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Gradient elasticity applied to a crack

Abstract: The aim of this paper is to investigate the stress and the displacement field of a crack within a robust version of gradient elasticity, focusing at the standard Mode I, II, III problems. Special treatment is attributed to the crack configuration near its tip, deriving the gradient elasticity results that are analogous to the classical asymptotical solutions near the crack tip.

Keywords: generalized functions; gradient elasticity; Helmholtz equation; non-singular crack fields.

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1 Introduction

In this work, we consider a continuum with a plane crack. The standard cases of Modes I, II, III are examined in classical elasticity in several ways, e.g., through the application of Fourier Transform [1] or the Westergaard stress functions [2]. In particular, the renowned classical first order approximations of linear elastic fracture mechanics are obtained by expanding the Westergaard stress functions in the Laurent series around the crack tip and only considering their first term. Nevertheless, the classical solutions suffer stress and strain singularities at the origin. By using a robust version of gradient elasticity, we modify the singular classical approximations in order to remove stress and strain singularities.

This study is divided into a displacement and a stress part. Both of these parts have already been examined to some extent in previous papers. More specifically, as far as displacement is concerned, Mode III has been studied in [3], which predicted a wedge crack profile for a Mode III crack. Another study [4] provided a full field asymptotic displacement expression in terms of a series expansion by proposing a new formula for the crack profile and

studying the corresponding gradient elasticity solution. Additionally, full field displacement solutions for Modes I, II, III have been provided in [1]. Some of these solutions may be questioned, in some sense, for their unclamped crack tip character. The non-singular stress part for Mode III has already appeared in the literature [5]. A simplified asymptotic non-singular stress solution for Mode I has been listed in [6], while its more complex, more accurate counterparts have been provided in [7, 8]. Further details can be found in [9]. We are complementing all these results by deriving appropriate formulae for a clamped crack tip displacement solution in gradient elasticity for all Modes I, II, III for both full field and asymptotical cases. Then, we incorporate the necessary modification to the shear stress derived in [8], in order to satisfy a stress-free crack surface boundary condition.

2 Governing equations

The governing equation of the gradient elasticity framework to be used is summarized by Eqs. (1a) and (1c):

$$(\mathbf{1}-c\Delta)\mathbf{u}=\mathbf{u}^0,$$

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{n}^2}=\mathbf{0};$$

$$(\mathbf{1}-c\Delta)\boldsymbol{\sigma}=\boldsymbol{\sigma}^0. \quad (1)$$

These equations provide the gradient elasticity displacement and stress field of a continuum in terms of the corresponding classical displacement $\mathbf{u}^0$ and stress $\boldsymbol{\sigma}^0$. Eq. (1a) results from the Ru-Aifantis theorem [3], and its solution is possible only by considering an extra displacement boundary condition like Eq. (1b), which has been proposed, among others, in [3, 4]. From a physical point of view, it corresponds to a zero hyper-traction boundary condition that has also been used for cracks in [10]. Meanwhile, Eq. (1c), which is closely related to non-local elasticity for certain kernels [11], has also been employed for the elastic or total stress. This equation has already been used in the literature for obtaining non-singular stress expressions for dislocations [12, 13], a Mode III crack near its tip in [5], and a Mode I crack [6–8].

3 Mode III crack

3.1 General considerations

We consider a line crack placed at $y=0$, $-\ell < x < \ell$ under antiplane shear τ_0 on the crack faces. By applying Fourier Transform, a gradient elasticity solution satisfying Eqs. (1a) and (1b) may be obtained [1], which has an unclamped crack tip character and is described by (see Figure 1):

$$u_z = \frac{-\tau_0 \ell}{\mu} \operatorname{signy} \int_0^\infty \left[\frac{J_1(\ell \xi)}{\xi} e^{-|y|\xi} - \frac{c \xi J_1(\ell \xi)}{1+c \xi^2} e^{-|y|\sqrt{\xi^2+1/c}} \right] \cos(x \xi) d\xi. \quad (2)$$

In [3], by restricting (1a) on the boundary $y=0$, the boundary values of Eq. (2) are obtained. In order to obtain a clamped crack tip solution, extra considerations for the displacement had to be taken. More specifically, the solution must satisfy the following additional boundary condition $u_z^{\text{gr}}(x, 0^+) = 0$, $|x| > \ell$. In view of the explanation provided in Section 5.2.2 on how these separate boundary conditions may coexist on the same boundary, we proceed to the results. The crack profile of Mode III is described by (Figure 2):

$$u_z(x, 0^+) = F(x) - [\cosh(x/\sqrt{c}) / \cosh(\ell/\sqrt{c})] F(\ell) \quad |x| < \ell, \quad (3)$$

where $F(x)$ is given by

$$F(x) = \frac{-\tau_0}{2\mu\sqrt{c}} \left[e^{\frac{x}{\sqrt{c}}} \int_{-\ell}^x \sqrt{\ell^2 - t^2} e^{\frac{t}{\sqrt{c}}} H(\ell - |t|) dt + e^{-\frac{x}{\sqrt{c}}} \int_x^\ell \sqrt{\ell^2 - t^2} e^{\frac{t}{\sqrt{c}}} H(\ell - |t|) dt \right].$$

