Research Article

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Reverse time migration for imaging periodic obstacles with electromagnetic plane wave

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Abstract: We propose novel reverse time migration (RTM) methods for the imaging of periodic obstacles using only measurements from the lower or upper side of the obstacle arrays at a fixed frequency. We analyze the resolution of the lower side and upper side RTM methods in terms of propagating modes of the Rayleigh expansion, Helmholtz–Kirchhoff equation and the distance of the measurement surface to the obstacle arrays, where the periodic structure leads to novel analysis. We give some numerical experiments to justify the competitive efficiency of our imaging functionals and the robustness against noises. Further, numerical experiments show sharp images especially for the vertical part of the periodic obstacle in the lower-RTM case, which is not shared by results for imaging bounded compactly supported obstacles.

Keywords: Reverse time migration, periodic structure, inverse scattering problem, resolution analysis

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1 Introduction

In this paper, we consider the inverse scattering problem of time-harmonic electromagnetic plane waves by periodic obstacles. For simplicity, we assume that the periodic structure is invariant in the $x_3$ direction and the scatterer is periodic in the $x_1$ direction, with periodicity $\Lambda$. Specifically, we have

$$y(x_1 + \Lambda, x_2) = y(x_1, x_2).$$

We further assume that $y(x) - 1$ is compactly supported in each $\Gamma$-periodic unit of $\mathbb{R}^2$, and the central unit is denoted by (Figure 1)

$$\Omega = \left\{ (x_1, x_2) \mid x_1 \in \left( -\frac{\Lambda}{2}, \frac{\Lambda}{2} \right), \ x_2 \in \mathbb{R} \right\}.$$

The left and right boundaries of $\Omega$ are denoted by $\partial\Omega_L, \partial\Omega_R$ respectively. We denote the scatterer in $\Omega$ by $D$, which is the support of $y(x) - 1$ in $\Omega$.

Let us consider the TE polarization in the following. That is to say, the electric field is along the $x_3$ direction and depends on $x_1, x_2$. The incident wave is given by

$$u_{\text{inc}}^\alpha(x) = e^{i(ax_1 - \beta x_2)},$$

where $a = k \cos \theta, \beta = \sqrt{k^2 - a^2} = k \sin \theta, k > 0$ is the wave number and $\theta \in (0, \pi)$ is the incident angle with respect to the $x_1$ direction. The total field $u_\alpha$ is governed by the Helmholtz equation

$$\Delta u_\alpha + k^2 y(x)u_\alpha = 0, \quad x \in \mathbb{R}^2,$$

where $u_\alpha$ is given by

$$u_\alpha = u_\alpha^s + u_\alpha^\text{inc}, \quad x \in \mathbb{R}^2.$$
The scattered wave \( u_s^\alpha \) is an \( \alpha \)-quasi-periodic function in the \( x_1 \) direction, namely,

\[
u_s^\alpha(x_1 + \Lambda, x_2) e^{-i\Lambda \alpha} = u_s^\alpha(x_1, x_2).
\]

Furthermore, \( u_s^\alpha \) satisfies the Rayleigh expansion condition which describes the radiation of the scattered wave when \( x_2 \to \pm \infty \),

\[
u_s^\alpha(x_1, x_2) = \sum_{m \in \mathbb{Z}} u_s^{\alpha_m} e^{i\alpha_m x_1} e^{i\beta_m x_2}, \quad x_2 \in \Omega^+_H, \\
u_s^\alpha(x_1, x_2) = \sum_{m \in \mathbb{Z}} u_s^{\alpha_m} e^{-i\alpha_m x_1} e^{-i\beta_m x_2}, \quad x_2 \in \Omega^-_H,
\]

where, for \( H > 0 \),

\[
\Omega^+_H = \left\{(x_1, x_2) \left| x_1 \in \left( -\frac{\Lambda}{2}, \frac{\Lambda}{2} \right), x_2 \geq H \right. \right\}, \\
\Omega^-_H = \left\{(x_1, x_2) \left| x_1 \in \left( -\frac{\Lambda}{2}, \frac{\Lambda}{2} \right), x_2 \leq -H \right. \right\},
\]

which correspond to the outer diffractive regions, with

\[
a_m = \alpha + \frac{2\pi}{\Lambda} m, \quad m \in \mathbb{Z}, \quad \text{and} \quad \beta_m = \begin{cases} \sqrt{k^2 - a_m^2}, & |a_m| \leq k, \\ i\sqrt{a_m^2 - k^2}, & |a_m| > k. \end{cases}
\]

We further assume that \( |a_m| \neq k \), that is, \( \alpha \) is not a Wood’s anomaly corresponding to \( k \). It is known that the scattering problem (1.1), (1.2) with Rayleigh expansion condition is well-posed for all but a countable set of \( k \) (see [4]), so in this paper, we always assume the wave number is chosen such that the scattering problem is well-posed. As \( \beta_m \) changes from real to imaginary as \( a_m \) passes \( k \), we define index sets \( B_a \) of terms corresponding to propagating plane waves and \( U_a \) corresponding to evanescent plane waves, which are

\[
B_a = \left\{n \in \mathbb{Z} \mid |a_n| < k \right\} \quad \text{and} \quad U_a = \left\{n \in \mathbb{Z} \mid |a_n| > k \right\}.
\]

The periodic scattering problem has long been an important topic in electromagnetic theory. It appears in extensive areas such as optics, photonics and phononics [1, 4]. Ever since Lord Rayleigh’s pioneering work in the early 20th century, a considerable amount of work has been done for the scattering problem of diffractive optics. Mathematically, the well-posedness of the above forward scattering problem is established, especially for the case of diffractive layers. See [19] for example. Numerically, in recent years, we have seen a rapid development of fast and reliable solvers. For instance, in the regime of boundary integral equation methods, [7] derived a scheme stemming from the free-space scattering problem with specially designed auxiliary density, while [9] overcome the slow convergence of the quasi-periodic Green function [22] by designing a special window function in their formulation. Furthermore, by formulating the Lippmann–Schwinger equation of quasi-periodic scattering, [20] uses the Fourier transformation to obtain a spectral Galerkin method. Moreover, in [13], an adaptive finite element PML method is developed, while in [24], an analysis on the transparent boundary condition of the scattering problem leads to the adaptive DtN method.
Having collected several aspects of the forward periodic scattering problem, we are ready to demonstrate the following inverse problem: given all \( u_{\alpha n} \), \( n \in B_\alpha \), with incident waves \( u_{\text{inc}}^{\alpha n} \), which are measured on \( \Gamma_h \) or \( \Gamma_{-h} \), \( h \geq 0 \) (Figure 1), reconstruct the boundary of the support of \( \gamma(x) - 1 \), where

\[
\Gamma_{\pm h} = \left\{ (x_1, x_2) \mid x_1 \in \left( -\frac{\Lambda}{2}, \frac{\Lambda}{2} \right), x_2 = \pm h \right\}.
\]

There has been numerous literature in the inverse problem community concerning the reconstruction of periodic structure; see [2, 5, 14] for the uniqueness theorems concerning the inverse problems in two and three dimensions. Further, starting from [19], etc., iterative reconstruction methods [6, 8, 15, 17, 18], as well as other two-step reconstruction methods [16] are designed and studied. Especially, see [4] for a comprehensive survey on the reconstruction methods for periodic grating profiles. As for the direct imaging methods on the periodic structures, there are a number of studies on diffractive periodic structures, such as factorization method [3], the linear sampling methods [25, 26]. We remark that, in the direct imaging methods proposed above, \( \alpha_n \) for \( n \in \mathbb{Z} \) is needed, and using only \( n \in B_\alpha \) results in the loss of information on the vertical part of the diffractive optics. In addition, for the reconstruction of periodically compactly supported obstacles, there has recently been literature on the design of a special indicator functional [23], where a comparison of the above direct imaging methods in the case of periodic scattering is included.

