

\[
\begin{align*}
\Delta \bar{p} + k^2\bar{p} &= 0 \quad \text{in } \Omega, \\
\frac{\partial \bar{p}}{\partial n} &= 0 \quad \text{on } \partial \Omega, \\
\Delta^* \bar{u} + \rho \omega^2 \bar{u} &= 0 \quad \text{in } R^2\setminus \Omega,
\end{align*}
\]

and the Kupradze radiation condition satisfied by \(\bar{u}\) (refer to [17, 26]), we obtain \(\bar{p} = 0\) in \(\Omega\) and \(\bar{u} = 0\) in \(R^2\setminus \Omega\).

Next, we define two potential functions \(\tilde{p} \in H^1_{\text{loc}}(R^2\setminus \Omega), \tilde{u} \in [H^1(\Omega_1)]^2\) similarly to (2.6), (2.7) in the expression forms,

\[
\begin{align*}
\tilde{p}(x) = \int_{\partial \Omega} \left[ a_0(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - y(b_0(y) \cdot n)\Phi(x, y) \right] ds(y), \quad x \in R^2\setminus \Omega. \\
\tilde{u}(x) = - \int_{\partial \Omega} \left[ \Gamma(x, y)(a_0(y)n) + (T, \Gamma(x, y))^T b_0(y) \right] ds(y) - \int_{\partial \Omega_0} \Gamma(x, y)c_0(y) ds(y), \quad x \in \Omega_1.
\end{align*}
\]

Combining with the jump relations on \(\partial \Omega\),

\[
\begin{align*}
\tilde{p}_+ - \tilde{p}_- &= a_0 \quad \text{on } \partial \Omega, \\
\frac{\partial \tilde{p}_+}{\partial n} - \frac{\partial \tilde{p}_-}{\partial n} &= y(b_0 \cdot n) \quad \text{on } \partial \Omega, \\
\tilde{u}_+ - \tilde{u}_- &= -b_0 \quad \text{on } \partial \Omega, \\
T\tilde{u}_+ - T\tilde{u}_- &= a_0 n \quad \text{on } \partial \Omega,
\end{align*}
\]

and boundary integral equation (2.5),

\[
\tilde{u}_+|_{\partial \Omega_0} = -S_{\Omega_0\Omega_0}(a_0n) - K_{\Omega_0\Omega_0}b_0 - S_{\Omega_0\partial \Omega_0}c_0 = 0,
\]

we find that \((\tilde{p}, \tilde{u}) \in H^1_{\text{loc}}(R^2\setminus \Omega) \times [H^1(\Omega_1)]^2\) satisfies the homogeneous problem (2.1),

\[
\begin{align*}
\Delta \tilde{p} + k^2\tilde{p} &= 0 \quad \text{in } R^2\setminus \Omega, \\
\Delta^* \tilde{u} + \rho \omega^2 \tilde{u} &= 0 \quad \text{in } \Omega_1, \\
\gamma(\tilde{u} \cdot n) - \frac{\partial \tilde{p}}{\partial n} &= 0 \quad \text{on } \partial \Omega, \\
T\tilde{u}_+ + \tilde{p}_n &= 0 \quad \text{on } \partial \Omega, \\
\tilde{u}_+ &= 0 \quad \text{on } \partial \Omega_0, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial \tilde{p}}{\partial r} - ik \tilde{p} \right) &= 0.
\end{align*}
\]


According to the uniqueness theorem (Theorem 2.1), we have \( \tilde{p} = 0 \) in \( R^2 \setminus \Omega \) and \( \tilde{u} = 0 \) in \( \Omega \). Hence, we conclude that \( a_0 = 0 \) and \( b_0 = 0 \).

Finally, we define a potential function \( \tilde{u} \) the same as (2.7) in \( \Omega_0 \) with \( a_0 = 0, b_0 = 0 \). Since \( \omega \) is not a Dirichlet eigenvalue of \( -\Delta^s \) in \( \Omega_0 \),

\[
\begin{cases}
\Delta^s \tilde{u} + \rho \omega^2 \tilde{u} = 0 & \text{in } \Omega_0, \\
\tilde{u} = 0 & \text{on } \partial \Omega_0,
\end{cases}
\]

we have \( \tilde{u} = 0 \) in \( \Omega_0 \). So \( c_0 = (T \tilde{u} - \tilde{u})|_{\partial \Omega_0} = 0 \). We complete the proof.

Note that boundary integral equation (2.3) is a scalar equation, while (2.4) and (2.5) are vector equations. If we transform the two vector equations into scalar equations (see [31]), it is a little difficult to analyse the obtained scalar system. So we consider the corresponding weak formulation of system (2.3)–(2.5), where the idea comes from [39].

The standard weak formulation takes the following form: for any

\[ (\tilde{a}, \tilde{b}, \tilde{c}) \in H^\frac{1}{2}(\partial \Omega) \times [H^\frac{1}{2}(\partial \Omega)]^2 \times [H^{-\frac{1}{2}}(\partial \Omega_0)]^2 =: \mathcal{H}, \]

find \( (a, b, c) \in \mathcal{H} \) satisfying

\[ A(a, b, c; \tilde{a}, \tilde{b}, \tilde{c}) = Q(\tilde{a}, \tilde{b}, \tilde{c}), \]

where the sesquilinear form \( A: \mathcal{H} \times \mathcal{H} \to \mathbb{R} \) is defined by

\[
A(a, b, c; \tilde{a}, \tilde{b}, \tilde{c}) = (-Na, \tilde{a}) + \gamma \left( \left( K' + \frac{1}{2}I \right)(b \cdot n), \tilde{a} \right) + \left( \frac{1}{2} I - K'_\omega_{\Omega_0} \right)(an, \tilde{b}) + (-Na_{\partial \Omega_0} b, \tilde{b}) - (K'_{\partial \Omega_0} c, \tilde{b}) + (S_{\partial \Omega_0_0} (an), \tilde{c}) + (K_{\partial \Omega_0} b, c) + (S_{\partial \Omega_0_0} c, \tilde{c}),
\]

and the linear functional \( Q: \mathcal{H} \to \mathbb{R} \) is defined by

\[ Q(\tilde{a}, \tilde{b}, \tilde{c}) = \left( \left( K' + \frac{1}{2}I \right)f_1, \tilde{a} \right) + \left( \frac{1}{2} I - K'_\omega_{\Omega_0} \right)f_2, \tilde{b} \right) + (S_{\partial \Omega_0_0} f_2, \tilde{c}). \]

Here and below, \( (\cdot, \cdot) \) denotes the conjugated duality pairing between \( H^\frac{1}{2}(\partial \Omega) \) and \( H^{-\frac{1}{2}}(\partial \Omega) \) or \( [H^\frac{1}{2}(\partial \Omega)]^2 \) and \( [H^{-\frac{1}{2}}(\partial \Omega_0)]^2 \) or \( [H^{-\frac{1}{2}}(\partial \Omega_0)]^2 \) defined by [9, Definition 7.1].

**Theorem 2.3.** The sesquilinear form \( A: \mathcal{H} \times \mathcal{H} \to \mathbb{R} \) in (2.8) satisfies \( A = A^0 + A^c \), where \( A^c \) is a compact sesquilinear form and \( A^0 \) satisfies

\[ A^0(a, b, c; a, b, c) \geq \mathcal{C}(\|a\|_{H^\frac{1}{2}(\partial \Omega)} + \|b\|_{H^\frac{1}{2}(\partial \Omega)}^2 + \|c\|_{H^{-\frac{1}{2}}(\partial \Omega_0)}^2), \]

for any \( (a, b, c) \in \mathcal{H} \), and \( \mathcal{C} \) is a positive constant.