To this end, we go a step further [9] by calculating the displacement field with boundary values given by Eq. (3). This is performed by solving a Dirichlet boundary value problem for the Helmholtz equation. Referring to Section 5.2.2 for further details, we list the

Fourier Transform of $F(x) - u_z(x, 0^+)$, which is equal to $\frac{2\sqrt{c} \cos(\ell x) F(\ell)}{1+c x^2} \left[1 + \tanh\left(\frac{\ell}{\sqrt{c}}\right) \right]$. The final solution (Figure 3) is given by:

$$u_z(x, y) = \frac{-\tau_0 \ell}{\mu} \operatorname{signy} \int_0^\infty \left[\frac{J_1(\ell \xi)}{\xi} e^{-|y|\xi} - \frac{c \xi J_1(\ell \xi) + k^{\text{III}} \cos(\ell \xi)}{1+c \xi^2} e^{-|y|\sqrt{\xi^2+1/c}} \right] \cos(x \xi) d\xi, \quad (4)$$

where k^{III} is given by

$$k^{\text{III}} = \frac{\ell}{\pi} \left[1 + \tanh\left(\frac{\ell}{\sqrt{c}}\right) \right] \int_{-1}^1 e^{\frac{\ell}{\sqrt{c}}(\xi-1)} \sqrt{1-\xi^2} d\xi.$$

3.2 Analysis of Mode III crack near the tip

It is an easy task to consider the case of a Mode III crack under antiplane shear at infinity. At this point we are interested in determining what happens near the crack tip. We consider the following stress-free crack surface boundary conditions: $\sigma_{xy}(x, 0^\pm) = \sigma_{yy}(x, 0^\pm) = \sigma_{zy}(x, 0^\pm) = 0$; $x \geq 0$. The solution for the displacement field in classical elasticity is given by:

$$u_x = 0 \quad u_y = 0 \quad u_z = A^* \sqrt{r} \cos(\theta/2); \\ A^* = 4K_{\text{III}}(1+\nu)/(E\sqrt{2\pi}) \quad 0 < \theta < 2\pi,$$

where E and K_{III} are the Young's modulus and the stress intensity factor, respectively. The gradient elasticity solution for the displacement field is obtained by the following boundary value problem for Eq. (1a):

$$(1-c\Delta)u_z = A^* \sqrt{r} \cos(\theta/2),$$

$$\frac{\partial^2 u_z^{\text{gr}}}{\partial y^2}(x, 0) = 0,$$

Figure 1 (A) Plot of Eq. (2) for $\tau_0 < 0$ in units of $-\tau_0 \ell / \mu$. (B) Plot of Eq. (2) from another perspective which denotes its unclamped crack tip character.

Figure 2 Plots of Eq. (3), classical elliptical profile and Eq. (2).

Figure 3 Plot of Eq. (4).

$$u_z^{\text{gr}}(x, 0) \stackrel{x \geq 0}{=} 0. \quad (5)$$

In [4], by following [3], Eq. (5a) is restricted on the line $y=0$ and the boundary value of the displacement $u_z^{\text{gr}}(x, 0)$ is calculated. Indeed for $y=0$, Eq. (5a) becomes $f(x) - cf''(x) = A^* H(x) \sqrt{x}$, where $H(x)$ is the Heaviside step function and its solution in Cartesian coordinates is given by:

$$u_z^{\text{gr}}(x, 0) = A^* f(x) = A^* H(x) \left\{ \sqrt{x} + \frac{\sqrt{\pi}}{4} c^{\frac{1}{4}} \left[e^{\frac{x}{\sqrt{c}}} \left(1 - \text{Erf} \left(\frac{\sqrt{x}}{c^{1/4}} \right) \right) - e^{\frac{x}{\sqrt{c}}} \left(\text{Erfi} \left(\frac{\sqrt{x}}{c^{1/4}} \right) - 1 \right) \right] \right\}. \quad (6)$$

Further calculation of the full field solution is performed in [4] in the form of a series expansion. We complement

this approach by providing the full field solution in integral form with boundary values given by Eq. (6). By referring to the calculation details of that Helmholtz equation Dirichlet problem shown in Section 5.2.2, we present the final result as follows (Figure 4):

$$u(x, y) = A^* \left[\sqrt{r} \cos \frac{\theta}{2} + \frac{y}{4} \int_0^\infty \frac{e^{2\xi/\sqrt{c}} \text{Erfc}(\sqrt{\xi}/c^{1/4}) - 1 - \text{Erfi}(\sqrt{\xi}/c^{1/4})}{\sqrt{\pi} c^{1/4} e^{\xi/\sqrt{c}} \sqrt{(x-\xi)^2 + y^2}} K_1 \left(\frac{\sqrt{(x-\xi)^2 + y^2}}{\sqrt{c}} \right) d\xi \right]. \quad (7)$$