As is known, the RTM method has a competitive resolution of the bounded obstacle if one obtains the full-aperture data [10–12]. Extending the RTM method to the case of unbounded surface scattering, [21] is able to reconstruct simultaneously the locally perturbed half-space and a compactly supported obstacle. On the other hand, using limited aperture data, an analysis of half-space RTM in [10] indicates that one can obtain partially the boundary of the obstacle, whose resolution is given by the Kirchhoff coefficient, which is closely connected with the opening of the scatterer.

The major contribution of this paper is an investigation of RTM method in an inverse periodic scattering problem. We analyze the RTM method with quasi-periodic data of measurements only from below or above the periodic array. The resolution analysis is based on the Helmholtz–Kirchhoff equation, propagating part of the Rayleigh expansion of the scattering wave and point spread function for a quasi-periodic scattering problem. Specially, we prove that the lower-side RTM imaging functional is positive, and the lower-side and upper-side RTM functional peak at the boundary of the scatterer. We also demonstrate numerically that, with partial data only from below the periodic array, one can find a clear image of the vertical part of the periodic array other than its lower part, and the imaging functional has the nice property of positivity, while with partial data only from above the periodic array, one can obtain a clear image of the horizontal part of the periodic array. We remark that the present work is not a trivial extension of the existing RTM methods since the periodic structure of the obstacle arrays is of importance to the imaging ability of our RTM functionals, which results in a different back-propagating Green function, specially designed cross-correlation for particular incident directions, and thus novel analysis on the RTM functionals.

The structure of the paper is as follows. To begin with, in Section 2, we investigate several preliminary tools for the resolution analysis of RTM functionals. By an investigation of the Helmholtz–Kirchhoff equation for the quasi-periodic scattering problem, we observe a natural point spread function corresponding to the propagating modes of the quasi-periodic Green function in terms of its spectral decomposition. In Section 3, where the RTM algorithm is proposed, the special structure of the point spread function leads to the form of cross-correlation between the incident waves of propagating modes and the back-propagation of the received data. Section 4 presents the resolution analysis regarding the imaging power of RTM functionals. We start with the lower RTM method which is constructed by measurements from below the obstacle, while the second part of the section gives a further analysis on the resolution of the upper RTM method. We extend our analysis to the sound-soft obstacle for the RTM methods in Section 5. Finally, in Section 6, the numerical experiments thus demonstrate the competitive imaging ability of our RTM functionals.
2 Preliminaries

2.1 Point spread function

We begin by recalling the quasi-periodic Green function [22] in \( \mathbb{R}^2 \),

\[
G_{qp}^{qp}(x, y) = \frac{i}{4} \sum_{n \in \mathbb{Z}} H_0^1(k|x - y_n|)e^{in\lambda x}.
\]  

(2.1)

Here, \( y_n = (y_{n1}, y_{n2}) = (y_1 + n\Lambda, y_2) \) and \( H_0^1(\cdot) \) is the zeroth-order Hankel function of the first kind. It is the solution to the equation

\[
\Delta G_{qp}^{qp}(x, y) + k^2 G_{qp}^{qp}(x, y) = -\sum_{n \in \mathbb{Z}} \delta_{y_1}(x_1) \delta_{y_2}(x_2)e^{in\lambda x}.
\]

The physical interpretation of (2.1) is the wave emitted by a periodic array of point sources, each of which is equipped with a phase shift in the \( x_1 \) direction.

Using the Poisson summation formula, we may obtain the spectral representation of the quasi-periodic Green function at non-Wood’s anomalies (see [4]),

\[
G_{qp}^{qp}(x, y) = \frac{i}{2\Lambda} \sum_{n \in \mathbb{Z}} \frac{1}{\beta_n} e^{i\alpha_n(x_1 - y_1) + i\beta_n(x_2 - y_2)}.
\]  

(2.2)

Now we can give the Helmholtz–Kirchhoff equation for the quasi-periodic Green function. To start with, we consider the case where the source points of the Green function are above the upper measurement surface.

**Theorem 2.1.** For \( y, z \in \Omega_{+h} = \{x = (x_1, x_2) | x \in \Omega, x_2 > -h, h > 0\} \), assume that \( a \) is not a Wood’s anomaly. Then we have the following Helmholtz–Kirchhoff equation:

\[
\int_{\Gamma_h} \frac{\partial G_{qp}^{qp}(x, y)}{\partial x_2} G_{qp}^{qp}(x, z) \, ds(x) - \int_{\Gamma_h} \frac{\partial G_{qp}^{qp}(x, z)}{\partial x_2} G_{qp}^{qp}(x, y) \, ds(x) = F_{\alpha}(y, z).
\]  

(2.3)

Here, \( \Gamma_{-h} = \{(x_1, x_2) | x_2 = -h, -\Lambda/2 < x_1 < \Lambda/2\} \) and

\[
F_{\alpha}(y, z) = \frac{i}{2\Lambda} \sum_{n \in B_\alpha} \frac{1}{\beta_n} e^{i\alpha_n(y_1 - z_1) - i\beta_n(y_2 - z_2)}.
\]  

(2.4)

**Proof.** To start with, we observe that, for \( y_2, z_2 \geq -h, |y_2 + h| = (y_2 + h), |y_2 - z_2| = (h + z_2) \). Further,

\[
\frac{i}{\beta_n} = \begin{cases} \frac{-i}{\beta_n}, & n \in B_\alpha, \\ \frac{i}{\beta_n}, & n \in U_\alpha. \end{cases}
\]

Thus the absolute value in the spectral form (2.2) of the quasi-periodic Green function can be exactly calculated. With the help of the orthogonality

\[
\int_{\Lambda_h} e^{i\alpha_n x_1} e^{-i\alpha_n x_1} \, ds(x) = \begin{cases} 0, & n \neq m, \\ \Lambda, & n = m, \end{cases}
\]

we obtain

\[
\int_{\Gamma_h} \frac{\partial G_{qp}^{qp}(x, y)}{\partial x_2} G_{qp}^{qp}(x, z) \, ds(x) = \frac{1}{4\Lambda^2} \int_{\Lambda_h} \sum_{n \in \mathbb{Z}} \frac{1}{\beta_n} e^{i\alpha_n(x_1 - y_1) + i\beta_n(y_2 + h)} \sum_{n \in \mathbb{Z}} e^{i\alpha_n(x_1 - z_1) + i\beta_n(z_2 + h)} \, ds(x)
\]

\[
= \frac{1}{4\Lambda^2} \sum_{n \in \mathbb{B}_\alpha} \frac{i}{\beta_n} e^{i\alpha_n(y_1 - z_1) - i\beta_n(y_2 - z_2)} + \sum_{n \in U_\alpha} \frac{i}{\beta_n} e^{i\alpha_n(y_1 - z_1) + i\beta_n(h + y_2 + z_2)} \),
\]

(2.5)

Thus, doing the subtraction, we obtain (2.3) and complete the proof. \( \square \)
Corollary 2.1. For \( y, z \in \Omega^L_h = \{ x = (x_1, x_2) \mid x \in \Omega, x_2 > -h, h > 0 \} \), assume that \( \alpha \) is not a Wood’s anomaly. Then we have the following asymptotic result:

\[
\int_{\Gamma_h} \frac{\partial G_{a}^{\alpha}(x,y)}{\partial x_2} G_{a}^{\alpha}(x,z) \, ds(x) = \frac{1}{2} F_{a}^{L}(y,z) + R_{a}^{L}(y,z; h),
\]

where \( |R_{a}^{L}(y,z; h)| = O(h^{-1}) \), \( |\nabla R_{a}^{L}(y,z; h)| = O(h^{-1}) \) as \( h \to \infty \).