**Proof.** From [24, Theorem 1.26] and [33, Chapter 10], we have

\[ -N = -N(i) + N^c, \quad -Na_{\partial \Omega_0} = -Na_{\partial \Omega_0}(i) + N^c_{\partial \Omega_0}, \quad S_{\partial \Omega_0_0} a_{\partial \Omega_0} = S_{\partial \Omega_0_0} a_{\partial \Omega_0}(i) + S^c_{\partial \Omega_0_0}, \]

where \( N(i), Na_{\partial \Omega_0}(i), S_{\partial \Omega_0_0}(i) \) are the boundary integral operators defined as \( N, Na_{\partial \Omega_0}, S_{\partial \Omega_0_0} \), respectively, with \( \omega \) replaced by \( i \) in the fundamental solution. Here,

\[ N^c: H^\frac{1}{2}(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega); \quad N^c_{\partial \Omega_0_0}: [H^\frac{1}{2}(\partial \Omega)]^2 \to [H^{-\frac{1}{2}}(\partial \Omega_0)]^2; \quad S^c_{\partial \Omega_0_0}: [H^{-\frac{1}{2}}(\partial \Omega_0)]^2 \to [H^\frac{1}{2}(\partial \Omega_0)]^2 \]

are compact operators and

\[
\begin{align*}
(-N(i)a, a) & \geq \mathcal{C}_1 \|a\|_{H^\frac{1}{2}(\partial \Omega)}^2 & & \text{for } a \in H^\frac{1}{2}(\partial \Omega), \\
(-N(i)b, b) & \geq \mathcal{C}_2 \|b\|_{H^\frac{1}{2}(\partial \Omega)}^2 & & \text{for } b \in [H^\frac{1}{2}(\partial \Omega)]^2, \\
(S_{\partial \Omega_0_0}(i)c, c) & \geq \mathcal{C}_3 \|c\|_{H^{-\frac{1}{2}}(\partial \Omega_0)}^2 & & \text{for } c \in [H^{-\frac{1}{2}}(\partial \Omega_0)]^2.
\end{align*}
\]
where $c_1$, $c_2$, $c_3$ are positive constants. We let
\[
A^0(a, b, c; \bar{a}, \bar{b}, \bar{c}) = (-N(i)a, \bar{a}) + (-N_\Omega a_\Omega(i)b, \bar{b}) + (S_{\Omega\Omega}(i)c, \bar{c}).
\]

Then $A^0$ satisfies (2.9).

Since the mappings $H^\frac{1}{2}(\partial \Omega) \to H^\frac{1}{2}(\partial \Omega), [H^\frac{1}{2}(\partial \Omega)]^2 \to [H^\frac{1}{2}(\partial \Omega)]^2$ are compact embeddings (see [33]), we have $I, K': H^\frac{1}{2}(\partial \Omega) \to H^\frac{1}{2}(\partial \Omega), I, K'_{\Omega\Omega}, [H^\frac{1}{2}(\partial \Omega)]^2 \to [H^\frac{1}{2}(\partial \Omega)]^2$ are compact operators. And $K'_{\Omega\Omega}, S_{\Omega\Omega}, K_{\Omega\Omega}$ are compact due to the fact that their integral kernels are continuous. Hence, $A^c = A - A^0$ is compact. Thus, we complete the proof.

\[\square\]

**Theorem 2.4.** Under the assumptions in Theorem 2.2, for any $f_1 \in H^{-\frac{1}{2}}(\partial \Omega), f_2 \in [H^\frac{1}{2}(\partial \Omega)]^2$, problem (2.1) has a unique solution $(p, u) \in H^1_{\text{loc}}(R^2 \setminus \overline{\Omega}) \times [H^1(\Omega)]^2$ satisfying that
\[
\|p\|_{H^1(B_R(\overline{\Omega}))} + \|u\|_{[H^1(\Omega)]^2} \leq C(\|f_1\|_{H^{-\frac{1}{2}}(\partial \Omega)} + \|f_2\|_{[H^\frac{1}{2}(\partial \Omega)]^2}),
\]
where $B_R$ is a disk of radius $R$ containing $\overline{\Omega}$ and the positive constant $C$ depends on $R$ but not on $f_1, f_2$.

## 3 The factorization method for interaction boundary

In this section, we consider the factorization method to recover the shape of $\partial \Omega$. Firstly, we introduce the following basic and important functional analysis result behind the factorization method. For a detailed proof, we can refer to [24, Theorem 2.15].

**Theorem 3.1.** Let $X \subset U \subset X^*$ be a Gelfand triple with a Hilbert space $U$ and a reflexive Banach space $X$ such that the embedding is dense. Furthermore, let $Y$ be a second Hilbert space, and let $F: Y \to Y, G: X \to Y$ and $M: X^* \to X$ be linear bounded operators such that $F = GMG^*$. We make the following assumptions:

1. $G$ is compact with dense range;
2. there exists $t \in [0, 2\pi]$ such that $\text{Re}[e^{it}M]$ has the form $\text{Re}[e^{it}M] = C + K$ with some compact operator $K$ and some self-adjoint and coercive operator $C: X^* \to X$, i.e. there exists $c > 0$ with
\[
\langle \varphi, C\varphi \rangle \geq c\|\varphi\|^2 \quad \text{for all } \varphi \in X^*;
\]
3. $\text{Im}(M)$ is non-negative or non-positive on $R(G^*) \subset X^*$, i.e. $\text{Im}(\varphi, M\varphi) \geq 0$ or $\text{Im}(\varphi, M\varphi) \leq 0$ for all $\varphi \in R(G^*)$;
4. (a) $M$ is injective or
   (b) $\text{Im}(M)$ is strictly positive or strictly negative on the closure $\overline{R(G^*)}$ of $R(G^*)$, i.e. $\text{Im}(\varphi, M\varphi) > 0$ or $\text{Im}(\varphi, M\varphi) < 0$ for all $\varphi \in \overline{R(G^*)}$ with $\varphi \neq 0$.

Then the operator $F_\# = |\text{Re}[e^{it}F]| + |\text{Im}(F)|$ is positive and the ranges of $G: X \to Y$ and $F_\#^{\frac{1}{2}}: Y \to Y$ coincide.

Then we introduce the acoustic Herglotz wave function (refer to [11, Definition 3.18]) with density $g \in L^2(S)$ defined by
\[
p_g(x) = \int_S e^{ik\theta \cdot x} g(\theta) \, ds(\theta), \quad x \in R^2, \tag{3.1}
\]
where $S = \{x \in R^2 : |x| = 1\}$ is the unit circle.

Define the far-field operator $F: L^2(S) \to L^2(S)$ by
\[
(Fg)(x) = \int_S p^{\infty}(\hat{x}, \theta)g(\theta) \, ds(\theta), \quad \hat{x} \in S,
\]
where $p^{\infty}(\hat{x}, \theta)$ is the far-field pattern of the radiation solution $p$ corresponding to the incident wave $p^i = e^{ikx \cdot \theta}$ of problem (2.1).

From the definition of the far-field operator, we know that $Fg$ is the far-field pattern corresponding to the incident field of the acoustic Herglotz wave function $p_g$ (refer to [11, Lemma 3.20]).
Assume that $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $\Omega$, we define the interior Dirichlet-to-Neumann map $\Lambda: H^\frac{1}{2}(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega)$ as $\Delta \theta + k^2 \theta = 0$ in $\Omega$, 
$$\theta = h \quad \text{on } \partial \Omega.$$ 

In the following, $\omega$ is said to be an interior transmission eigenvalue if there exists a non-trivial pair $(p, u) \in H^1(\Omega) \times [H^1(\Omega)]^2$ that satisfies the system

$$
\begin{aligned}
\Delta p + k^2 p &= 0 \quad \text{in } \Omega, \\
\Delta^* u + \rho \omega^2 u &= 0 \quad \text{in } \Omega_1, \\
\gamma u \cdot n - \frac{\partial p}{\partial n} &= 0 \quad \text{on } \partial \Omega, \\
Tu + pn &= 0 \quad \text{on } \partial \Omega, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial p}{\partial r} - ikp \right) &= 0.
\end{aligned}
$$

(3.2)

**Assumption 3.2.** In the subsequent arguments, we need the following assumptions: $k^2$ is neither a Dirichlet nor a Neumann eigenvalue of $-\Delta$ in $\Omega$; $\omega$ is neither a Dirichlet eigenvalue of $-\Delta^*$ in $\Omega_0$ nor a Jones frequency nor an interior transmission eigenvalue.