4 Mode I crack

4.1 General considerations

Analogous results can be obtained for a Mode I crack. In [1], the gradient elasticity solution $\tilde{\mathbf{u}}$ was given for a Mode I crack under the boundary conditions $\sigma_{yy}(x, 0) = -\sigma_0$, $|x| < \ell$ and Eq. (1b). That solution has an unclamped crack tip character. In order to obtain a clamped crack tip solution, extra considerations have to be taken for the displacement field. Thus, we introduce the additional boundary conditions given by:

$$\begin{aligned} u_y(x, 0) &= 0, \quad |x| > \ell, \\ u_x(x, 0) &= u_x^0(x, 0), \quad |x| < \ell. \end{aligned} \quad (8)$$

Equation (8a) is the clamped crack boundary condition. Here, Eq. (8b) is adopted because in the case of a Mode I crack under biaxial stress, $u_x^0(x, 0^\pm)$ is zero on the crack surface, a result that we can apply in the gradient elasticity case as well. Similar to Mode III, we add to $\tilde{\mathbf{u}}$ an appropriate solution $\tilde{\tilde{\mathbf{u}}}$ of the homogeneous Helmholtz equation, such that $\mathbf{u} = \tilde{\mathbf{u}} + \tilde{\tilde{\mathbf{u}}}$ satisfies Eq. (8). Referring to the calculation details provided in the Appendix (6), we present the final solution as follows:

Figure 4 (A) classical solution, (B) gradient elasticity crack profile/Eq. (6), (C) full field solution/Eq. (7).

$$\begin{aligned}
u_x &= \frac{-\sigma_0 \ell}{2\mu} \int_0^\infty \left[-1 + 2\nu + |y|\xi - 2c\xi^2 \right] \frac{J_1(\ell\xi)}{\xi} e^{-|y|\xi} + \frac{c\xi(3-2\nu+2c\xi^2)J_1(\ell\xi) + k_x^1 \sin(\ell\xi)}{1+c\xi^2} e^{-|y|\sqrt{\frac{1}{c}+\xi^2}} \sin(x\xi) d\xi, \\
u_y &= \frac{\sigma_0 \ell \operatorname{sign} y}{-2\mu} \int_0^\infty \left[2(1-\nu-c\xi^2) + |y|\xi \right] \frac{J_1(\ell\xi)}{\xi} e^{-|y|\xi} + \frac{2c\xi(\nu+c\xi^2)J_1(\ell\xi) + k_y^1 \cos(\ell\xi)}{1+c\xi^2} e^{-|y|\sqrt{\frac{1}{c}+\xi^2}} \cos(x\xi) d\xi, \\
k_x^1 &= \frac{(-1+2\nu)\ell}{\pi} \int_1^\infty e^{\frac{\ell}{\sqrt{c}}(1-\xi)} \sqrt{\xi^2-1} d\xi, \\
k_y^1 &= \frac{2(-1+\nu)\ell}{\pi} \left[1 + \tanh\left(\frac{\ell}{\sqrt{c}}\right) \right] \int_1^\infty e^{\frac{\ell}{\sqrt{c}}(\xi-1)} \sqrt{1-\xi^2} d\xi.
\end{aligned}$$

4.2 Analysis of Mode I crack near the tip

4.2.1 Stress field

All the above may be applied in the case of a plane strain Mode I crack at an infinite medium under a biaxial stress σ_∞ . The following stress-free crack surface boundary conditions apply: $\sigma_{xy}(x,0)=\sigma_{yy}(x,0)=\sigma_{zy}(x,0)=0$; $x \leq 0$. The gradient elasticity solution of Eq. (1c) for the Mode I stresses has recently appeared in literature [7–9]. However, the shear stress formula derived in [8] has to be modified, so that the traction-free crack surface boundary condition to be satisfied. By calculating an appropriate solution of the homogeneous Helmholtz equation, we obtain the desired modification as follows (Figure 5):

$$\begin{aligned}
\sigma_{xy} &= \frac{K_I}{4\sqrt{2\pi}r^{5/2}} \left[r^2 \sin \frac{\theta}{2} \left(-1 + e^{\frac{r}{\sqrt{c}}} \right) \right. \\
&\quad \left. + \sin \frac{5\theta}{2} \left(-6c + r^2 + 2e^{\frac{r}{\sqrt{c}}} (3c + 3\sqrt{c}r + r^2) \right) \right] + \sigma_{\text{hom}}, \quad (9)
\end{aligned}$$

$$\begin{aligned}
\sigma_{\text{hom}} &= \frac{K_I}{2\sqrt{2\pi}} \frac{3y}{\pi c^{1/4}} \int_{-\infty}^0 |s|^{5/2} \frac{e^s}{\sqrt{-s}} \left[\frac{1}{2} \frac{1}{s} + \frac{1}{s^2} \right] \\
&\quad K_I \left[\sqrt{\left(\frac{x}{\sqrt{c}} - s \right)^2 + \left(\frac{y}{\sqrt{c}} \right)^2} ds \right]. \quad (10)
\end{aligned}$$