Proof. Denote

\[
R_{a}^{L}(y,z; h) = \frac{1}{4\Lambda} \sum_{n \in U_a} \frac{i}{\beta_n} e^{i\alpha_n(y_1-z_1) - |\beta_n(-2h-(y_2+z_2))|}, \quad h > 0, \quad \text{and} \quad \beta_n = \min \{|i\beta_n| \mid n \in U_a\}.
\]

From (2.5), we obtain

\[
4\Lambda |R_{a}^{L}(y,z; h)| = \left| \sum_{n \in U_a} \frac{i}{\beta_n} e^{i\alpha_n(y_1-z_1) - |\beta_n(-2h-(y_2+z_2))|} \right|
\]

\[
= \left| \sum_{n \in U_a} \frac{1}{\sqrt{\alpha_n^2 - k^2}} e^{-i\alpha_n^2 k^2(2h+(y_2+z_2))} e^{i\alpha_n(y_1-z_1)} \right|
\]

\[
\leq 2 \left( \frac{1}{\beta_{\alpha}} e^{-\beta_{\alpha}(2h+z_2+y_2)} + \int_{0}^{\infty} \frac{1}{k} e^{-\sqrt{s^2-k^2}(2h+(y_2+z_2))} \, ds \right)
\]

\[
\leq 2 \left( \frac{1}{\beta_{\alpha}} e^{-\beta_{\alpha}(2h+z_2+y_2)} + \int_{0}^{\infty} \frac{1}{\sqrt{t^2+k^2}} \, dt \right)
\]

\[
\leq 2 \left( \frac{1}{\beta_{\alpha}} e^{-\beta_{\alpha}(2h+z_2+y_2)} + \int_{0}^{\infty} \frac{1}{k} e^{-t(2h+(y_2+z_2))} \, dt \right)
\]

\[
\leq 2 \left( \frac{1}{\beta_{\alpha}} e^{-\beta_{\alpha}(2h+z_2+y_2)} + \frac{1}{k} \right)
\]

Similarly, we may prove the estimate for \( |\nabla R_{a}^{L}(y,z)| \) and thus complete the proof of the lemma. \( \square \)

Thus, it is reasonable to hope that the cross-correlation between the quasi-periodic equation and the conjugate of its \( x_2 \) derivative would produce a good approximation of \( F_{a}^{L}(y,z) \), whose imaginary part has the form of a point spread function that peaks at \( y = z \) and decays as \( y \) leaves \( z \). In Section 4, we shall see that this quasi-periodic point spread function reflects the imaging ability of the lower RTM functional \( \gamma_L(z) \).

Following a proof similar to Theorem 2.1, we obtain the cross-correlation between the quasi-periodic Green function for \( y, z \in \Omega^L_h \).

Theorem 2.2. For \( y, z \in \Omega^L_h = \{ x = (x_1, x_2) \mid x \in \Omega, x_2 < h, h > 0 \} \), assume that \( \alpha \) is not a Wood’s anomaly. Then we have the following Helmholtz–Kirchhoff equation:

\[
\int_{\Gamma_h} \frac{\partial G_{a}^{\alpha}(x,y)}{\partial x_2} G_{a}^{\alpha}(x,z) \, ds(x) = -F_{a}^{L}(y,z).
\]

Here, \( \Gamma_h = \{(x_1, x_2) \mid x_2 = h, -\Lambda/2 < x_1 < \Lambda/2\} \) and

\[
F_{a}^{L}(y,z) = \frac{i}{2\Lambda} \sum_{n \in U_a} \frac{1}{\beta_n} e^{i\alpha_n(y_1-z_1) + |\beta_n(y_2+z_2)|}. \tag{2.6}
\]

Corollary 2.2. For \( y, z \in \Omega^L_h = \{ x = (x_1, x_2) \mid x \in \Omega, x_2 < h, h > 0 \} \), assume that \( \alpha \) is not a Wood’s anomaly. Then we have the following asymptotic result:

\[
\int_{\Gamma_h} \frac{\partial G_{a}^{\alpha}(x,y)}{\partial x_2} G_{a}^{\alpha}(x,z) \, ds(x) = -\frac{1}{2} F_{a}^{L}(y,z) + R_{a}^{L}(y,z; h),
\]

where \( |R_{a}^{L}(y,z; h)| = O(h^{-1}) \), \( |\nabla R_{a}^{L}(y,z; h)| = O(h^{-1}) \) as \( h \to \infty \).
The difference of \( t_+ - t_- \) in (2.6) and (2.4) leads to a sharper point spread function for RTM functionals. Namely, if we consider the representations

\[
F_1^t(y, z) = \frac{i}{2\Lambda} \sum_{n \in \mathbb{Z}} \frac{1}{\tilde{B}_n} e^{i\alpha_n(y-z_1)} \cos \beta_n(y_2 - z_2), \\
F_2^t(y, z) = \frac{i}{2\Lambda} \sum_{n \in \mathbb{Z}} \frac{1}{\tilde{B}_n} e^{i\alpha_n(y-z_1)} \sin \beta_n(y_2 - z_2),
\]

we have

\[
F_1^q(y, z) = F_1^t(y, z) + iF_2^t(y, z), \quad F_2^q(y, z) = F_1^t(y, z) - iF_2^t(y, z).
\]

Now, using the spectral expansion of the quasi-periodic Green function, we obtain the next theorem.

**Theorem 2.3.** Assume that \( |a_n| \neq k, n \in \mathbb{Z}, |y - z| \neq m\Lambda, m \in \mathbb{Z}. \) Then we have the following result:

\[
F_1^q(y, z) = \sum_{n \in \mathbb{Z}} \frac{i}{4\Lambda} f_0^q(k|y - z_n|)e^{i\alpha_n},
\]

where \( f_0^q(k|x - y|) \) is the Bessel function of the first kind.

**Proof.** Being aware of

\[
G_{a}^{qp}(z, y) = \sum_{n \in \mathbb{Z}} \frac{i}{4} H_0^1(k|z - y_n|)e^{i\alpha_n} = \sum_{n \in \mathbb{Z}} \frac{i}{4} H_0^1(k|y - z_n|)e^{i\alpha_n} = \sum_{n \in \mathbb{Z}} \frac{i}{4} H_0^1(k|y - z_n|)e^{-i\alpha_n},
\]

since \( a_n \neq k \) and \( |y - z| \neq n\Lambda \) for all \( n \in \mathbb{Z} \), the series is convergent. Then we have

\[
G_{a}^{qp}(y, z) - \overline{G_{a}^{qp}(z, y)} = \sum_{n \in \mathbb{Z}} \frac{i}{4} H_0^1(k|y - z_n|)e^{i\alpha_n} + \sum_{n \in \mathbb{Z}} \frac{i}{4} H_0^1(k|z - y_n|)e^{i\alpha_n}
\]

\[
= \frac{i}{4} \sum_{n \in \mathbb{Z}} (H_0^1(k|y - z_n|) + H_0^1(k|z - y_n|))e^{i\alpha_n}
\]

\[
= \sum_{n \in \mathbb{Z}} \frac{i}{2} f_0^q(k|y - z_n|)e^{i\alpha_n}.
\]

On the other hand,

\[
G_{a}^{qp}(y, z) - \overline{G_{a}^{qp}(z, y)} = \frac{i}{2\Lambda} \sum_{n \in \mathbb{Z}} \frac{1}{\tilde{B}_n} e^{i\alpha_n(y_1-z_1+i|\beta_n|y_2-z_2)} + \frac{1}{\tilde{\beta}_n} e^{i\alpha_n(y_1-z_1-i|\beta_n|y_2-z_2)}
\]

\[
= \frac{i}{\Lambda} \sum_{n \in \mathbb{Z}} \frac{1}{\tilde{B}_n} e^{i\alpha_n(y_1-z_1)} \cos \beta_n(y_2 - z_2) = 2F_1^q(y, z),
\]

which is exactly what we have asserted. \( \square \)

We remark that the right-hand side of (2.8) also appeared in [23] as the point spread function. Equation (2.8) shows that \( F_1^q \) has an imaging power similar to the Bessel function of the first kind. We further include Figure 2 of the point spread functions \( F_1^q \) and \( F_1^t \). From Figure 2 (a) and (b), which correspond to \( \text{Im}(F_1^q) \), and Figure 2 (c) and (d), which correspond to \( \text{Im}(F_1^t) \), it is clear that the point spread function \( F_1^q \) has behavior similar to \( F_0 \). This similarity is reflected both theoretically and numerically in Sections 4 and 5.