Next, we define a data to pattern operator $G: H^\frac{1}{2}(\partial \Omega) \to L^2(S)$ in a special way under Assumption 3.2 which is motivated by [25]. For $\psi \in H^\frac{1}{2}(\partial \Omega)$, we define $G\psi = p^{\text{fo}}$, where $p^{\text{fo}}$ is the far-field pattern of the solution $p$ to the problem

$$
\begin{aligned}
\Delta p + k^2 p &= 0 \quad \text{in } R^2 \setminus \overline{\Omega}, \\
\Delta^* u + \rho \omega^2 u &= 0 \quad \text{in } \Omega_1, \\
\gamma u \cdot n - \frac{\partial p}{\partial n} &= \Lambda \psi \quad \text{on } \partial \Omega, \\
Tu + pn &= -\psi n \quad \text{on } \partial \Omega, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial p}{\partial r} - ikp \right) &= 0.
\end{aligned}
$$

(3.3)

We see that $(p, u)$ is a unique solution of problem (2.1) with special boundary data $f_1 = \Lambda \psi \in H^{-\frac{1}{2}}(\partial \Omega)$, $f_2 = -\psi n \in [H^\frac{1}{2}(\partial \Omega)]^2$.

Then we introduce the acoustic Herglotz wave operator $H: L^2(S) \to H^\frac{1}{2}(\partial \Omega)$ by

$$(Hg)(x) = p_g(x)|_{\partial \Omega} = \int_S e^{ik\theta} g(\theta) \, ds(\theta), \quad x \in \partial \Omega.$$  

Recalling the definition of $\Lambda$, we have $\Lambda Hg = \frac{\partial p_g}{\partial n}|_{\partial \Omega}$. By the superposition principle and the definition of $G$ and $F$, we obtain the relation

$$F = GH.$$  

(3.4)

**Theorem 3.3.** Under Assumption 3.2, the far-field operator $F: L^2(S) \to L^2(S)$ has the factorization $F = GB^* G^*$. Here, $B: H^{-\frac{1}{2}}(\partial \Omega) \to H^\frac{1}{2}(\partial \Omega)$ is given by $B\phi = (q - \tilde{q})|_{\partial \Omega}$, where $\tilde{q}$ is defined by (3.5) and $q$ is defined by (3.6).

**Proof.** From the definition of $H$ in (3.3), we can calculate the adjoint operator $H^*: H^{-\frac{1}{2}}(\partial \Omega) \to L^2(S)$ by

$$(H^* \varphi)(\tilde{x}) = \int_{\partial \Omega} \varphi(y) e^{-ikx \cdot y} \, ds(y), \quad \varphi \in H^{-\frac{1}{2}}(\partial \Omega).$$

By the asymptotic behavior of the fundamental solution, we can see that $H^* \varphi$ is the far-field pattern of the single-potential function $\tilde{q}(x)$ defined by

$$\tilde{q}(x) = \int_{\partial \Omega} \varphi(y) \Phi(x, y) \, ds(y), \quad x \in R^2 \setminus \partial \Omega.$$  

(3.5)
We know that \( \tilde{q} \) is a radiation solution of the Helmholtz equation \( \Delta \tilde{q} + k^2 \tilde{q} = 0 \) in \( R^2 \setminus \partial \Omega \). And from the jump relationship of single-potential, we have \( \varphi = (\frac{\partial \tilde{q}}{\partial n} - \frac{\partial \tilde{q}}{\partial n})|_{\partial \Omega} \) and \( \tilde{q} \mid_{\partial \Omega} = \tilde{q} - \tilde{q} \mid_{\partial \Omega} \).

Since \( \omega \) is not an interior transmission eigenvalue, for this \( \varphi \in H^{\frac{1}{2}}(\partial \Omega) \), we can determine a solution \((q, v) \in H^1(\Omega) \times [H^1(\Omega)]^2 \) of the problem

\[
\begin{aligned}
\Delta q + k^2 q &= 0 \quad \text{in} \, \Omega, \\
\Delta^* v + \rho \omega^2 v &= 0 \quad \text{in} \, \Omega_1, \\
yv \cdot n - \frac{\partial \tilde{q}}{\partial n} &= -\varphi \quad \text{on} \, \partial \Omega, \\
Tv + qn &= 0 \quad \text{on} \, \partial \Omega, \\
v &= 0 \quad \text{on} \, \partial \Omega_0.
\end{aligned}
\] (3.6)

Note that the well-posedness of the above problem (3.6) can be achieved similarly to [34, Theorem 3.3].

Combining with (3.5) and (3.6), we conclude that \((\tilde{q}, v)\) satisfies problem (3.2) with the boundary conditions

\[
y v \cdot n - \frac{\partial \tilde{q}}{\partial n} = -\varphi + \frac{\partial q}{\partial n} - \frac{\partial \tilde{q}}{\partial n} = \frac{\partial q}{\partial n} - \frac{\partial \tilde{q}}{\partial n} = \Lambda(q - \tilde{q}) \quad \text{on} \, \partial \Omega, \\
Tv + qn = -(\varphi v + \tilde{q} n) = -(q - \tilde{q}) \quad \text{on} \, \partial \Omega, \\
v &= 0 \quad \text{on} \, \partial \Omega_0.
\]

We define an operator \( B : H^{\frac{1}{2}}(\partial \Omega) \to H^{\frac{1}{2}}(\partial \Omega) \) which is given by \( B\varphi = (q - \tilde{q})|_{\partial \Omega} \). By the definition of \( G \), we conclude that \( GB\varphi = \tilde{q}^{\text{co}} = H^* \varphi \) (a factor \( \frac{\rho \omega}{\sqrt{\rho \omega^2 + k^2}} \) is omitted for simplicity). Then we have \( H = B^* G^* \). Combining with relation (3.4), we obtain \( F = GB^* G^* \). The proof is completed. \( \square \)

**Theorem 3.4.** Assume that \( k^2 \) and \( \omega \) satisfy Assumption 3.2. Then

1. \( \text{Im}(B) \) is strictly positive on \( H^{-\frac{1}{2}}(\partial \Omega), \) i.e. \( \text{Im}(\varphi, B\varphi) > 0 \) for all \( \varphi \in H^{-\frac{1}{2}}(\partial \Omega) \) with \( \varphi \neq 0 \),
2. the operator \( B \) is the sum of a coercive operator \( B^0 \) and a compact operator \( B^c \).

**Proof.** (1) For any \( \varphi \in H^{-\frac{1}{2}}(\partial \Omega) \), let \( \tilde{q} \) be defined as (3.5), and \((q, v)\) be the solution of (3.6). Then

\[
(\varphi, B\varphi) = (\varphi, q - \tilde{q}) = \int_{\partial \Omega} \varphi \tilde{q} \, ds - \int_{\partial \Omega} \varphi \bar{q} \, ds = \int_{\partial \Omega} \left( \frac{\partial \tilde{q}}{\partial n} - yv \cdot n \right) \bar{q} \, ds + \int_{\partial \Omega} \left( \frac{\partial \tilde{q}}{\partial n} - \frac{\partial q}{\partial n} \right) \tilde{q} \, ds
\]

\[
= \int_{\Omega} \left( |\nabla q|^2 - k^2 |q|^2 \right) \, dx - y \int_{\partial \Omega} \left( \varphi - \bar{q} n \right) \, ds + \int_{\partial \Omega} \frac{\partial \tilde{q}}{\partial n} \, ds
\]

\[
- \int_{B_R} \left( |\nabla \tilde{q}|^2 - k^2 |\tilde{q}|^2 \right) \, dx - \int_{\Omega} \left( |\nabla \tilde{q}|^2 - k^2 |\tilde{q}|^2 \right) \, dx,
\] (3.7)

where \( B_R \) is a disk centered at the origin with radius \( R \) such that \( \overline{\Omega} \subset B_R \). Using the boundary conditions and the Betti formula, we have

\[
- y \int_{\partial \Omega} \left( \varphi v + \tilde{q} n \right) \, ds = y \int_{\partial \Omega} \left( v \cdot \bar{T} \, ds + \int_{\Omega} \left( E(v, \bar{v}) - \rho \omega |v|^2 \right) \, dx \right) = y \int_{\Omega} \left( E(v, \bar{v}) - \rho \omega |v|^2 \right) \, dx.
\]