4.2.2 Simplified gradient elasticity stress field

In order to obtain a simpler expression for the stress field inspired from the simple structure of the corresponding Mode III solutions [5], we propose the following simplification $\sigma^{\text{sim}} = \sigma^0(1 - e^{-r/\sqrt{c}})$ of the stress formulae [6], which still remains exact in the case of σ_{zz} (Figure 6):

$$\sigma_{xx}^{\text{sim}} = \frac{K_I}{\sqrt{2\pi}r} \left[\cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right] (1 - e^{-\frac{r}{\sqrt{c}}}),$$

$$\sigma_{yy}^{\text{sim}} = \frac{K_I}{\sqrt{2\pi}r} \left[\cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right] (1 - e^{-\frac{r}{\sqrt{c}}}), \quad (11)$$

$$\sigma_{xy}^{\text{sim}} = \frac{K_I}{\sqrt{2\pi}r} \left(\sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right) (1 - e^{-\frac{r}{\sqrt{c}}}), \quad (12)$$

$$\sigma_z^{\text{sim}} = \frac{2K_I \nu}{\sqrt{2\pi}r} \cos \frac{\theta}{2} (1 - e^{-\frac{r}{\sqrt{c}}}).$$

Next, we compare σ^{sim} and σ to determine the extent to which Eqs. (11) and (12a) could be acceptable for engineering applications. First, we note that both σ^{sim} and σ coincide on the crack surface. Second, by using a series expansion, we find that σ_{xx}^{sim} and σ_{xx} coincide around $\theta = \pm\pi/5$, $\pm 3\pi/5$ where the approximation $\cos(5\theta/2) > 0$ applies. Additionally, by using numerical means, we find that σ_{xy}^{sim} and σ_{xy} coincide among others around $r = \sqrt{c}$, $\theta = \pm 0.89\pi$. Details may be found in the Appendix (5).

4.2.3 Displacement field

The displacement field for Mode I $u(x,y)$ in gradient elasticity may be computed by using the Ru-Aifantis theorem and solving Eq. (1a) under the higher order extra boundary condition $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{n}^2} \Big|_{y=0} = \mathbf{0}$ and the restrictions that apply to the classical solution displacement field $u_y(x,0)=0$, $u_x(-x,0)=0$ for $x \geq 0$. We recall the classical elasticity solution, given by:

$$\mathbf{u}^0 = \frac{K_I \sqrt{r}(1+\nu)}{2E\sqrt{2\pi}} \begin{Bmatrix} (2\kappa-1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \\ (2\kappa+1) \sin \frac{\theta}{2} - \sin \frac{3\theta}{2} \end{Bmatrix}, \quad -\pi < \theta < \pi, \quad \kappa = 3-4\nu.$$

Figure 5 (A) Gradient elasticity stress σ_{xy} , (B) Singular classical stress σ_{xy} .

Figure 6 Plots of simplified relations for σ_{xx} , σ_{xy} and σ_{yy} , respectively.

First, we find an arbitrary solution $\tilde{\mathbf{u}}$ of Eq. (1a). Following [7, 9], we consider the following:

$$\tilde{\mathbf{u}} = \frac{K_I(1+\nu)\sqrt{r}}{E\sqrt{2\pi}} \begin{Bmatrix} \left(\kappa - \frac{1}{2} \right) \cos \frac{\theta}{2} + \frac{2c-r^2-2(c+\sqrt{cr})e^{-r/\sqrt{c}}}{2r^2} \cos \frac{3\theta}{2} \\ \left(\kappa + \frac{1}{2} \right) \sin \frac{\theta}{2} + \frac{2c-r^2-2(c+\sqrt{cr})e^{-r/\sqrt{c}}}{2r^2} \sin \frac{3\theta}{2} \end{Bmatrix}.$$

The next step is the conversion of the initial boundary value problem to a Dirichlet boundary value problem [3] by restricting Eq. (1a) on the boundary $y=0$ and by computing $\mathbf{u}(x, 0)$. By considering Eqs. (6) and (7), we find that:

$$\mathbf{u}(x, 0) = \frac{2(1+\nu)K_I}{E\sqrt{2\pi}} \begin{Bmatrix} (1-2\nu)f(x) \\ 2(1-\nu)f(-x) \end{Bmatrix}.$$

Subsequently we define $\tilde{\tilde{\mathbf{u}}}(x, y)$ to be the field whose components are solutions to the homogeneous Helmholtz equation with $\tilde{\tilde{\mathbf{u}}}(x, 0) = (\tilde{\mathbf{u}}_x(x, 0) - u_x(x, 0), \tilde{\mathbf{u}}_y(x, 0) - u_y(x, 0))$. Obviously, the Mode I displacement field is equal to $\tilde{\mathbf{u}} - \tilde{\tilde{\mathbf{u}}}$. The boundary values of $\tilde{\tilde{\mathbf{u}}}$ are given by:

$$\tilde{\tilde{\mathbf{u}}}(x, 0) = \frac{K_I(1+\nu)}{E\sqrt{2\pi}} \begin{Bmatrix} \left\{ \frac{c-e^{-x/\sqrt{c}}(c+\sqrt{cx})}{x^{3/2}} - \frac{(\kappa-1)\sqrt{\pi}}{4} c^{\frac{1}{4}} \left[e^{\frac{x}{\sqrt{c}}} \left(1 - \operatorname{Erf} \left(\frac{\sqrt{x}}{c^{1/4}} \right) \right) - e^{\frac{x}{\sqrt{c}}} \left(\operatorname{Erfi} \left(\frac{\sqrt{x}}{c^{1/4}} \right) + 1 \right) \right] \right\} H(x) \\ \left\{ \frac{-c+e^{x/\sqrt{c}}(c-\sqrt{cx})}{(-x)^{3/2}} - \frac{(\kappa+1)\sqrt{\pi}}{4} c^{\frac{1}{4}} \left[e^{\frac{-x}{\sqrt{c}}} \left(1 - \operatorname{Erf} \left(\frac{\sqrt{-x}}{c^{1/4}} \right) \right) - e^{\frac{-x}{\sqrt{c}}} \left(\operatorname{Erfi} \left(\frac{\sqrt{-x}}{c^{1/4}} \right) + 1 \right) \right] \right\} H(-x) \right\}.$$