The second basic ingredient is the following Lippmann–Schwinger equation.

**Theorem 2.4.** Denoting by \( D \) the compact support of \( y(x) - 1 \) in the periodic cell \( \Omega \), the \( a \)-quasi-periodic solution to the Helmholtz equation satisfies the following quasi-periodic Lippmann–Schwinger equation:

\[
u_a^q(x) = \int_D k^2(y(y) - 1)G_a^{qp}(x, y)u_a(y) \, dy
\]

for all \( x \in \Omega \setminus D. \)
Proof. With (1.1), (1.2), we have
\[
\Delta u_\alpha^s(y) + k^2 u_\alpha^s(y) = k^2(1 - \gamma(y)) u_\alpha(y).
\]
For any \( x \in \Omega \) outside of \( D \), take a small ball \( B_\rho(x) \) of radius \( \rho \) that is contained in \( \Omega \). Further, bound the outside of \( D \) with a rectangular section \( \partial \Omega_0 \), whose upper and lower bounds are given by \( \Gamma_{L_0} \) with \( h \geq H \). The region included in the section is denoted by \( \Omega_0 \). Thus, multiplying both sides of the equations by \( G_{\alpha}^{qp}(x, y) \) and integrating over the \( \Omega_0 \setminus B_\rho(x) \), we have
\[
\int_{\Omega_0 \setminus B_\rho(x)} G_{\alpha}^{qp}(x, y)(\Delta u_\alpha^s(y) + k^2 u_\alpha^s(y)) \, dy = \int_D k^2(1 - \gamma(y)) G_{\alpha}^{qp}(x, y) u_\alpha(y) \, dy.
\]
Using Green’s second formula, we obtain
\[
LHS = \int_{\partial(\Omega_0 \setminus B_\rho(x))} G_{\alpha}^{qp}(x, y) \frac{\partial u_\alpha^s(y)}{\partial v(y)} - u_\alpha^s(y) \frac{\partial G_{\alpha}^{qp}(x, y)}{\partial v(y)} \, ds(y).
\]
We denote by \( RHS = I_1 + I_2 \) the integration on the two boundaries in (2.9) respectively. Thus, on \( \partial \Omega_0 \),
\[
I_1 = \left( \int_{\Gamma_{L_0}} - \int_{\Gamma_{R_0}} \right) G_{\alpha}^{qp}(x, y) \frac{\partial u_\alpha^s(y)}{\partial v(y)} - u_\alpha^s(y) \frac{\partial G_{\alpha}^{qp}(x, y)}{\partial v(y)} \, ds(y).
\]
Observing that \( G_{\alpha}^{qp}(x, y) \) is \(-\alpha\)-quasi-periodic in \( y_1 \), we have that the difference of integration along the left and right boundaries in (2.10) vanishes, and we are left with
\[
I_1 = \int_{\Gamma_{L_0}} G_{\alpha}^{qp}(x, y) \frac{\partial u_\alpha^s(y)}{\partial y_2} - u_\alpha^s(y) \frac{\partial G_{\alpha}^{qp}(x, y)}{\partial y_2} \, ds(y).
\]
Using the spectral representation of $G^{\text{pp}}_a(x, y)$ and the Rayleigh expansion of $u^s_\alpha$ on $\Gamma_{z, h}$, since $x \in \Omega_0$, $x_2 \in (-h, h)$, $h \geq H$, we have

$$
\left\{ \begin{array}{l}
G^{\text{pp}}_a(x, y) \frac{\partial u^s_\alpha(y)}{\partial y_2} - u^s_\alpha(y) \frac{\partial G^{\text{pp}}_a(x, y)}{\partial y_2} = ds(y) \\
\quad = \frac{i}{2\Lambda} \left( \sum_{m=\mathbb{Z}} \frac{1}{\beta_m} e^{i\alpha_1(x-y_1) + i\beta_m(h-x_2)} \right) \left( \sum_{m=\mathbb{Z}} i\beta_m u^s_{m,a} e^{i\alpha_1 y_1 + i\beta_m h} \right) ds(y) \\
\quad \quad - \sum_{m=\mathbb{Z}} \left( \sum_{n=\mathbb{Z}} \left( \sum_{m=\mathbb{Z}} \frac{1}{\beta_n} e^{i\alpha_1(x-y_1) + i\beta_n(h-x_2)} \right) \left( \sum_{m=\mathbb{Z}} u^s_{m,a} e^{i\alpha_1 y_1 - i\beta_n h} \right) \right) ds(y) = 0,
\end{array} \right.
$$

Similarly, we have

$$
\left\{ \begin{array}{l}
G^{\text{pp}}_a(x, y) \frac{\partial u^s_\alpha(y)}{\partial y_2} - u^s_\alpha(y) \frac{\partial G^{\text{pp}}_a(x, y)}{\partial y_2} = ds(y) \\
\quad = \frac{i}{2\Lambda} \left( \sum_{m=\mathbb{Z}} \frac{1}{\beta_m} e^{i\alpha_1(x-y_1) + i\beta_m(h-x_2)} \right) \left( \sum_{m=\mathbb{Z}} i\beta_m u^s_{m,a} e^{i\alpha_1 y_1 + i\beta_m h} \right) ds(y) \\
\quad \quad - \sum_{m=\mathbb{Z}} \left( \sum_{n=\mathbb{Z}} \left( \sum_{m=\mathbb{Z}} \frac{1}{\beta_n} e^{i\alpha_1(x-y_1) + i\beta_n(h-x_2)} \right) \left( \sum_{m=\mathbb{Z}} u^s_{m,a} e^{i\alpha_1 y_1 - i\beta_n h} \right) \right) ds(y) = 0.
\end{array} \right.
$$

Thus, $I_1 = 0$ and

$$
I_2 = - \int_{\partial B_p(x)} G^{\text{pp}}_a(x, y) \frac{\partial u^s_\alpha(y)}{\partial y_2} - u^s_\alpha(y) \frac{\partial G^{\text{pp}}_a(x, y)}{\partial y_2} ds(y).
$$

Recalling (2.1), the singular term of $G^{\text{pp}}_a(x, y)$, $y \in \partial B_p(x)$ as $\rho \to 0$ is $G(x, y) = \frac{i}{4} H^1_0(k|x-y|)$. Thus,

$$
I_2 = - \int_{\partial B_p(x)} G(x, y) \frac{\partial u^s_\alpha(y)}{\partial y_2} - u^s_\alpha(y) \frac{\partial G(x, y)}{\partial y_2} ds(y).
$$

By the singularity of the fundamental solution to the free-space Green function, letting $\rho \to 0$, $I_2 = -u^s_\alpha(x)$

Further, we introduce the following function spaces:

$$
H^1_{a, \text{pp}}(\Omega) = \{ u \in H^1(\Omega) \mid u(x_1 + \Lambda, x_2) e^{-i\Lambda a} = u(x_1, x_2) \},
$$

with induced norm from $H^1(\Omega)$, and

$$
L^2_{a, \text{pp}}(\Omega) = \{ u \in L^2(\Omega) \mid u(x_1 + \Lambda, x_2) e^{-i\Lambda a} = u(x_1, x_2) \},
$$

with induced norm from $L^2(\Omega)$. Now we further include the well-posedness of an $\alpha$-quasi-periodic scattering solution to the Helmholtz equation, whose proof is similar to that of [25].