Taking the imaginary part of (3.7) and letting \( R \) tend to infinity yield

\[
\text{Im}(\varphi, B\varphi) = \text{Im} \left( \lim_{R \to \infty} \int_{\partial B_R} \frac{\partial \tilde{q}}{\partial n} \, ds \right) = k \lim_{R \to \infty} \int_{\partial B_R} |\tilde{q}|^2 \, ds \geq \frac{1}{8\pi} \| \tilde{q}^{\text{co}} \|_{L^2(S)}^2 \geq 0.
\]

Now let \( \text{Im}(\varphi, B\varphi) = 0 \); then \( \tilde{q}^{\text{co}} = 0 \). Based on [9, Theorem 4.1], we conclude that \( \tilde{q} = 0 \) in \( R^2 \setminus \overline{\Omega} \), and so \( q \mid_{\partial \Omega} = \tilde{q} - q \mid_{\partial \Omega} = 0 \). Since \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \), we have \( q = 0 \) in \( \Omega \). Consequently, \( \varphi = (\frac{\partial \tilde{q}}{\partial n} - \frac{\partial q}{\partial n})|_{\partial \Omega} = 0 \). So we have shown the strictly positive property \( \text{Im}(\varphi, B\varphi) > 0 \) for \( \varphi \in H^{-\frac{1}{2}}(\partial \Omega) \) with \( \varphi \neq 0 \).

(2) From the definition of single-potential operator \( S \) (refer to [24, Lemma 1.14]), we have \( \tilde{q} \mid_{\partial \Omega} = S\varphi \). We define a mapping \( M : H^{-\frac{1}{2}}(\partial \Omega) \to H^{\frac{1}{2}}(\partial \Omega) \) by \( M\varphi = q \mid_{\partial \Omega} \); then \( B\varphi = M\varphi - S\varphi \).
We consider $\gamma$ and $\omega$ as being two independent parameters and write $M = M(\gamma, \omega)$ to indicate this dependence. Similarly, we write $S = S(k)$ to indicate the dependence on $k$. Note that $\omega = ic_0$ is equivalent to $k = i$. We decompose $B = M(0, ic_0) - S(i) + [M(\gamma, \omega) - M(0, ic_0)] - [S(k) - S(i)]$. Next, we will show that $B^0 = M(0, ic_0) - S(i)$ is compact. $B^c = [M(\gamma, \omega) - M(0, ic_0)] - [S(k) - S(i)]$ is compact.

Firstly, we consider $B^0 = M(0, ic_0) - S(i)$. From the above calculation in (3.7), we have

$$(\varphi, B^0 \varphi) = \|q\|^2_{H^1(\Omega)} - \|\tilde{q}\|^2_{H^1(\mathbb{R}^2)}.$$  

Since $y = 0$, $k = i$, then $\varphi = \frac{\partial q}{\partial n}\big|_{\partial \Omega}$ and

$$\|q\|^2_{H^1(\mathbb{R}^2)} = (\varphi, \tilde{q}) = \int_{\partial \Omega} \frac{\partial q}{\partial n} \tilde{q} \, ds = \int_{\partial \Omega} \left( \nabla \tilde{q} \cdot \tilde{q} \right) \, dx \leq \|q\|_{H^1(\Omega)} \|\nabla \tilde{q}\|_{L^2(\mathbb{R}^2)} \leq \|q\|_{H^1(\Omega)} \|\tilde{q}\|_{H^1(\mathbb{R}^2)}.$$  

Therefore, $\|\tilde{q}\|_{H^1(\mathbb{R}^2)} \leq \|q\|_{H^1(\Omega)}$; thus $(\varphi, B^0 \varphi) \geq 0$, that is, $B^0$ is non-negative. It remains to show for this part that $B^0$ is coercive.

Assume, on the contrary, that this is not the case. Then there exists a sequence $\varphi_j \in H^{-\frac{1}{2}}(\partial \Omega)$ with $\|\varphi_j\|_{H^{-\frac{1}{2}}(\partial \Omega)} = 1$ for all $j$ and $(\varphi_j, B^0 \varphi_j) \to 0$ as $j \to \infty$.

Let $q_j \in H^1(\Omega)$ and $\tilde{q}_j \in H^1(\mathbb{R}^2)$ be the corresponding fields. Then $\|q_j\|^2_{H^1(\Omega)} - \|\tilde{q}_j\|^2_{H^1(\mathbb{R}^2)} \to 0$ as $j \to \infty$. Furthermore,

$$\|q_j - \tilde{q}_j\|^2_{H^1(\Omega)} = \|q_j\|^2_{H^1(\Omega)} + \|\tilde{q}_j\|^2_{H^1(\Omega)} - 2 \text{Re}(q_j, \tilde{q}_j)_{H^1(\Omega)}$$  

which contradicts $\|\varphi\|_{H^{-\frac{1}{2}}(\partial \Omega)} = 1$.

Secondly, consider $B^c = [M(\gamma, \omega) - M(0, ic_0)] - [S(k) - S(i)]$. Referring to [24, Lemma 1.14], we know $S(k) - S(i)$ is compact, and we only need to show the compactness of $M(\gamma, \omega) - M(0, ic_0)$. For $\varphi \in H^{-\frac{1}{2}}(\partial \Omega)$, we denote the fields corresponding to $(\gamma, \omega)$ and $(0, ic_0)$ by $(q, v)$ and $(q_i, v_i)$, respectively.

Therefore, the difference $(q - q_i, v - v_i)$ satisfies the source problem

$$
\begin{aligned}
\Delta q + k^2 q &= 0 \quad \text{in } \Omega, \\
\Delta^* v + \rho \omega^2 v &= 0 \quad \text{in } \Omega_1, \\
yv \cdot n - \frac{\partial q}{\partial n} &= -\varphi \quad \text{on } \partial \Omega, \\
Tv + qn &= 0 \quad \text{on } \partial \Omega, \\
v &= 0 \quad \text{on } \partial \Omega_0.
\end{aligned}
$$

Therefore, the difference $(q - q_i, v - v_i)$ satisfies the source problem [34]

$$
\begin{aligned}
\Delta (q - q_i) - (q - q_i) &= -(k^2 + 1)q \quad \text{in } \Omega, \\
\Delta^* (v - v_i) - \rho \omega^2 (v - v_i) &= -\rho (\omega^2 + c_0^2) v \quad \text{in } \Omega_1, \\
y(v - v_i) \cdot n - \frac{\partial q}{\partial n} &= -y(v_i \cdot n) \quad \text{on } \partial \Omega, \\
Tv(v - v_i) + (q - q_i)n &= 0 \quad \text{on } \partial \Omega, \\
v - v_i &= 0 \quad \text{on } \partial \Omega_0.
\end{aligned}
$$

The mapping $(q, v, (v_i \cdot n))_{|\partial \Omega} \to (q - q_i, v - v_i)$ from the source $(q, v)$ and the boundary data $(v_i \cdot n)_{|\partial \Omega}$ on the right-hand side of (3.8) to the solution $(q - q_i, v - v_i)$ is bounded from $L^2(\Omega) \times [L^2(\Omega_1)]^2 \times H^{-\frac{1}{2}}(\partial \Omega)$ into $H^1(\Omega) \times [H^1(\Omega_1)]^2$ (see [34]). Since the embedding map

$$H^1(\Omega) \times [H^1(\Omega_1)]^2 \times H^{-\frac{1}{2}}(\partial \Omega) \to L^2(\Omega) \times [L^2(\Omega_1)]^2 \times H^{-\frac{1}{2}}(\partial \Omega)$$
is compact, we have \((q, v, (v \cdot n)|_{\partial \Omega}) \rightarrow (q - q_i, v - v_i)\) from the source \(q, v\) and the boundary data \((v \cdot n)|_{\partial \Omega}\) to the solution \((q - q_i, v - v_i)\) is compact from \(H^1(\Omega) \times [H^1(\Omega)]^2 \times H^2(\partial \Omega)\) to \(H^1(\Omega) \times [H^1(\Omega)]^2\).