It follows that the desired solution for the Mode I displacement field for $y>0$ is given by:

$$\mathbf{u}(x, y) = \tilde{\mathbf{u}}(x, y) - \frac{y}{\pi\sqrt{c}} \int_{-\infty}^{+\infty} \frac{\tilde{\tilde{\mathbf{u}}}(x, 0)}{\sqrt{(x-\xi)^2 + y^2}} K_1 \left(\frac{\sqrt{(x-\xi)^2 + y^2}}{\sqrt{c}} \right) d\xi.$$

The solution for arbitrary y is obtained by extending the components of $\mathbf{u}$ according to the symmetry of Mode I crack, i.e., odd extension of u_y , even for u_x .

5 Mode II crack

5.1 General considerations

Analogous results can be provided for a Mode II crack. In [1], the gradient elasticity solution $\tilde{\mathbf{u}}$ is given for a Mode II crack under the boundary conditions $\sigma_{yx}(x, 0) = -\tau_o$, $|x| < \ell$ and Eq. (1b). That solution has an unclamped crack tip character. In order to obtain a clamped crack tip solution, extra considerations must be taken for the displacement field. Thus, we introduce additional boundary conditions given by:

$$\begin{aligned} u_x(x, 0) &= 0, |x| > \ell, \\ u_y(x, 0) &= u_y^0(x, 0), |x| < \ell. \end{aligned} \quad (13)$$

Here, Eq. (13a) is the clamped crack boundary condition and Eq. (13b) is analogous to Eq. (8b). Similar to Mode I, we obtain the following solutions:

$$\begin{aligned} u_x &= \frac{\tau_\infty \ell \operatorname{sign} y}{2\mu} \int_0^\infty \left[2(1-\nu+c\xi^2) - \xi|y| \right] \frac{J_1(\ell\xi)}{\xi} e^{-|y|\xi} \\ &\quad + \frac{2c\xi(-2+\nu-c\xi^2)J_1(\ell\xi) + k_x^\Pi \cos(\ell\xi)}{1+c\xi^2} e^{-|y|\sqrt{\frac{1}{c}+\xi^2}} \cos(x\xi) d\xi, \\ u_y &= \frac{\tau_\infty \ell}{2\mu} \int_0^\infty \left[(1-2\nu-2c\xi^2+\xi|y|) \frac{J_1(\ell\xi)}{\xi} e^{-|y|\xi} \right. \\ &\quad \left. + \frac{c\xi(1+2\nu+2c\xi^2)J_1(\ell\xi) - k_y^\Pi \sin(\ell\xi)}{1+c\xi^2} e^{-|y|\sqrt{\frac{1}{c}+\xi^2}} \right] \sin(x\xi) d\xi, \\ k_x^\Pi &= \frac{2(-1+\nu)\ell}{\pi} \left[1 + \tanh\left(\frac{\ell}{\sqrt{c}}\right) \right] \int_1^\infty e^{\frac{\ell}{\sqrt{c}}(\xi-1)} \sqrt{1-\xi^2} d\xi, \\ k_y^\Pi &= \frac{(-1+2\nu)\ell}{\pi} \int_1^\infty e^{\frac{\ell}{\sqrt{c}}(1-\xi)} \sqrt{\xi^2-1} d\xi. \end{aligned}$$

5.2 Analysis of Mode II crack near the tip

A similar analysis may be performed for a plane strain mode II crack under uniform shear τ_∞ at infinity. Once again, the problem is solved within classical elasticity by calculating the appropriate Westergaard stress function. The classical displacement field $\mathbf{u}^0$ is described by:

$$\begin{aligned} u_x^0 &= \frac{K_{II}(1+\nu)\sqrt{r}}{E\sqrt{2\pi}} \left(\kappa + \frac{3}{2} \right) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2}, \\ u_y^0 &= \frac{K_{II}(1+\nu)\sqrt{r}}{E\sqrt{2\pi}} - \left(\kappa - \frac{3}{2} \right) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2}. \end{aligned}$$