**Theorem 2.5.** Assume that $\gamma(x) - 1 \in L^{\infty}(\Omega)$ is periodic in $x_1$ and $f(x) \in L^2_{a, \text{pp}}(\Omega)$ is $\alpha$-quasi-periodic in $x_1$, with period $\Lambda$. Both of them are supported in $D$ for $x \in \Omega$. Moreover, $w^s_\alpha$ is the $\alpha$-quasi-periodic scattering solution to the Helmholtz equation

$$
\Delta w^s_\alpha(x) + k^2 \gamma(x) w^s_\alpha(x) = f(x)
$$

with Rayleigh expansion condition, namely,

$$
w^s_\alpha = \begin{cases} 
\sum_{m=\mathbb{Z}} w^s_{m,a} e^{i\alpha_1 x_1 + i\beta_m x_2}, & x_2 \in \Omega^+_{H}, \\
\sum_{m=\mathbb{Z}} w^s_{m,a} e^{i\alpha_1 x_1 - i\beta_m x_2}, & x_2 \in \Omega^-_{H},
\end{cases}
$$

(2.11)

Then we have, for some constant $C$ that is dependent on $k, |D|$,

$$
\| w^s_\alpha \|_{L^2_{a, \text{pp}}(\Omega)} \leq C \| f \|_{L^2_{a, \text{pp}}(D)}.
$$

The final ingredient to our resolution analysis is that the propagating part of the wave carries the major contribution of the cross-correlation.
Theorem 2.6. Let a compactly supported region \( D \subset \Omega \) be given. For any \( w_a^s \in H^{1,q}_a(\Omega \setminus D) \) satisfying the Helmholtz equation
\[
\Delta w_a^s(x) + k^2 w_a^s(x) = 0 \quad \text{in } \Omega \setminus D,
\]
and the Rayleigh expansion condition \((2.11)\), taking the clock-wise direction, we have
\[
- \text{Im} \int_{\partial D} \frac{\partial w_a^s(y)}{\partial v(y)} w_a^s(y) \, ds(y) = \Lambda \sum_{n \in B_a} \beta_n(|w_{n,a}^{s+}|^2 + |w_{n,a}^{s-}|^2).
\]

Proof. Take \( \Gamma_h \) and \( \Gamma_{-h} \) that are the line sections above and below \( H, -H \) in the Rayleigh expansion condition.

We take a rectangular region \( \Omega_h \) containing \( D \), with \( \Gamma_h, \Gamma_{-h} \) the upper and lower horizontal part, while the left and right boundaries are located on \( \partial \Omega_L \) and \( \partial \Omega_R \). Using partial integration and that \( \Delta w_a^s + k^2 w_a^s = 0 \) on \( \Omega_h \setminus D \), we have
\[
\int_{\partial D} \frac{\partial w_a^s(y)}{\partial v(y)} w_a^s(y) \, ds(y) = \int_{\Omega_h \setminus D} |\nabla w_a^s|^2 - k^2 |w_a^s|^2 \, dy + \int_{\partial \Omega_h} \frac{\partial w_a^s(y)}{\partial v(y)} w_a^s(y) \, ds(y).
\]
Namely,
\[
\text{Im} \int_{\partial D} \frac{\partial w_a^s(y)}{\partial v(y)} w_a^s(y) \, ds(y) = \text{Im} \int_{\partial \Omega_h} \frac{\partial w_a^s(y)}{\partial v(y)} w_a^s(y) \, ds(y).
\]
For the integral on the right-hand side, since \( w_a^s(y) \) is \( \alpha \)-quasi-periodic, the integrand inside the integral is periodic in \( y_1 \) direction; thus the left-hand part and the right-hand part of the section vanish due to quasi-periodicity. Then
\[
\text{Im} \int_{\partial D} \frac{\partial w_a^s(y)}{\partial v(y)} w_a^s(y) \, ds(y) = \text{Im} \int_{\Gamma_h \cup \Gamma_{-h}} \frac{\partial w_a^s(y)}{\partial v(y)} w_a^s(y) \, ds(y).
\]
For the integral on \( \Gamma_h \), since \( v(y) = (0, 1) \) using the Rayleigh expansion, we obtain
\[
- \text{Im} \left( \int_{\Gamma_h} \frac{\partial w_a^s(y)}{\partial v(y)} w_a^s(y) \, ds(y) \right) = - \text{Im} \left( \int_{\Gamma_h} \frac{\partial w_a^s(y)}{\partial y_2} w_a^s(y) \, ds(y) \right) = \Lambda \text{Im} \sum_{n \in \mathbb{Z}} \beta_n |w_{n,a}^{s+}|^2 = \Lambda \sum_{n \in B_a} \beta_n |w_{n,a}^{s+}|^2,
\]
where the last equality is due to the fact that \( \beta_n \) are real if \( n \in B_a \). Similarly, noticing the change of direction in \( v(y) = (0, -1) \), we have
\[
- \text{Im} \left( \int_{\Gamma_{-h}} \frac{\partial w_a^s(y)}{\partial v(y)} w_a^s(y) \, ds(y) \right) = \text{Im} \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial w_a^s(y_1, -h)}{\partial y_2} w_a^s(y) \, ds(y) \right) = \Lambda \sum_{n \in B_a} \beta_n |w_{n,a}^{s-}|^2.
\]
This completes the proof.

3 The RTM method

We are now ready to propose the following RTM imaging functionals:
\[
J_U(z) = \text{Im} \sum_{n \in B_a, \Gamma_h} \int_{\Gamma_h} \frac{i}{\beta_n} u_{a_n}^{inc}(z) \frac{\partial G_a^{qp}(x_r, z)}{\partial x_2} u_{a_n}(x_r) \, ds(x_r),
\]
which is named the upper RTM functional, and
\[
J_L(z) = - \text{Im} \sum_{n \in B_a, \Gamma_{-h}} \int_{\Gamma_{-h}} \frac{i}{\beta_n} u_{a_n}^{inc}(z) \frac{\partial G_a^{qp}(x_r, z)}{\partial x_2} u_{a_n}(x_r) \, ds(x_r),
\]
which is named the lower RTM functional. We remark that, due to the \( \alpha \)-quasi-periodicity of \( u_{a_n}^{inc} \) and the \( -\alpha \)-quasi-periodicity of \( G_a^{qp} \), both \( J_U \) and \( J_L \) are naturally periodic in \( z_1 \) direction. Here, we take \( J_L(z) \) for instance to explain the functional as a two-step algorithm.
Algorithm 3.1. Given the data \( u_{a_n}^s(x_r) \), which is the measurement of scattered field on \( \Gamma_h \), at points
\[
x_r \in \Gamma_h = \{(x_1,-h) \mid x_1 \in \left(-\frac{\Lambda}{2},\frac{\Lambda}{2}\right), \; h > 0\}, \quad r = 1, \ldots, N_r, \quad \text{for all } a_n \in B_a,
\]
the following holds.

1. Back propagation: For all \( r = 1, \ldots, N_r, a_n, n \in B_a \), compute
\[
v_{a_n}(z) = \frac{1}{{N_r}} \sum_{r=1}^{N_r} \frac{\partial G_{a_n}^{qp}(x_r, z)}{\partial x_2} u_{a_n}^s(x_r).
\]

2. Cross-correlation: For all \( z \in \Omega \), calculate
\[
\hat{J}_L(z) = -\text{Im} \sum_{n \in B_a} i \frac{\beta_n}{\hat{\beta}_n} u_{a_n}^\text{inc}(z) v_{a_n}(z).
\]

If we use the measurement data on \( \Gamma_h \), we will have the upper RTM algorithm similarly. It is seen that \( \hat{J}_L(z) \) is an approximation of the continuous integral (3.2).

4 Resolution analysis

In this section, we analyze the resolution of the proposed RTM methods. Firstly, we consider the resolution for the lower RTM algorithm.