Furthermore, the solution mapping \(\varphi \rightarrow (q, v, (v \cdot n)|_{\partial \Omega})\) is bounded from \(H^{-2}(\Omega)\) into

\[ H^1(\Omega) \times [H^1(\Omega)]^2 \times H^2(\partial \Omega), \]

and the trace map \((q - q_i)|_{\Omega} \rightarrow (q - q_i)|_{\partial \Omega}\) is bounded from \(H^1(\Omega)\) to \(H^\frac{1}{2}(\partial \Omega)\). We conclude that \(\varphi \rightarrow (q - q_i)|_{\partial \Omega}\) is compact from \(H^{-\frac{1}{2}}(\partial \Omega)\) into \(H^\frac{1}{2}(\partial \Omega)\). The proof is completed.

\[ \square \]

**Theorem 3.5.** Assume that \(k^2\) and \(\omega\) satisfy Assumption 3.2. Then the data to pattern operator \(G\) is compact and has a dense range in \(L^2(S)\).

**Proof.** The proof of compactness is similar to \([24]\); we omit it here for brevity. Next, we will show that the adjoint operator \(G^*\) is injective, which implies that \(G\) has a dense range in \(L^2(S)\).

From the definition of \(G\), we calculate the adjoint operator \(G^*\) as follows (refer to \([40, Lemma 3.2]\)). Assume that \((p, u)\) is the solution of (3.2) with \(\psi\), and \((\tilde{p}, \tilde{u})\) is the solution of (3.2) with \(\tilde{p}_g\), the corresponding total field is denoted by \(\tilde{p}^t = \tilde{p} + \tilde{p}_g\).

Based on Green’s representation theorem, we have

\[
p(x) = \int_{\partial \Omega} \left[ p(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \frac{\partial p}{\partial n}(y) \Phi(x, y) \right] ds(y), \quad x \in R^3 \setminus \Omega.
\]

Then the corresponding far-field pattern \(p^{\infty}\) is given by (a factor \(\frac{\sin \theta}{\sqrt{\lambda \pi k}}\) is omitted for simplicity)

\[
p^{\infty}(d) = \int_{\partial \Omega} \left[ p(y) \frac{\partial e^{-ikd\cdot y}}{\partial n(y)} - \frac{\partial p}{\partial n}(y) e^{-ikd\cdot y} \right] ds(y).
\]

Recalling the definition of \(G\) and the representation of \(p_g\) in (3.1), we have

\[
\langle G\psi, g \rangle = \int_S p^{\infty}(d) \overline{g(d)} \, ds(d) = \int_{\partial \Omega} \left[ p(y) \frac{\partial \tilde{p}_g}{\partial n} - \frac{\partial p}{\partial n}(y) \overline{\tilde{p}_g} \right] ds(y) = \int_{\partial \Omega} \left[ p(y) \frac{\partial \tilde{p}^t}{\partial n} - \frac{\partial p}{\partial n}(y) \tilde{p}^t \right] ds(y).
\]

The last equality is obtained by the fact that \(p\) and \(\tilde{p} = \tilde{p}^t - \overline{\tilde{p}_g}\) are radiation solutions to the Helmholtz equation in \(R^3 \setminus \Omega\). Next, we will use the boundary conditions in (3.2) to gain an explicit characterization of \(G^*\).

\[
\langle G\psi, g \rangle = \int_{\partial \Omega} \left[ p(y) (y \tilde{u} \cdot n) - (yu \cdot n - \Lambda \psi) \tilde{p}^t \right] ds(y) = \int_{\partial \Omega} \left[ y \tilde{u} \cdot (\psi y - Tu) + yu \cdot T\tilde{u} + \Lambda \psi \tilde{p}^t \right] ds(y)
\]

\[
\begin{align*}
= -y \int_{\partial \Omega} (\tilde{u} \cdot n) \psi ds(y) + \int_{\partial \Omega} \Lambda \tilde{p}^t \psi ds(y) + y \int_{\partial \Omega} (u \cdot T\tilde{u} - \tilde{u} \cdot T\tilde{u}) ds(y) + y \int_{\partial \Omega} (u \cdot \Lambda \Delta \tilde{u} - \tilde{u} \cdot \Lambda \Delta \tilde{u}) dy \\
= \int_{\partial \Omega} (-y \tilde{u} \cdot n + \Lambda \tilde{p}^t) \psi ds(y) = \langle \psi, G^* g \rangle.
\end{align*}
\]

The boundary conditions on \(\partial \Omega\) are used in the first and second equalities. In the third equality, we have used \([40, Lemma 3.1]\) and the Betti formula. In the fourth equality, the boundary conditions on \(\partial \Omega_0\) and the Navier equation are used. Hence,

\[
G^* g = (-y \tilde{u} \cdot n + \Lambda \tilde{p}^t)|_{\partial \Omega}.
\]

Finally, we will show that \(G^*\) is injective. Let \(G^* g = 0\); then \((-y \tilde{u} \cdot n + \Lambda \tilde{p}^t)|_{\partial \Omega} = 0\). Recalling the boundary conditions on \(\partial \Omega\) and the facts that \(\tilde{p}^t = \tilde{p} + \overline{\tilde{p}_g}\), \(\Lambda \tilde{p}_g = \frac{\partial \tilde{p}_g}{\partial n}\), we have

\[
\frac{\partial \tilde{p}}{\partial n} = y \tilde{u} \cdot n - \frac{\partial \tilde{p}_g}{\partial n} = \Lambda (\tilde{p} + \overline{\tilde{p}_g}) - \frac{\partial \tilde{p}_g}{\partial n} = \Lambda \tilde{p} \quad \text{on} \ \partial \Omega.
\]

(3.9)

Since \(k^2\) is not a Dirichlet eigenvalue of \(-\Delta\) in \(\Omega\), we can find a unique solution \(\tilde{p} \in H^1(\Omega)\) satisfying

\[
\begin{cases}
\Delta \tilde{p} + k^2 \tilde{p} = 0 & \text{in} \ \Omega, \\
\tilde{p} = \tilde{p} & \text{on} \ \partial \Omega.
\end{cases}
\]
In this section, we consider the following problem which has been deeply investigated in [25]:

The explicit characterization of the unknown penetrable obstacle \( \Omega \) of the equation is an eigenvalue of 2. Assume that we complete the proof of the theorem.

(1) Assume firstly that \( \partial p/\partial n \mid_{\partial \Omega} = \Lambda \hat{p} + \partial \hat{p}/\partial n \mid_{\partial \Omega} \), that is,

\[
\hat{p} = \hat{\rho}, \quad \frac{\partial \hat{p}}{\partial n} = \frac{\partial \hat{\rho}}{\partial n} \quad \text{on} \ \partial \Omega.
\]

Define \( P = \hat{p} \in \Omega, P = \hat{p} \in R^2 \setminus \Omega. \) Then \( P \) is an entire radiating solution of the Helmholtz equation in \( R^2 \), implying that \( P = 0 \) in \( R^2 \). Consequently, \( \hat{p} \mid_{\partial \Omega} = 0 = \frac{\partial \hat{p}}{\partial n} \mid_{\partial \Omega}, \) and \((\overline{\rho}, \hat{u})\) satisfies the homogeneous transmission problem

\[
\begin{align*}
\Delta \overline{\rho} + k^2 \overline{\rho} &= 0 \quad \text{in} \ \Omega, \\
\Delta^* \hat{u} + \rho \omega^2 \hat{u} &= 0 \quad \text{in} \ \Omega_1, \\
g \hat{u} \cdot n - \frac{\partial \overline{\rho}}{\partial n} &= 0 \quad \text{on} \ \partial \Omega, \\
T \hat{u} + \overline{\rho} n &= 0 \quad \text{on} \ \partial \Omega_0, \\
\hat{u} &= 0 \quad \text{on} \ \partial \Omega_0.
\end{align*}
\]

Under the assumption that \( \omega \) is not an interior transmission eigenvalue, we have \( \overline{\rho} = 0 \) in \( \Omega \). By the analyticity, \( \overline{\rho} = 0 \) in \( R^2 \), which implies that \( g = 0. \) The proof is completed.