5.2.1 Stress field

By applying Eq. (1c) similar to Modes I and III, we obtain for the stress field the following formulae after appropriate modification of the σ_{yy} component with the σ_{hom} term described by Eq. (10):

$$\begin{aligned} \sigma_{xx} &= \frac{-K_{II}}{4\sqrt{2\pi r^{5/2}}} \left[7r^2 \sin \frac{\theta}{2} (1-e^{-r/\sqrt{c}}) \right. \\ &\quad \left. + \sin \frac{5\theta}{2} [-6c+r^2+2e^{-r/\sqrt{c}}(3c+3\sqrt{cr}+r^2)] \right], \end{aligned}$$

$$\begin{aligned} \sigma_{xy} &= \frac{K_{II}}{4\sqrt{2\pi r^{5/2}}} \left[3r^2 \cos \frac{\theta}{2} (1-e^{-r/\sqrt{c}}) \right. \\ &\quad \left. + \cos \frac{5\theta}{2} [-6c+r^2+2e^{-r/\sqrt{c}}(3c+3\sqrt{cr}+r^2)] \right], \\ \sigma_{yy} &= \frac{K_{II}}{4\sqrt{2\pi r^{5/2}}} \left\{ r^2 \sin \frac{\theta}{2} (-1+e^{-r/\sqrt{c}}) \right. \\ &\quad \left. + \sin \frac{5\theta}{2} [-6c+r^2+2e^{-r/\sqrt{c}}(3c+3\sqrt{cr}+r^2)] \right\} + \sigma_{\text{hom}}, \\ \sigma_{zz} &= \frac{-2K_{II}\nu}{\sqrt{2\pi r}} \sin \frac{\theta}{2} (1-e^{-r/\sqrt{c}}). \end{aligned}$$

5.2.2 Displacement field

Similar to Mode I, we obtain the displacement for Mode II. Then, we solve $(1-c\Delta)\mathbf{u}=\mathbf{u}^0$ under the extra boundary condition, $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{n}^2} \Big|_{y=0} = \mathbf{0}$, and the restrictions $u_x(x, 0)=0$, $u_y(-x, 0)=0$ for $x \geq 0$ by considering $\tilde{\mathbf{u}}$, $\tilde{\tilde{\mathbf{u}}}$ in analogy to the Mode I case. The result is given by:

$$\begin{aligned} \tilde{\mathbf{u}} &= \frac{K_{II}(1+\nu)\sqrt{r}}{E\sqrt{2\pi}} \left\{ \left(\kappa + \frac{3}{2} \right) \sin \frac{\theta}{2} + \frac{-2c+r^2+2(c+\sqrt{cr})e^{-r/\sqrt{c}}}{2r^2} \sin \frac{3\theta}{2} \right\}, \\ \tilde{\tilde{\mathbf{u}}} &= \frac{K_{II}(1+\nu)}{E\sqrt{2\pi}} \left\{ \left[\left(\kappa + \frac{3}{2} \right) \frac{-4c+2x^2+4\sqrt{c}(\sqrt{c}-x)e^{x/\sqrt{c}}}{4x^2} \right] \sqrt{-x}H(-x) - 4(1+\kappa)f(-x) \right\}, \\ &\quad \left\{ \left[\left(\kappa - \frac{3}{2} \right) \frac{-4c+2x^2+4\sqrt{c}(\sqrt{c}+x)e^{-x/\sqrt{c}}}{4x^2} \right] \sqrt{x}H(x) - 4(1-\kappa)f(x) \right\}, \end{aligned}$$

which describe the displacement field of a Mode I crack $\tilde{\mathbf{u}}, \tilde{\tilde{\mathbf{u}}}$ for $y > 0$:

$$\tilde{\tilde{\mathbf{u}}}(\mathbf{x}, y) = \tilde{\mathbf{u}}(\mathbf{x}, y) - \frac{y}{\pi\sqrt{c}} \int_{-\infty}^{+\infty} \frac{\tilde{\mathbf{u}}(\mathbf{x}, 0)}{\sqrt{(x-\xi)^2+y^2}} K_1 \left(\frac{\sqrt{(x-\xi)^2+y^2}}{\sqrt{c}} \right) d\xi.$$

Thus, the solution for arbitrary y is obtained by extending the components of $\mathbf{u}$ according to the symmetry of Mode II crack, i.e., odd extension of u_x , even for u_y .

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Appendix

Appendix 1: Boundary value problem of the Helmholtz equation

In general, a certain Dirichlet boundary value problem of the two-dimensional Helmholtz equation, Eq. (1a), with boundary condition $u(x, 0^+) = f(x)$ can be solved if we first calculate an arbitrary solution of Eq. (1a) and then solve a Dirichlet boundary value problem of the homogeneous Helmholtz equation $(1 - c\Delta)u = 0$. Indeed, if $\tilde{u}$ is a solution of Eq. (1a) and $\tilde{u}$ is a solution of $(1 - c\Delta)u = 0$ with $\tilde{u}(x, 0^+) := \tilde{u}(x, 0^-) - f(x)$, then the requested solution is obviously $u = \tilde{u} - \tilde{u}$.