4.1 The resolution for lower RTM

Theorem 4.1. The lower RTM functional has the following resolution analysis:
\[
\hat{J}_L(z) = \Lambda^2 \text{Im} \int_\mathcal{D} k^2 (1 - \gamma(y))(F_{\alpha}^n(y, z) + \overline{v_{a_n}^s(y, z)}F_{\alpha}^n(y, z)) \, dy + O(h^{-1}),
\]
where \( v_{n,a}^s \) are the Rayleigh coefficients to the scattering solution \( v_{a}^s \) to
\[
\Delta v_{a}^s(y, z) + k^2 \gamma(y)v_{a}^s(y, z) = k^2 (1 - \gamma(y))F_{\alpha}^n(y, z)
\]
with the Rayleigh scattering condition, for \( n \in B_a \). Further, in terms of the Rayleigh coefficients, we have
\[
\hat{J}_L(z) = \Lambda^2 \sum_{n \in B_a} \beta_n (|v_{n,a}^{\text{inc}}(z)|^2 + |v_{n,a}^s(z)|^2) + O(h^{-1}).
\]

Proof. Recall the lower RTM functional for all \( z \in \Omega^{-}_{h}, \)
\[
\hat{J}_L(z) = -\text{Im} \sum_{n \in B_a} i \frac{1}{\hat{\beta}_n} u_{a_n}^\text{inc}(z) \frac{\partial G_{a_n}^{qp}(x_r, z)}{\partial x_2} u_{a_n}^s(x_r) \, ds(x_r).
\]

Using Theorem 2.3 and Corollary 2.1, we have
\[
\int_{\Gamma_{-h}} \frac{\partial G_{a_n}^{qp}(x_r, z)}{\partial x_2} u_{a_n}^s(x_r) \, ds(x_r) = \int_{\Gamma_{-h}} \frac{\partial G_{a_n}^{qp}(x_r, z)}{\partial x_2} \left( \int_\mathcal{D} k^2 (1 - \gamma(y))G_{a_n}^{qp}(x_r, y) \, dy \right) \, ds(x_r)
\]
\[
= k^2 (1 - \gamma(y)) \int_{\Gamma_{-h}} \frac{\partial G_{a_n}^{qp}(x_r, z)}{\partial x_2} G_{a_n}^{qp}(x_r, y) \, ds(x_r) \, dy
\]
\[
= k^2 (1 - \gamma(y)) \int_\mathcal{D} \left( \frac{1}{2} F_{\alpha}^n(z, y) + R_{\alpha}^n(z, y; h) \right) \, dy.
\]
Here, $|R_0^L(z, y; \hbar)| \leq C(h^{-1})$. We obtain
\[
\mathcal{J}_L(z) = \Lambda \operatorname{Im} \int_D k^2(1 - \gamma(y)) F_{\alpha}^L(z, y) \sum_{n \in \mathbb{Z}} \left( \frac{i}{2 \alpha \beta_n} u_{\alpha n}^{inc}(z) \overline{u_{\alpha n}(y)} \right) dy + O(h^{-1}).
\]

Now we introduce
\[
v_\alpha(y, z) = \sum_{n \in \mathbb{Z}} \frac{i}{2 \alpha \beta_n} u_{\alpha n}(y) u_{\alpha n}^{inc}(z).
\]

It follows that
\[
\sum_{n \in \mathbb{Z}} \frac{i}{2 \alpha \beta_n} u_{\alpha n}^{inc}(z) \overline{u_{\alpha n}(y)} = - \sum_{n \in \mathbb{Z}} \frac{i}{2 \alpha \beta_n} u_{\alpha n}(y) u_{\alpha n}^{inc}(z) = - v_\alpha(y, z).
\]

Further, since
\[
F_{\alpha}^L(z, y) = \sum_{n \in \mathbb{Z}} \frac{i}{2 \alpha \beta_n} e^{i \alpha_n(z_1 - y_1) - i \beta_n(z_2 - y_2)} = - \sum_{n \in \mathbb{Z}} \frac{i}{2 \alpha \beta_n} e^{i \alpha_n(y_1 - z_1) - i \beta_n(y_2 - z_2)} = - F_{\alpha}^L(z, y),
\]

it follows
\[
\mathcal{J}_L(z) = \Lambda \operatorname{Im} \int_D k^2(1 - \gamma(y)) F_{\alpha}^L(z, y) v_\alpha(y, z) dy + O(h^{-1}).
\]

Recalling that $u_{\alpha n}(y) = u_{\alpha n}^{inc}(y) + u_{\alpha n}^s(y)$, if we further introduce
\[
v_\alpha^{s}(y, z) = v_\alpha(y, z) - F_{\alpha}^L(z, y) = \sum_{n \in \mathbb{Z}} \frac{i}{2 \alpha \beta_n} u_{\alpha n}(y) u_{\alpha n}^{inc}(z) - \sum_{n \in \mathbb{Z}} \frac{i}{2 \alpha \beta_n} u_{\alpha n}^{inc}(y) u_{\alpha n}^{inc}(z)
\]
\[
= \sum_{n \in \mathbb{Z}} \frac{i}{2 \alpha \beta_n} u_{\alpha n}(y) u_{\alpha n}^{inc}(z),
\]

we observe that it is the solution to
\[
\Delta_y v_\alpha^{s}(y, z) + k^2 v_\alpha^{s}(y, z) = k^2(1 - \gamma(y)) v_\alpha(y, z).
\]

Thus it is the $\alpha$-quasi-periodic solution to
\[
\Delta v_\alpha^{s}(y, z) + k^2 \gamma(y) v_\alpha^{s}(y, z) = k^2(1 - \gamma(y)) F_{\alpha}^L(y, z)
\]

and satisfies the Rayleigh expansion condition
\[
v_\alpha^{s}(y, z) = \sum_{m \in \mathbb{Z}} v_{m, \alpha}^{s}(z) e^{i \alpha_n y_1 + i \beta_n y_2}, \quad y_2 \in \Omega^+_H,
\]
\[
\sum_{m \in \mathbb{Z}} v_{m, \alpha}^{s}(z) e^{i \alpha_n y_1 - i \beta_n y_2}, \quad y_2 \in \Omega^-_H.
\]

Eventually, we have
\[
\mathcal{J}_L(z) = \Lambda \operatorname{Im} \int_D k^2(1 - \gamma(y)) (F_{\alpha}^L(z, y) + v_\alpha^{s}(y, z)) F_{\alpha}^L(z, y) dy + O(h^{-1}),
\]

which is
\[
\mathcal{J}_L(z) = \Lambda \operatorname{Im} \int_D k^2(1 - \gamma(y)) v_\alpha^{s}(y, z) F_{\alpha}^L(z, y) dy + O(h^{-1}).
\]

Since
\[
\int_D k^2(1 - \gamma(y)) F_{\alpha}^L(z, y) v_\alpha^{s}(y, z) dy = \operatorname{Im} \int_D (\Delta v_\alpha^{s}(y, z) + k^2 \gamma(y) v_\alpha^{s}(y, z)) v_\alpha^{s}(y, z) dy
\]
\[
= \operatorname{Im} \int_D \Delta v_\alpha^{s}(y, z) v_\alpha^{s}(y, z) dy
\]
\[
= - \operatorname{Im} \int_D \frac{\partial v_\alpha^{s}(y, z)}{\partial \gamma(y)} v_\alpha^{s}(y, z) dy,
\]

(4.2)
by Theorem 2.6, we obtain
\[
\mathcal{J}_L(z) = -\Lambda \operatorname{Im} \int_{\partial D} \frac{\partial v_d^+(y,z)}{\partial y} v_d^-(y,z) \, ds(y) + O(h^{-1}) = \Lambda^2 \sum_{n \in B_a} \beta_n (|v_{n,a}^+(z)|^2 + |v_{n,a}^-(z)|^2) + O(h^{-1}).
\]

\[\square\]

### 4.2 The resolution for upper RTM

For \( \mathcal{J}_U(z) \), following steps similar to the proof of Theorem 4.1 until (4.1), we arrive at the following result.