**Theorem 3.6.** For any \( z \in R^2 \), define \( \phi_z(\hat{x}) = e^{-ik\cdot\hat{x}} \) for \( \hat{x} \in \mathcal{S} \). Then \( z \in \Omega \) if and only if \( \phi_z \in R(G) \), where \( R(G) \) denotes the range of \( G \).

**Proof.** (1) Assume firstly that \( z \in \Omega \). Define \( \psi(x) = \Phi(x, z), x \in \partial \Omega \). Then \( p(x) = \Phi(x, z) \in R^2 \setminus \Omega \) and \( u = 0 \) in \( \Omega_1 \) solve transmission problem (3.2) for this choice of \( \psi \). Therefore, \( G\psi = \phi_z \) since \( \phi_z \) is the far-field pattern of \( \Phi(x, y) \).

(2) Now assume that \( z \notin \Omega \) and there exists \( \psi \) such that \( G\psi = \phi_z \). Let \( (p, u) \) be the corresponding fields with \( \psi \). According to the Rellich’s lemma and the unique continuation principle, we see that \( p(x) = \Phi(x, z) \) for \( x \in R^2 \setminus (\Omega \cup \{z\}) \).

However, if \( z \in R^2 \setminus \Omega \), then \( \|p\|_{H^1(U(z))} < \infty \) and \( \|\Phi(x, z)\|_{H^1(U(z))} = \infty \), where \( U(z) \) is a small domain in \( R^2 \setminus \Omega \) centered at point \( z \); if \( z \in \partial \Omega \), then \( p \mid_{\partial \Omega} \in H^1(\partial \Omega) \) and \( \Phi(x, z) \mid_{\partial \Omega} \notin H^1(\partial \Omega) \). This is a contradiction. So we complete the proof of the theorem.

**Theorem 3.7.** Assume that \( k^2 \) is neither a Dirichlet nor a Neumann eigenvalue of \( -\Delta \) in \( \Omega \), \( \omega \) is neither a Dirichlet eigenvalue of \( -\Delta^* \) in \( \Omega_0 \) nor a Jones frequency nor an interior transmission eigenvalue. For any \( z \in R^2 \), define \( \phi_z(\hat{x}) = e^{-ik\cdot\hat{x}} \) for \( \hat{x} \in \mathcal{S} \). Then

\[
z \in \Omega \iff \phi_z \in R(F^{-1}_t) \iff \left\{ \sum_{n=1}^{\infty} \frac{|(\phi_z, \varphi_n)_{L^2(\mathcal{S})}|^2}{|\lambda_n|} \right\}^{-1} > 0,
\]

where \( \{\lambda_n, \varphi_n\} \) is the eigensystem of the operator \( F_t = |\text{Re}(F)| + |\text{Im}(F)| \).

**Remark 3.8.** The explicit characterization of the unknown penetrable obstacle \( \Omega \) in terms of the solvability of the equation

\[
(F^{-1}_t g)(\hat{x}) = \Phi^{\omega}(\hat{x}, z) = \frac{e^{i\omega/4}}{\sqrt{8\pi k}} e^{-ik\cdot\hat{x}}.
\]

### 4 The mixed reciprocity relation

In this section, we consider the following problem which has been deeply investigated in [25]:

\[
\begin{align*}
\Delta p_0^t + k^2 p_0^t &= 0 \quad \text{in} \ R^2 \setminus \Omega, \\
\Delta^* u_0 + \rho \omega^2 u_0 &= 0 \quad \text{in} \ \Omega, \\
g u_0 \cdot n - \frac{\partial p_0^t}{\partial n} &= 0 \quad \text{on} \ \partial \Omega, \\
T u_0 + p_0^t n &= 0 \quad \text{on} \ \partial \Omega.
\end{align*}
\]
Here, \( p^s = p^i + p_0 \) is the total field which is the superposition of the given incident plane wave \( p^i = e^{ikx-d} \) and the scattered wave \( p_0 \). And \( p_0 \) satisfies the Sommerfeld radiation condition, i.e.

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial p_0}{\partial r} - ikp_0 \right) = 0, \quad r = |x|.
\]

We let \((G_k(x, z), G_\omega(x, z))\) be the Green function under the fixed direction \( \alpha \) for the above problem, that is, \( G_k(x, z) - \Phi(x, z) \) is a smooth solution of the Helmholtz equation, \( G_\omega(x, z) - \Gamma(x, z) \alpha \) is a smooth solution of the Navier equation ([20]), and let \( G_k^\omega(\hat{x}, z) \) be the corresponding far-field pattern.

Let \( p^l(z, -\hat{x}) = p(z, -\hat{x}) + p^i(z, -\hat{x}) \) be the total field at \( z \in R^2 \setminus \overline{\Omega} \) with the incidence direction \(-\hat{x} \in S\), and let \( u_0(z, -\hat{x}) \) be the total displacement field at \( z \in \Omega \). We have the following mixed reciprocity relation (a similar result has been shown in [6, Theorem 2.1]).

**Theorem 4.1** (Mixed reciprocity relation). For \( \hat{x} \in S \), we have

\[
G_k^\omega(\hat{x}, z) = \begin{cases} 
    p^l_0(z, -\hat{x}), & z \in R^2 \setminus \overline{\Omega}, \\
    yu_0(z, -\hat{x}) \cdot \alpha, & z \in \Omega,
\end{cases} \tag{4.2}
\]

where \((p^l_0(\cdot, -\hat{x}), u_0(\cdot, -\hat{x}))\) solves problem (4.1) and \( \alpha \) is a fixed unit vector.

**Proof.** We divide the proof into two cases.

**Case one:** \( z \in R^2 \setminus \overline{\Omega} \). Since the difference \( G_k(\cdot, z) - \Phi(\cdot, z) \) is a smooth radiation solution of the Helmholtz equation in \( R^2 \setminus \overline{\Omega} \), then, for \( x \in R^2 \setminus \overline{\Omega} \), we have

\[
G_k(x, z) - \Phi(x, z) = \int_{\partial \Omega} \left[ G_k(y, z) - \Phi(y, z) \right] \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) - \int_{\partial \Omega} \left[ \frac{\partial G_k(y, z)}{\partial n(y)} - \frac{\partial \Phi(y, z)}{\partial n(y)} \right] \Phi(x, y) ds(y). \tag{4.3}
\]

For \( x, z \in R^2 \setminus \overline{\Omega} \), we can apply Green’s theorem to \( \Phi(x, y) \) and \( \Phi(y, z) \) in \( \Omega \) to obtain

\[
0 = \int_{\partial \Omega} \left[ \frac{\partial \Phi(x, y)}{\partial n(y)} - \frac{\partial \Phi(y, z)}{\partial n(y)} \right] \Phi(x, y) ds(y). \tag{4.4}
\]

Adding the above two equations (4.3) and (4.4) yields

\[
G_k(x, z) - \Phi(x, z) = \int_{\partial \Omega} \left[ G_k(y, z) - \Phi(y, z) \right] \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) - \int_{\partial \Omega} \left[ \frac{\partial G_k(y, z)}{\partial n(y)} - \frac{\partial \Phi(y, z)}{\partial n(y)} \right] \Phi(x, y) ds(y). \tag{4.5}
\]

Since \( G_k(\cdot, z) - \Phi(\cdot, z) \) and \( p_0(\cdot, -\hat{x}) \) are smooth radiation solution of the Helmholtz equation in \( R^2 \setminus \overline{\Omega} \), we can again use Green’s theorem and arrive at

\[
0 = \int_{\partial \Omega} \left[ \frac{\partial [G_k(y, z) - \Phi(y, z)]}{\partial n(y)} - \frac{\partial p_0(y, -\hat{x})}{\partial n(y)} [G_k(y, z) - \Phi(y, z)] \right] ds(y). \tag{4.6}
\]

