Ambarzumyan-type theorem for the impulsive Sturm–Liouville operator

Abstract: We prove that if the Neumann eigenvalues of the impulsive Sturm–Liouville operator $-D^2 + q$ in $L^2(0, \pi)$ coincide with those of the Neumann Laplacian, then $q = 0$.

Keywords: Impulsive Sturm–Liouville operator, eigenvalue, Ambarzumyan theorem

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1 Introduction

Ambarzumyan’s theorem [1] says that if $q$ is a real-valued sufficiently smooth potential, and $\lambda_n = n^2$ for all $n = 0, 1, 2, \ldots$, then $q = 0$. Here, $\lambda_n$ are the eigenvalues of the problem

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad 0 < x < \pi, \quad y'(0) = y'(\pi) = 0.$$ 

This result is of interest because, in general, $q(x)$ is not uniquely determined by the spectrum. For example, two spectra for two various boundary conditions determine $q(x)$ (see [2]). Various generalizations of Ambarzumyan’s theorem can be found in [3, 5, 6, 9, 11, 14–16, 18] and other works.

The purpose of this paper is to state and prove a generalization of the above result for the following impulsive Sturm–Liouville operator $L(q)$:

$$Ly := -y''(x) + q(x)y(x) = \lambda \rho(x)y(x), \quad x \in I_{\pi/2} := \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right),$$

with the jump conditions

$$\begin{align*}
y\left(\frac{\pi}{2} + 0\right) &= ay\left(\frac{\pi}{2} - 0\right), \\
y'\left(\frac{\pi}{2} + 0\right) &= a^{-1}y'\left(\frac{\pi}{2} - 0\right),
\end{align*}$$

and the Neumann conditions

$$y'(0) = y'(\pi) = 0,$$

where

$$\rho(x) = \begin{cases} 
1, & 0 < x < \frac{\pi}{2}, \\
a^2, & \frac{\pi}{2} < x < \pi,
\end{cases} \quad 0 < a < 1,$$

$a > 0$ and $q(x)$ is a real-valued function in $L^2(0, \pi)$. Note that the domain $D(L(q))$ of the operator $L(q)$ is

$$D(L(q)) = \{y \in L^2(0, \pi) \mid y' \in AC_{loc}(I_{\pi/2}), \text{ly} \in L^2(0, \pi), \text{y satisfying (1.2) and (1.3))}\}.$$
Such problems are related to discontinuous material characters of a intermediary [7, 8, 12]. Generally, to determine the potential it is necessary to specify two spectra of the problems with different boundary conditions [2, 17]. It is well known that the problem $L(q)$ is self-adjoint and semi-bounded from below, and has a countable set $\{\lambda_n(q)\}_{n \geq 0}$ of eigenvalues with $\lambda_0(q) < \lambda_1(q) < \cdots < \lambda_n(q) < \lambda_{n+1}(q) < \cdots \to \infty$, which are real and simple [13].

Our result is the following theorem.

**Theorem 1.1.** If $\lambda_n(q) = \lambda_n(0)$ for all $n = 0, 1, 2, \ldots$, then $q = 0$ a.e. in $L^2(0, \pi)$.

In the next section, we will prove Theorem 1.1.

## 2 Proof

The proof consists of five steps.

**Step 1:** Prove that the characteristic function of the problem $L(q)$ is uniquely determined by its eigenvalues $\lambda_n(q)$. Firstly, set $y_1(x) = y(x)$ and $y_2(x) = y(\pi - x)$ for $0 < x < \frac{\pi}{2}$. Then the Sturm–Liouville operator $L(q)$ can be rewritten as, for $0 < x < \frac{\pi}{2}$,

\[
\begin{align*}
l_1y_1 &= -y_1''(x) + q_1(x)y_1(x) = \lambda y_1(x), \\
l_2y_2 &= -y_2''(x) + q_2(x)y_2(x) = \lambda y_2(x),
\end{align*}
\]

with

\[y'_1(0) = 0 = y'_2(0)\]

and

\[y_2(\frac{\pi}{2}) = ay_1(\frac{\pi}{2}), \quad y'_2(\frac{\pi}{2}) + a^{-1}y'_1(\frac{\pi}{2}) = 0,\]

where

\[q_1(x) = q(x)|_{[0, \frac{\pi}{2}]} \quad \text{and} \quad q_2(x) = q(\pi - x)|_{[0, \frac{\pi}{2}]}.
\]

Let $l_1\varphi_1 = \lambda \varphi_1$ and $l_2\varphi_2 = \lambda \varphi_2$ with the respective initial conditions

\[
\varphi_1(0) = 1, \quad \varphi'_1(0) = 0 \quad \text{and} \quad \varphi_2(0) = 1, \quad \varphi'_2(0) = 0.
\]