**Theorem 4.2.** The upper RTM functional has the following representation:

\[
\mathcal{J}_U(z) = \Lambda \operatorname{Im} \int_D k^2 (1 - y(y))(\overline{F_a^U(y,z) + v_a^+(y,z)}) F_a^U(y,z) \, dy + O(h^{-1}),
\]

where \( v_a^s \) is the \( a \)-quasi-periodic scattering solution

\[
\Delta_y v_a^s(y,z) + k^2 (1 - y(y)) v_a^s(y,z) = k^2 (1 - y(y)) F_a^U(y,z)
\]

with the Rayleigh expansion condition.

From (4.3), we see that the decaying property of \( \operatorname{Im}(F_a^U) \) and \( \operatorname{Im}(F_a^U) \) as \( z \) leaves \( \partial D \) gives that \( \mathcal{J}_U(z) \) has the decaying property as \( z \) leaves \( \partial D \), in the probing area \( \Omega_a \). On the other hand, as \( z \to \partial D \), we denote the main part of \( \mathcal{J}_U(z) \) by

\[
\hat{\mathcal{J}}_U(z) = \Lambda \operatorname{Im} \int_D k^2 (1 - y(y))(\overline{F_a^U(y,z) + v_a^+(y,z)}) F_a^U(y,z) \, dy.
\]

Recalling

\[
F_a^U(y,z) = F_a^1(y,z) + i F_a^2(y,z), \quad F_a^1(y,z) = F_a^1(y,z) - i F_a^2(y,z),
\]

we let \( v_a^i(y,z), i = 1,2, n \in B_a, \) be the \( a \)-quasi-periodic scattering solution that corresponds to the equations

\[
\Delta_y v_a^i(y,z) + k^2 (1 - y(y)) v_a^i(y,z) = k^2 (1 - y(y)) F_a^U(y,z)
\]

and satisfies the Rayleigh expansion condition

\[
v_a^i(y,z) = \begin{cases} 
\sum_{m \in \mathbb{Z}} v_{m,a}^+(z) e^{i a_m y}, & y \in \Omega_a^+, \\
\sum_{m \in \mathbb{Z}} v_{-m,a}^-(z) e^{-i a_m y}, & y \in \Omega_a^-.
\end{cases}
\]

It follows that

\[
\hat{\mathcal{J}}_U(z) = \Lambda \operatorname{Im} \int_D k^2 (1 - y(y)) (\overline{F_a^1 + i F_a^2} + \overline{v_a^+} + i \overline{v_a^-}) (F_a^1 + i F_a^2) \, dy
\]

\[
= \Lambda \operatorname{Im} \int_D k^2 (1 - y(y)) F_a^1 \overline{v_a^+} \, dy + R_1(z) + R_2(z).
\]

Here, we have

\[
R_1(z) = -\Lambda \operatorname{Im} \int_D (1 - y(y)) F_a^2(y,z) \overline{v_a^+(y,z)} \, dy,
\]

\[
R_2(z) = \Lambda \operatorname{Im} \int_D ik^2 (1 - y(y)) (\overline{v_a^+} F_a^1 + \overline{v_a^-} F_a^2 + \overline{F_a^1} F_a^0 + \overline{F_a^2} F_a^0) \, dy.
\]

The definition of \( F_a^0 \) in (2.7) indicates that \( F_a^0 = \mathcal{O}(|z_2 - y_2|) \) as \( y \to z \). Then, with the help of Theorem 2.5, we have \( v_a^s (\mathcal{O}(|z_2 - y_2|) \) as \( y \to z \). Since the integral can be converted to an integral on \( \partial D \) as in (4.2) in the proof of Theorem 4.1, we know that \( R_1(z), R_2(z) \to 0 \) as \( z \) approaches \( \partial D \).

Thus the property of \( \mathcal{J}_U(z) \) as \( z \) approaches the boundary of \( D \) is reflected in \( v_a^1(z) \), which peaks at the boundary with behavior similar to that of \( \operatorname{Im}(F_a^1(z)) \).
5 Extensions to sound-soft case

Our RTM functionals $\mathcal{J}_L(z)$, $\mathcal{J}_C(z)$ can also be applied to the case of detecting sound-soft periodic array. Namely, we are given $u_{\alpha_n}^{\text{inc}}$ as above, and the scattered field $u_{\alpha_n}^s$ is given by

$$\Delta u_{\alpha_n}^s + k^2 u_{\alpha_n}^s = 0 \quad \text{in } \mathbb{R}^2 \setminus \hat{D},$$
$$u_{\alpha_n}^s = -u_{\alpha_n}^{\text{inc}} \quad \text{on } \partial D,$$

with $a_n$-quasi-periodicity in $x_1$ direction, and satisfies the Rayleigh expansion condition. We can obtain the following resolution results on the two RTM functionals for the case of sound-soft periodic scattering problem.

**Theorem 5.1.** Let the lower RTM functional be given by (3.2), and let $\psi(y, z)$ be the solution to the following problem:

$$\Delta \psi(y, z) + k^2 \psi(y, z) = 0 \quad \text{for } y \in \mathbb{R}^2 \setminus \hat{D}, \quad \psi(y, z) = -I^L_{a_n}(y, z) \quad \text{for } y \in \partial D,$$

with the Rayleigh expansion condition

$$\psi(y, z) = \sum_{y_2 \in \Omega_H} \left\{ \begin{array}{ll} \sum_{m \in \mathbb{Z}} \psi_n^+(z)e^{i\alpha_n y_1 + i\beta_n y_2}, & y_2 \in \Omega_H^+, \\ \sum_{m \in \mathbb{Z}} \psi_n^-(z)e^{i\alpha_n y_1 - i\beta_n y_2}, & y_2 \in \Omega_H^-. \end{array} \right.$$

We have the following result:

$$\mathcal{J}_L(z) = 2\Lambda^2 \sum_{n \in \mathbb{Z}} \beta_n (|\psi_n^+(z)|^2 + |\psi_n^-(z)|^2) + O(h^{-1}).$$

**Proof.** We recall the Green representation formula for sound-soft obstacle scattering,

$$u_{\alpha_n}^s(x_r) = \int_{\partial D} u_{\alpha_n}^x(y)\frac{\partial G^{qp}(x_r, y)}{\partial v(y)} + \frac{\partial u_{\alpha_n}^s(y)}{\partial v(y)} G^{qp}(x_r, y) \, ds(y).$$

Thus, using Corollary 2.1, we obtain

$$\int_{\partial D} \frac{\partial G(x_r, z)}{\partial x_2}u_{\alpha_n}^s(x_r) \, ds(x_1) = \int_{\partial D} u_{\alpha_n}^x(y)\left(\frac{\partial}{\partial v(y)} \frac{\partial}{\partial x_2} (-\frac{1}{2} F_a^L(y, z) + R_{a_n}^L(z, y; h)) - \frac{\partial u_{\alpha_n}^s(y)}{\partial v(y)} \left(-\frac{1}{2} F_a^L(y, z) + R_{a_n}^L(z, y; h)\right) \right) \, ds(y)$$

$$= -\frac{1}{2} \int_{\partial D} u_{\alpha_n}^x(y)\frac{\partial F_a^L(y, z)}{\partial v(y)} - \frac{\partial u_{\alpha_n}^s(y)}{\partial v(y)} F_a^L(y, z) \, ds(y) + R_f(z; h).$$

Here,

$$R_f(z; h) = \int_{\partial D} \frac{\partial R_{a_n}^L(z, y; h)}{\partial v(y)} - \frac{\partial u_{\alpha_n}^s(y)}{\partial v(y)} R_{a_n}^L(z, y; h) \, ds(y).$$

Using Corollary 2.1 once again, we obtain $|R_f(z; h)| = O(h^{-1})$, $h \to \infty$. Eventually, we obtain

$$\mathcal{J}_L(z) = -\Lambda \Im \int_{\partial D} \frac{\partial F_a^L(y, z)}{\partial v(y)} - \frac{\partial \psi(y, z)}{\partial v(y)} F_a^L(y, z) \, ds(y) + O(h^{-1}),$$

where $\psi(y, z)$ is given by (5.1). Using the sound-soft boundary condition, we obtain

$$\mathcal{J}_L(z) = -2\Lambda \Im \int_{\partial D} \psi(y, z) \frac{\partial \psi(y, z)}{\partial v(y)} \, ds(y) + O(h^{-1}).$$

Now, using Theorem 2.6, we obtain the desired resolution analysis. 