Using Green’s representation formula and equality (4.6), we have

\[
p_0(z, -\hat{x}) = \int_{\partial \Omega} \left[ p_0(y, -\hat{x}) \frac{\partial \Phi(y, z)}{\partial n(y)} - \frac{\partial p_0(y, -\hat{x})}{\partial n(y)} \Phi(y, z) \right] ds(y)
\]

\[
= \int_{\partial \Omega} \left[ p_0(y, -\hat{x}) \frac{\partial G_k(y, z)}{\partial n(y)} - \frac{\partial p_0(y, -\hat{x})}{\partial n(y)} G_k(y, z) \right] ds(y)
\]

\[
= \int_{\partial \Omega} \left[ p^l_0(y, -\hat{x}) \frac{\partial G_k(y, z)}{\partial n(y)} - \frac{\partial p^l_0(y, -\hat{x})}{\partial n(y)} G_k(y, z) \right] ds(y)
\]

\[
- \int_{\partial \Omega} \left[ p^i(y, -\hat{x}) \frac{\partial G_k(y, z)}{\partial n(y)} - \frac{\partial p^i(y, -\hat{x})}{\partial n(y)} G_k(y, z) \right] ds(y).
\]
Applying the transmission conditions and the Betti formula, we can see that

\[
\int_{\partial \Omega} \left[ p^I(y, \mathbf{x}) \frac{\partial G_k(y, z) \omega}{\partial n(y)} - \frac{\partial p^I_0(y, \mathbf{x}) \omega}{\partial n(y)} G_k(y, z) \right] ds(y)
= \int_{\partial \Omega} \left[ p^I_0(y, \mathbf{x})_0(y, z) \omega \cdot n_y u_0(y, \mathbf{x}) \cdot n \right] ds(y)
= \gamma \int_{\partial \Omega} \left[ u_0(y, \mathbf{x}) \cdot T G_\omega(y, z) - G_\omega(y, z) \cdot T u_0(y, \mathbf{x}) \right] ds(y)
= \gamma \int_{\partial \Omega} \left[ u_0(y, \mathbf{x}) \cdot \Delta^* G_\omega(y, z) - G_\omega(y, z) \cdot \Delta^* u_0(y, \mathbf{x}) \right] dy = 0.
\]

Hence, combining with (4.5), we obtain

\[
p_0(z, \mathbf{x}) = \int_{\partial \Omega} \left[ \frac{\partial p^I(y, \mathbf{x}) \omega}{\partial n(y)} G_k(y, z) - p^I(y, \mathbf{x}) \frac{\partial G_k(y, z)}{\partial n(y)} \right] ds(y)
= \int_{\partial \Omega} \left[ \frac{\partial \Phi^\omega(\mathbf{x}, y)}{\partial n(y)} G_k(y, z) - \Phi^\omega(\mathbf{x}, y) \frac{\partial G_k(y, z)}{\partial n(y)} \right] ds(y)
= G_k^\omega(\mathbf{x}, z) - \Phi^\omega(\mathbf{x}, z) = G_k^\omega(\mathbf{x}, z) - p^I(z, \mathbf{x}),
\]
that is, \( G_k^\omega(\mathbf{x}, z) = p^I_0(z, \mathbf{x}) \) for \( z \in R^3 \backslash \Omega \).

Case two: \( z \in \Omega \). Now the Green function \( G_k(x, z) \) is a smooth radiating solution to the Helmholtz equation in \( R^3 \backslash \Omega \). Then its far-field pattern has the form

\[
G_k^\omega(\mathbf{x}, z) = \int_{\Omega} \left[ G_k(y, z) \frac{\partial \Phi^\omega(\mathbf{x}, y)}{\partial n(y)} - \frac{\partial G_k(y, z)}{\partial n(y)} \Phi^\omega(\mathbf{x}, y) \right] ds(y)
= \int_{\Omega} \left[ G_k(y, z) \frac{\partial p^I(y, \mathbf{x}) \omega}{\partial n(y)} - \frac{\partial G_k(y, z)}{\partial n(y)} p^I(y, \mathbf{x}) \right] ds(y).
\]

Using the Betti representation and the symmetrical property of \( \Gamma(x, y) \), we have

\[
u_0(z, \mathbf{x}) \cdot a = \int_{\partial \Omega} \left[ [\Gamma(y, z) T u_0(y, \mathbf{x}) - (T_y \Gamma(y, z))^T u_0(y, \mathbf{x})] \cdot a \right] ds(y)
= \int_{\partial \Omega} \left[ [\Gamma(y, z) a] \cdot T u_0(y, \mathbf{x}) - T_y[\Gamma(y, z) a] \cdot u_0(y, \mathbf{x}) \right] ds(y).
\]

Since the difference \( G_\omega(\cdot, z) - \Gamma(\cdot, z) a \) and \( u_0(\cdot, \mathbf{x}) \) are smooth solutions of the Navier equation in \( \Omega \), then

\[
\int_{\partial \Omega} \left[ [G_\omega(y, z) - \Gamma(y, z) a] \cdot T u_0(y, \mathbf{x}) - T_y[G_\omega(y, z) - \Gamma(y, z) a] \cdot u_0(y, \mathbf{x}) \right] ds(y) = 0.
\]

Adding the above equality into (4.9) and applying the transmission conditions,

\[
\int_{\partial \Omega} \left[ G_\omega(y, z) \cdot T u_0(y, \mathbf{x}) - T_y G_\omega(y, z) \cdot u_0(y, \mathbf{x}) \right] ds(y)
= \int_{\partial \Omega} \left[ -G_\omega(y, z) \cdot (p^I_0(y, \mathbf{x}) \omega) + G_k(y, z) \frac{\partial u_0(y, \mathbf{x})}{\partial n(y)} \right] ds(y)
= \frac{1}{\gamma} \int_{\partial \Omega} \left[ G_k(y, z) \frac{\partial p^I(y, \mathbf{x}) \omega}{\partial n(y)} - p^I(y, \mathbf{x}) \frac{\partial G_k(y, z)}{\partial n(y)} \right] ds(y). \tag{4.10}
\]
The last equality is obtained owing to the fact that $G_k(\cdot, z)$ and $p_0(\cdot, -\hat{x})$ are smooth radiation solutions of the Helmholtz equation in $R^2 \setminus \overline{\Omega}$, where $p'_0 = p_0 + p^I$.

Recalling the representation of $G^\alpha_k(x, z)$ in (4.8), we conclude that $G^\alpha_k(x, z) = y u_0(z, -\hat{x}) \cdot a$. The proof is completed.

Define the far-field operator $F_0 : L^2(S) \to L^2(S)$ by

$$(F_0 g)(\hat{x}) = \int_S p_0^\alpha(\hat{x}, \theta) g(\theta) \, ds(\theta), \quad \hat{x} \in S.$$  

According to [25, Theorem 1.1], we know the scattering matrix $S = I + \frac{i}{4\pi} F_0$ is unitary, i.e. $S^* S = S S^* = I$.

In the following, for $z \in \Omega$, we can establish a close relation between the interior elastic field and the matrix $S$.

**Remark 4.2.** Referring to the paper [6], the factorization method is applied to reconstruct the embedded obstacle of acoustic scattering problem. Based on the following theorem, we may gain a satisfied factorization formulation of $F - F_0$ after clearly defining the data to pattern operator $\hat{G}$ to the difference problem between (1.1) and (4.1). So, in the future, we want to use the factorization method to recover the buried object.

**Theorem 4.3.** For all $z \in \Omega$, $\hat{x} \in S$, we have $u_0(z, -\hat{x}) \cdot a = (S[I u_0(z, \cdot) \cdot a])(\hat{x})$.