Then we get (see [4, 10])

\[
\begin{align*}
\varphi_1(\frac{\pi}{2}, \lambda) &= \cos(\frac{\sqrt{\lambda}}{2}\pi) + [q_1]\frac{\sin(\frac{\sqrt{\lambda}}{2}\pi)}{\sqrt{\lambda}} + \frac{\psi_1(\lambda)}{\sqrt{\lambda}}, \\
\varphi'_1(\frac{\pi}{2}, \lambda) &= -\sqrt{\lambda}\sin(\frac{\sqrt{\lambda}}{2}\pi) + [q_1]\frac{\sin(\frac{\sqrt{\lambda}}{2}\pi)}{\sqrt{\lambda}} + \psi_2(\lambda), \\
\varphi_2(\frac{\pi}{2}, \lambda) &= \cos(\frac{\sqrt{\lambda}}{2}a\pi) + [q_2]\frac{\sin(\frac{\sqrt{\lambda}}{2}a\pi)}{\sqrt{\lambda}a} + \frac{\psi_3(\lambda)}{\sqrt{\lambda}}, \\
\varphi'_2(\frac{\pi}{2}, \lambda) &= -\sqrt{\lambda}a\sin(\frac{\sqrt{\lambda}}{2}a\pi) + [q_2]\frac{\sin(\frac{\sqrt{\lambda}}{2}a\pi)}{\sqrt{\lambda}a} + \psi_4(\lambda),
\end{align*}
\]

where

\[
\psi_j(\lambda) = o(e^{\frac{2\pi}{\sqrt{\lambda}}j}), \quad j = 1, 2, \\
\psi_j(\lambda) = o(e^{\frac{\pi a}{\sqrt{\lambda}}j}), \quad j = 3, 4,
\]

\[[q_j] = \frac{1}{2} \int_0^{\frac{\pi}{2}} q_j(x) dx, \quad j = 1, 2.
\]

Define the characteristic function

\[
\Phi_q(\lambda) = a\varphi_1(\frac{\pi}{2}, \lambda)\varphi'_2(\frac{\pi}{2}, \lambda) + a^{-1}\varphi'_1(\frac{\pi}{2}, \lambda)\varphi_2(\frac{\pi}{2}, \lambda).
\]
\[ \Phi_q(\lambda) = \Phi_0(\lambda) + A \cos \frac{\sqrt{\lambda} \pi}{2}(1 + a) + B \cos \frac{\sqrt{\lambda} \pi}{2}(a - 1) + \psi_0, \]  
\tag{2.1} \]

where
\[ \psi_0 = o(e^{\frac{1+|\lambda|}{2n}}), \]
\[ \Phi_0(\lambda) = \frac{aa - a^{-1}}{2} \sqrt{\lambda} \sin \frac{\sqrt{\lambda} \pi}{2}(1 + a) - \frac{aa - a^{-1}}{2} \sqrt{\lambda} \sin \frac{\sqrt{\lambda} \pi}{2}(a - 1), \]
\[ A = \frac{a[q_2]}{2} + \frac{a[a[q_1] + a^{-1}[q_2] - a^{-1}[q_1]]}{2} + \frac{aa[a[q_1] + a^{-1}[q_2]]}{2}, \]
\[ B = \frac{a[q_2]}{2} - \frac{aa[a[q_1] + a^{-1}[q_2] - a^{-1}[q_1]]}{2}. \]

Note that the eigenvalues \( \lambda_n(q) \) of \( L(q) \) coincide with the zeros of \( \Phi_q(\lambda) \), and the eigenvalues \( \lambda_n(0) \) of \( L(0) \) coincide with the zeros of \( \Phi_0(\lambda) \).

It follows from (2.1) that \( \Phi_q(\lambda) \) is entire in \( \lambda \) of order \( \frac{1}{2} \), and consequently by Hadamard’s factorization theorem, \( \Phi_q(\lambda) \) is uniquely determined up to a multiplicative constant by its zeros:
\[ \Phi_q(\lambda) = C(q)\lambda \prod_{n; \lambda_n(q)\neq 0} \left(1 - \frac{\lambda}{\lambda_n(q)}\right). \]

For convenience, we introduce
\[ \Psi(\lambda) = -\frac{2}{aa + a^{-1}} \Phi_q(\lambda) = \tilde{C}(q)\lambda \prod_{n; \lambda_n(q)\neq 0} \left(1 - \frac{\lambda}{\lambda_n(q)}\right), \]

where \( \tilde{C}(q) = -\frac{2C(q)}{aa + a^{-1}}. \) Then
\[ \Psi(\lambda) = \sqrt{\lambda} \sin \frac{\sqrt{\lambda} \pi}{2}(1 + a) + \frac{aa - a^{-1}}{aa + a^{-1}} \sqrt{\lambda} \sin \frac{\sqrt{\lambda} \pi}{2}(1 - a) + o(e^{\frac{|\tau|}{2n}}), \]  
\tag{2.2} \]

where \( |\tau| = \sqrt{\lambda} \).