□
Using a similar technique, we may obtain the analysis for $I_U(z)$.

**Theorem 5.2.** Let the upper RTM functional be given by (3.1), and let $\psi(y, z)$ be the solution to the following problem:

$$\Delta_y \psi(y, z) + k^2 \psi(y, z) = 0 \quad \text{for} \quad y \in \mathbb{R}^2 \setminus \tilde{D}, \quad \psi(y, z) = -F^U_0(y, z) \quad \text{for} \quad y \in \partial D.$$

We have the following result:

$$J_U(z) = -\Lambda \operatorname{Im} \left\{ \psi(y, z) \frac{\partial F^U_0(y, z)}{\partial \nu(y)} - F^U_0(y, z) \frac{\partial \overline{\psi}(y, z)}{\partial \nu(y)} \right\} dy + O(h^{-1}).$$

### 6 Numerical result

In this section, we test several cases of the periodic scattering objects to demonstrate the imaging ability of our imaging functionals $J_L(z), J_U(z)$.

The probing area of our numerical experiment is

$$\Omega_0 = \left\{ (z_1, z_2) \left| |z_1| \leq \frac{\Lambda}{2}, |z_2| \leq \frac{\Lambda}{2} \right. \right\}.$$

In the following experiments, we choose $\Lambda = 2\pi$. The probing area is discretized by $101 \times 101$ equally distributed points, and the number of receivers on $\Gamma_{+h}$, where $h = 7$, is $N_y = 101$. Since the structure of our RTM functionals has periodicity in the $z_1$ direction with period $\Lambda$, the reconstruction in this single period reflects the reconstruction for the periodic array. To get the synthetic data of the quasi-periodic scattered wave, we use the MPSPACK based on a modified Nyström method proposed by [7]. For the calculation of the quasi-periodic Green function explicitly used in the indicator function, we follow the Ewald method and the procedure introduced in [1] to obtain a fast simulation.

The refractive index of our numerical experiment for a penetrable obstacle is $\gamma(x) = 1.5$. The boundaries of the obstacles that are used in our numerical experiments are listed below, where $t \in [0, 2\pi), \rho > 0$.

- **Circle:** $z_1 = \rho \cos(t), z_2 = \rho \sin(t)$.
- **Kite:** $z_1 = \rho(1.1 \cos(t) + 0.625 \cos(2t) - 0.625), z_2 = \rho(1.5 \sin(t))$.
- **Peanut:** $z_1 = \cos(t) + \rho \cos(3t), z_2 = \sin(t) + \rho \sin(3t)$.

We remark that $\alpha = k \cos \theta$ and $\theta$ is the incident angle. In our numerical examples, the incoming angles are chosen as $\theta = \frac{\pi}{4} + \frac{m \pi}{7}, m \in \mathbb{Z}$. In Examples 1 and 2, for Figure 3 to Figure 6, (a) corresponds to $\theta = \frac{\pi}{2}$ and $m = 0$, that is, $\alpha = 0$, which is the vertical incident direction. Panel (b) corresponds to the average of the sum for imaging functionals of five different incident angles with $m = 0, \pm 1, \pm 2, \pm 3, \pm 4$. Panel (c) corresponds to the average of the sum for imaging functionals of nine different incident angles with $m = 0, \pm 1, \pm 2, \pm 3, \pm 4$.

**Example 1.** In this example, we consider the imaging of penetrable periodic circles with radius $\rho = 0.8$ at $k = 5.2\pi$ by our RTM functionals. Figure 3 shows the imaging quality of $J_L$, which demonstrates that the imaging functional has positive values and peaks at the boundary of the scatterer. Figure 4 shows the imaging results of $J_U$. It is clear from the pictures that one can get better imaging result as the number of $a$ increases. We remark that the imaging result of $J_L$ is sharp especially for the vertical part of the boundary, while for $J_U$, the horizontal part of the circle is imaged clearly.

**Example 2.** In this example, we consider the imaging of sound-soft periodic kite arrays. Here, $\rho = 0.6$ and $k = 4.68\pi$. Figure 5 shows the imaging results of $J_L$, which has positive values and captures the non-convexity of the vertical part clearly, and with enough $a$’s, even the upper part of the obstacle array is obtained. This confirms our resolution analysis for the lower RTM method (5.2). Figure 6 shows the imaging results of $J_U$. We can find that the upper horizontal part of the sound-soft periodic kite can be reconstructed clearly. With more $a$’s, the imaging quality is also sharper with fewer false images.
Figure 3: Reconstruction by lower RTM for penetrable circle.

Figure 4: Reconstruction by upper RTM for penetrable circle.
Figure 5: Reconstruction by lower RTM for sound-soft kite.

Figure 6: Reconstruction by upper RTM for sound-soft kite.
| $\mu$  | $\sigma$  | $|U^s_a|_F$  | $|V^{noise}_a|_F$  |
|------|----------|-------------|-----------------|
| 0.100000  | 0.255456  | 0.347392   | 0.084532        |
| 0.200000  | 0.510912  | 0.347392   | 0.170810        |
| 0.400000  | 1.021823  | 0.347392   | 0.341842        |
| 0.600000  | 1.532735  | 0.347392   | 0.508574        |

Table 1: Different levels of average signal and noises for lower RTM averaging over nine different $\alpha$’s.

Figure 7: Reconstruction by lower RTM for penetrable peanut with noise levels 10%, 20%, 40%, 60%.

Example 3. In this example, we consider the stability of our RTM functionals with respect to the complex additive Gaussian random noise as in [10] on the peanut-like scatterer with $\rho = 0.2$ at $k = 4.2\pi$. Since there are $N_r$ measured data on $\Gamma_h$ (or $\Gamma_{-h}$) for any $n \in B_a$, the received data form an $N_r \times |B_a|$ matrix for each $a$; we name it $U^s_a$. Thus we introduce the additive Gaussian noise as follows:

$$U^s_{a,noise} = U^s_a + V^{noise}_a,$$

where $V^{noise}_a$ is the Gaussian noise of mean zero with standard deviation of $\mu$ multiplied by the maximum of the data $|U^s_a|$, $V^{noise}_a = \mu \cdot \max |U^s_a| (\epsilon_1 + i\epsilon_2)$.

Here, $\epsilon_j \sim \mathcal{N}(0, 1)$ for the real ($j = 1$) and imaginary part ($j = 2$). The noise level is calculated as

$$|V^{noise}_a|_F^2 = \frac{1}{N_r|B_a|} \sum |V^{noise}_a(x_r)|^2,$$
and $\sigma = \mu \cdot \max |U_{\alpha}^s|$, while the received data level is calculated as

$$|U_{\alpha}^{\text{noise}}|_F = \frac{1}{N_r |B^{x}} \sum |U_{\alpha}^{s}(x_r)|^2$$

for each $\alpha$, and the arithmetic mean is taken over all nine $\alpha$’s. The result are listed in Table 1 and Table 2.

Here, we use images of nine different $\alpha$’s, and (a) is the image of noise level of 10%, while (b)–(d) correspond to noise levels of 20% to 60%. Here, Figure 7 shows the imaging quality of $J_L$ of the vertical part of the penetrable periodic peanut. Figure 8 shows the imaging quality of $J_U$ of the horizontal part of the penetrable peanut. The experiments demonstrate that, even with large amount of additive noise in the received data, the imaging functional still gives the image of the boundary of the obstacle arrays.

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References