The solution of a boundary value problem on the upper half-plane of Eq. (1a) with $u(x, 0^+) = b(x)/2$ is possible by using generalized functions [14]. A quick derivation of that solution follows. By taking into account that for the sufficiently smooth functions that we are concerned with, the generalized derivative is the sum of the classical derivative and a discontinuous Dirac term $\partial_i f = \{\partial_i f\} + [f]_s \cdot n_i \delta_s$, we consider the odd with respect to y extension of u on the whole plane $u_1(x, y) = \text{sign}(y)u(x, |y|)$ which is a solution of the equation on the whole plane with the exception in the line $y = 0$. Then, by applying the operator $1 - c\Delta$ to u_1 according to the generalized functions formalism, we obtain $(1 - c\Delta)u_1 = \{(1 - c\Delta)u_1\} - cb(x)\delta'(y)$ and thus $(1 - c\Delta)u_1 = -cb(x)\delta'(y)$. By applying the Fourier Transform [14] on the latter, we are able to deduce the formula

$$u_1(x, y) = \frac{y}{2\pi\sqrt{c}} \int_{-\infty}^{+\infty} \frac{b(\xi)}{\sqrt{(x-\xi)^2 + y^2}} K_1\left(\frac{\sqrt{(x-\xi)^2 + y^2}}{\sqrt{c}}\right) d\xi. \quad \text{This}$$

has an alternative integral representation in the case that $b(x)$ is an even function. Then, with the help of Fourier Transform [14], we obtain the following [9]:

$$u_1(x, y) = \text{sign}(y) \int_{-\infty}^{+\infty} f(s) \cos(sx) e^{-|y|\sqrt{\frac{1}{c} + s^2}} ds, \quad (14)$$

where $f(s) = \frac{1}{4\pi} F[b(x)](s)$.

Similarly, when $b(x)$ is odd then we have:

$$u_1(x, y) = \text{sign}(y) \int_{-\infty}^{+\infty} f(s) \sin(sx) e^{-|y|\sqrt{\frac{1}{c} + s^2}} ds, \quad (15)$$

where $f(s) = \frac{-i}{4\pi} F[b(x)](s)$.

Appendix 2: Displacement field tuning in gradient elasticity

A question arises as to how two different boundary conditions for Eqs. (1b) and (3) may coexist in the same boundary value problem. If this matter is viewed under the generalized functions theory [14], the answer is clear. The condition in Eq. (1b) is not complete unless the smoothness considerations of the requested solution boundary values are taken into account. Any differential operator within the theory of generalized functions, as shown in the example in Section 5.2.1, generates extra terms that account for any loss of smoothness (discontinuities included), apart from the classical derivative. Thus, by applying the Fourier transform on the equation $f(x) - cf''(x) = H(x)\sqrt{x}$ we impose an implicit boundary condition, i.e., that $f(x)$ is smooth. Hence, the resulting solution and its first derivative are expected to be continuous. Thus, before the application of Fourier Transform, smoothness considerations have to be accounted for. The latter should clarify whether extra Dirac terms accounting for loss of smoothness should be introduced in the source member of $f(x) - cf''(x) = H(x)\sqrt{x}$ or not. It turns out that the boundary condition in Eq. (1b) may still be satisfied if a discontinuity of $\partial_x u_y$ at the crack tip is considered, that is, a wedge like profile.

Appendix 3: Revisiting [4] supplemented by a series formula

In [4], parabolic coordinates and Hermite polynomials $H_n(x)$ are employed for the solution of the homogeneous Helmholtz equation in the form of a series expansion. However, a typo occurred in the coordinate expression. Particularly, in Eq. (15), in place of $\sqrt{2}$, 2 appears, that is:

$$\begin{aligned} x &= \frac{\sqrt{c}}{2}(\alpha^2 - \beta^2), \quad y = \sqrt{c}\alpha\beta, \quad \alpha = \sqrt{2c^{-1/4}}\sqrt{r} \cos \frac{\theta}{2}, \\ \beta &= \sqrt{2c^{-1/4}}\sqrt{r} \sin \frac{\theta}{2}. \end{aligned} \quad (16)$$

The aforementioned oversight is also propagated in the first term of the displacement solution, which is described by the following equation:

$$u_z^{\text{gr}}(x, y) = A^* c^{1/4} (\alpha/2) [\sqrt{2} - e^{-(\alpha^2 + \beta^2)/2} [1 - \sqrt{\pi} \beta e^{\beta^2} \text{erfc}(\beta)]].$$

Figure 7 Plot of crack profile near the tip in the (red) classical case and in the cases (blue) $n=1$; (green) $n=5$; (magenta) $n=9$; (cyan) gradient elasticity profile, Eq. (6). All approximations generate the same parabola-like crack profile with a strain singularity, which is removed only in the limit $n \rightarrow \infty$.

We revisit [4] by giving the full solution in polar coordinates as follows:

$$u_z^{\text{gr}}(x, y) = A^* \sqrt{r} \cos \frac{\theta}{2} + A^* c^{1/4} e^{-\frac{r}{\sqrt{c}}} \\ + \sum_{n=0}^{\infty} \frac{d_n H_n \left(c^{-1/4} \sqrt{2r} \cos \frac{\theta}{2} \right) H_{n-1} \left(c^{-1/4} \sqrt{2r} \sin \frac{\theta}{2} \right)}{2^{n+2} n! H_{n-1}(0)} \\ d_n = \int_{-\infty}^{\infty} H_n(t) e^{-t^2} \left[-\text{sign}t + e^{t^2} \text{erfc} \left(\frac{|t|}{\sqrt{2}} \right) \text{sign}t - \text{erfi} \left(\frac{|t|}{\sqrt{2}} \right) \right] dt.$$

The first three non-trivial terms of the solution are obtained for $n=1, 5, 9$. In [4], only the first term of the solution is presented (Figure 7).