Let \( \lambda = -\kappa^2 (\kappa > 0) \). Then \( \sqrt{\lambda} = i\kappa \). Substituting \( \lambda \) into (2.2) yields that for large \( \kappa \),
\[ -\tilde{C}(q)\kappa^2 \prod_{n; \lambda_n(q)\neq 0} \left(1 + \frac{\kappa^2}{\lambda_n(q)}\right) = -\kappa^2 e^{\frac{i\pi}{2}}[1 + o(1)], \]
\[ \tilde{C}(q) = \frac{1}{2} \lim_{\kappa \to \infty} \left[e^{\frac{i\pi}{2}}\left(\kappa \prod_{n; \lambda_n(q)\neq 0} \left(1 + \frac{\kappa^2}{\lambda_n(q)}\right)\right)^{-1}\right]. \]

So
\[ \Phi_q(\lambda) = -\frac{\tilde{C}(q)}{2} (aa + a^{-1}) \lambda \prod_{n; \lambda_n(q)\neq 0} \left(1 - \frac{\lambda}{\lambda_n(q)}\right). \]

Namely, \( \Phi_q(\lambda) \) and \( \Phi_0(\lambda) \) are uniquely determined by \( \{\lambda_n(q)\}_{n \geq 0} \) and \( \{\lambda_n(0)\}_{n \geq 0} \), respectively.

Step 2: Deduce that \( |q_1| + a^2[q_2] = 0 \). From the assumption that \( \lambda_n(q) = \lambda_n(0), n \geq 0 \), we get \( \Phi_q(\lambda) = \Phi_0(\lambda) \), which implies that
\[ A \cos \frac{\sqrt{\lambda} \pi}{2}(1 + a) + B \cos \frac{\sqrt{\lambda} \pi}{2}(a - 1) + \psi_0(\lambda) \equiv 0. \]  
\tag{2.3} \]

Taking \( \sqrt{\lambda} = \frac{2n + 1}{1 + \pi}, n \geq 0 \), in (2.3), we obtain
\[ B \cos \frac{\alpha - 1}{\alpha+1} \left(n \pi + \frac{\pi}{2}\right) + \psi_0 \left(\frac{2n + 1}{1 + \alpha}\right) - M = 0. \]

Note that \( \psi_0\left(\frac{2n + 1}{1 + \pi}\right) = o(1) \) as \( n \to \infty \). So \( B = 0 \). Once we get \( B = 0 \), then \( A = 0 \). This yields \( A + B = 0 \), so
\[ |q_1| + a^2[q_2] = 0. \]
Step 3: Verify that $\lambda_0(0) = 0$. Obviously, $\Phi_0(0) = 0$, so $0 \in \{\lambda_n(0)\}$. In particular, when $q = 0$ a.e. in $L^2(0, \pi)$, for the operator $L(0)$ we have

$$
\lambda_0(0) = \inf_{y \in D(L(0))} \frac{\int_{I_{0\pi}} -y''(x)y(x)dx}{\int_{I_{0\pi}} \rho(x)|y(x)|^2 dx}.
$$

Therefore,

$$
\lambda_0(0) = \inf_{y \in D(L(0))} \frac{\int_{I_{0\pi}} |y'(x)|^2 dx}{\int_{I_{0\pi}} \rho(x)|y(x)|^2 dx} \geq 0.
$$

Namely, the operator $L(0)$ is non-negative, and $0 \in \lambda_0(0)$, so we have $\lambda_0(0) = 0$.

Step 4: Find the eigenfunction corresponding to the least eigenvalue $\lambda_0(q)$. From the variational definition of $\lambda_0(q)$, and the assumption $\lambda_0(q) = \lambda_0(0) = 0$, we get

$$
0 = \lambda_0(q) = \inf_{y \in D(L(q))} \frac{\int_{I_{0\pi}} (-y''(x)y(x) + q(x)|y(x)|^2)dx}{\int_{I_{0\pi}} \rho(x)|y(x)|^2 dx}.
$$

Taking

$$
y_0(x) = \begin{cases} 
1, & 0 < x < \frac{\pi}{2}, \\
a, & \frac{\pi}{2} < x < \pi,
\end{cases}
$$

we find that

$$
\int_{I_{0\pi}} (|y_0'(x)|^2 + q(x)|y_0(x)|^2)dx = 2([q_1] + a^2[q_2]).
$$

Together with $[q_1] + a^2[q_2] = 0$ in step 2, we conclude that $y_0(x)$ achieves its infimum of the functional in the variational formula, which follows that $y_0(x)$ is an eigenfunction corresponding to the eigenvalue $\lambda_0(q) = 0$.

Step 5: Verify that $q(x) = 0$ a.e. in $L^2(0, \pi)$. From equation (1.1) with $\lambda = 0$ and $y(x) = y_0(x)$, we conclude that $q(x) = 0$ a.e. in $L^2(0, \pi)$.

The proof of Theorem 1.1 is complete.

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References