**Proof.** For $z \in \Omega$, $\gamma \in R^2 \setminus \overline{\Omega}$, take $R$ large enough such that $\Omega$ and $\gamma$ are contained in $B_R$, where $B_R$ is a disk centered at the origin with radius $R$. Note that both $G_k(\cdot, z) - \overline{G_k(\cdot, z)}$ and $G_k(\cdot, z) - \Phi(\cdot, z)$ are smooth solutions of the Helmholtz equation in $R^2 \setminus \Omega$. Thus, by Green’s representation theorem,

$$G_k(y, z) - \overline{G_k(y, z)} = \int_{\partial B_R} \left\{ \frac{\partial [G_k(x, z) - \overline{G_k(x, z)}]}{\partial n(x)} \Phi(x, y) - \frac{\partial \Phi(x, y)}{\partial n(x)} [G_k(x, z) - \overline{G_k(x, z)}] \right\} ds(x)$$

for $y \in B_R \setminus \overline{\Omega}$. Further, by Green’s second theorem,

$$0 = \int_{\partial B_R} \left\{ \frac{\partial [G_k(x, z) - \overline{G_k(x, z)}]}{\partial n(x)} [G_k(x, y) - \Phi(x, y)] - \frac{\partial [G_k(x, y) - \Phi(x, y)]}{\partial n(x)} [G_k(x, z) - \overline{G_k(x, z)}] \right\} ds(x)$$

$$- \int_{\partial \Omega} \left\{ \frac{\partial [G_k(x, z) - \overline{G_k(x, z)}]}{\partial n(x)} [\Phi(x, y)] - \frac{\partial \Phi(x, y)}{\partial n(x)} [G_k(x, z) - \overline{G_k(x, z)}] \right\} ds(x).$$

Adding the above two identities yields

$$G_k(y, z) - \overline{G_k(y, z)} = \int_{\partial B_R} \left\{ \frac{\partial [G_k(x, z) - \overline{G_k(x, z)}]}{\partial n(x)} [G_k(x, y) - \Phi(x, y)] - \frac{\partial [G_k(x, y) - \Phi(x, y)]}{\partial n(x)} [G_k(x, z) - \overline{G_k(x, z)}] \right\} ds(x)$$

$$- \int_{\partial \Omega} \left\{ \frac{\partial [G_k(x, z) - \overline{G_k(x, z)}]}{\partial n(x)} [G_k(x, y) - \Phi(x, y)] - \frac{\partial \Phi(x, y)}{\partial n(x)} [G_k(x, z) - \overline{G_k(x, z)}] \right\} ds(x).$$

Applying Green’s second formula to the functions $G_k(\cdot, z) - \overline{G_k(\cdot, z)}$ and $G_k(\cdot, y)$ in $\Omega$, and recalling the fact that both $G_k(\cdot, y)$ and $G_k(\cdot, z)$ are smooth radiating solutions in $R^2 \setminus \overline{B_R}$, we have

$$G_k(y, z) - \overline{G_k(y, z)} = \int_{\partial B_R} \left\{ \frac{\partial G_k(x, y)}{\partial n(x)} G_k(x, z) - \frac{\partial \overline{G_k(x, z)}}{\partial n(x)} \overline{G_k(x, y)} \right\} ds(x)$$

$$+ \frac{i}{4\pi} \int_S \frac{G^\alpha_k(\theta, z) G^\alpha_k(\theta, y)}{s} \, ds(\theta). \quad (4.11)$$
Using the representation of \(u_0(z, \tilde{x}) \cdot \alpha\) given in (4.10), we have

\[
\overline{u_0(z, \tilde{x}) \cdot \alpha} = \frac{1}{i} \int_{\partial \Omega} \left[ G_k(y, z) \frac{\partial p^i(y, \tilde{x})}{\partial n(y)} - p^i(y, \tilde{x}) \frac{\partial G_k(y, z)}{\partial n(y)} \right] ds(y)
\]

\[
= \frac{1}{i} \int_{\partial \Omega} \left[ G_k(y, z) \frac{\partial p^i(y, \tilde{x})}{\partial n(y)} - p^i(y, \tilde{x}) \frac{\partial G_k(y, z)}{\partial n(y)} \right] ds(y).
\]

Then

\[
u_0(z, \tilde{x}) \cdot \alpha - \overline{u_0(z, \tilde{x}) \cdot \alpha} = \frac{1}{i} \int_{\partial \Omega} \left\{ \left[ G_k(y, z) \frac{\partial p^i(y, \tilde{x})}{\partial n(y)} - p^i(y, \tilde{x}) \frac{\partial G_k(y, z)}{\partial n(y)} \right] \right\} ds(y).
\]

Applying the representation \(G_k(y, z) - \overline{G_k(y, z)}\) in (4.11) and the reciprocity relation \(G_k^\alpha(\tilde{x}, z) = \gamma u_0(z, \tilde{x}) \cdot \alpha\) for \(z \in \Omega\) in (4.2), we have

\[
u_0(z, \tilde{x}) \cdot \alpha - \overline{u_0(z, \tilde{x}) \cdot \alpha} = \frac{1}{i} \int_{\partial \Omega} \left\{ \left[ G_k^\alpha(\tilde{x}, z) G_k^\alpha(\tilde{x}, y) \right] \frac{\partial p^i(y, \tilde{x})}{\partial n(y)} - p^i(y, \tilde{x}) \frac{\partial G_k^\alpha(\tilde{x}, y)}{\partial n(y)} \right\} ds(y)
\]

\[
= \frac{1}{i} \int_{\partial \Omega} \left\{ \left[ G_k^\alpha(\tilde{x}, z) G_k^\alpha(\tilde{x}, y) \right] \frac{\partial p^i(y, \tilde{x})}{\partial n(y)} - p^i(y, \tilde{x}) \frac{\partial G_k^\alpha(\tilde{x}, y)}{\partial n(y)} \right\} ds(y)
\]

Recalling the first equality in (4.7), when \(z \in R^3 \setminus \overline{\Omega}\), we have

\[
p_0(z, \tilde{x}) = \int_{\partial \Omega} \left[ \frac{\partial p^i(y, \tilde{x})}{\partial n(y)} G_k(y, z) - p^i(y, \tilde{x}) \frac{\partial G_k(y, z)}{\partial n(y)} \right] ds(y),
\]

\[
p_0^\alpha(\tilde{x}, \tilde{\tau}) = \int_{\partial \Omega} \left[ \frac{\partial p^i(y, \tilde{x})}{\partial n(y)} G_k^\alpha(y, \tilde{\tau}) - p^i(y, \tilde{x}) \frac{\partial G_k^\alpha(y, \tilde{\tau})}{\partial n(y)} \right] ds(y).
\]

Hence, using [34, Lemma 2.2], we obtain

\[
u_0(z, \tilde{x}) \cdot \alpha - \overline{u_0(z, \tilde{x}) \cdot \alpha} = \frac{i}{4\pi} \int_{\partial \Omega} \left\{ \left[ u_0(z, \tilde{\tau}) \cdot \alpha \right] p_0^\alpha(\tilde{x}, \tilde{\tau}) \right\} ds(\tilde{\tau})
\]

\[
= \frac{i}{4\pi} \int_{\partial \Omega} \left\{ \left[ u_0(z, \tilde{\tau}) \cdot \alpha \right] p_0^\alpha(\tilde{x}, \tilde{\tau}) \right\} ds(\tilde{\tau})
\]

\[
= \frac{i}{4\pi} \int_{\partial \Omega} \left\{ \left[ u_0(z, \tilde{\tau}) \cdot \alpha \right] p_0^\alpha(\tilde{x}, \tilde{\tau}) \right\} ds(\tilde{\tau}),
\]

that is,

\[
u_0(z, \tilde{x}) \cdot \alpha = u_0(z, \tilde{x}) \cdot \alpha + \frac{i}{4\pi} \int_{\partial \Omega} \left( F_0 \left[ u_0(z, \cdot) \cdot \alpha \right] \right) ds(\tilde{\tau}) = \left( I + \frac{i}{4\pi} F_0 \right) \left[ u_0(z, \cdot) \cdot \alpha \right] \tilde{x} = (\delta[u_0(z, \cdot) \cdot \alpha]) \tilde{x}.
\]

The proof is completed. 

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References