Appendix 4: Analysis of Mode I normal stresses into series

The $O(r^{5/2})$ expansions of the exact stresses are listed below:

$$\sigma_{xx} = \frac{K_I \sqrt{r}}{4\sqrt{2\pi c}} \left[3 \cos \left(\frac{\theta}{2} \right) - \frac{\left[6 \cos \left(\frac{\theta}{2} \right) - \cos \left(\frac{5\theta}{2} \right) \right] r}{4\sqrt{c}} \right], \\ \sigma_{yy} = \frac{K_I \sqrt{r}}{4\sqrt{2\pi c}} \left[5 \cos \left(\frac{\theta}{2} \right) - \frac{\left[10 \cos \left(\frac{\theta}{2} \right) + \cos \left(\frac{5\theta}{2} \right) \right] r}{4\sqrt{c}} \right].$$

Table 1 Numerical estimate of gradient elasticity simplified stress σ_{xy}^{sim} .

$r-\theta$	-0.84π	-0.6π	0	0.2π	0.39π	0.9π
$\sqrt{c}$	0.44	1.14	0	0.44	-0.33	-0.18
$2\sqrt{c}$	0.43	1.11	0	0.42	-0.32	-0.17
$3\sqrt{c}$	0.38	0.99	0	0.39	-0.29	-0.15

Table 2 Numerical estimate of gradient elasticity shear stress σ_{xy} .

$r-\theta$	-0.84π	-0.6π	0	0.2π	0.39π	0.9π
$\sqrt{c}$	0.32	0.55	0	-0.02	-0.31	-0.19
$2\sqrt{c}$	0.39	0.72	0	0.09	-0.32	-0.23
$3\sqrt{c}$	0.38	0.76	0	0.16	-0.30	-0.21

The corresponding $O(r^{5/2})$ expansions for Eq. (11a,b) are:

$$\sigma_{xx}^{\text{sim}} = \frac{K_I \sqrt{r}}{4\sqrt{2\pi c}} \left[3 \cos \left(\frac{\theta}{2} \right) + \cos \left(\frac{5\theta}{2} \right) - \left[6 \cos \left(\frac{\theta}{2} \right) + 2 \cos \left(\frac{5\theta}{2} \right) \right] \frac{r}{4\sqrt{c}} \right], \\ \sigma_{yy}^{\text{sim}} = \frac{K_I \sqrt{r}}{4\sqrt{2\pi c}} \left[5 \cos \left(\frac{\theta}{2} \right) - \cos \left(\frac{5\theta}{2} \right) - \left[10 \cos \left(\frac{\theta}{2} \right) - 2 \cos \left(\frac{5\theta}{2} \right) \right] \frac{r}{4\sqrt{c}} \right].$$

Appendix 5: Numerical values of Mode I shear stresses

In Tables 1 and 2, a simple numerical comparison in units of $K_I / (4\sqrt{2\pi c}^{1/4})$ is provided for the exact gradient elasticity shear stress solution given by Eq. (9), and its simplification given by Eq. (12a). Here, we used Mathcad 2001 as a numerical estimation tool.

Appendix 6: Calculation of u_x^{gr} in Mode I

We need to calculate u_x^{gr} from the equation using the following:

$$(1-c\Delta)u_x^{\text{gr}} = u_x^0; \quad \frac{\partial^2 u_x^{\text{gr}}}{\partial y^2}(x, 0^+) = 0, \\ u_x^{\text{gr}}(x, 0^+) = u_x^0(x, 0^+) = \frac{\sigma_o(1-2\nu)}{2\mu} x, \quad |x| < \ell$$

We recognize u_x^{gr} as the sum $u_x^{\text{gr}} = \tilde{u}_x + \tilde{\tilde{u}}_x$ where $\tilde{u}_x$ and $\tilde{\tilde{u}}_x$ are defined by the equations:

$$- (1-c\Delta)\tilde{u}_x = u_x^0, \quad \tilde{u}_x(x, 0^+) = f(x) + \frac{\sigma_o(1-2\nu)}{2\mu} x, \\ f(x) - cf''(x) = f^0(x)$$

$$\text{where } f^0(x) = u_x^0 - \frac{\tau_o \nu}{\lambda} x = -\frac{\tau_o \nu}{\lambda} \text{sign}(x) H(|x| - \ell) \sqrt{x^2 - \ell^2}$$

$$-(1-c\Delta)\tilde{u}_x=0, \quad \tilde{u}_x(x,0^+)=d_1 e^{|x+\ell|/\sqrt{c}}+d_2 e^{|x-\ell|/\sqrt{c}}$$

where $d_1=-d_2$, such that the following relation

$$\tilde{u}_x(x,0^+)=-f(x)=-\frac{a}{\sqrt{c}}\sinh\left(\frac{x}{\sqrt{c}}\right) \text{ applies for } |x|<\ell \text{ with } a=\frac{\sigma_0(1-2\nu)}{2\mu}\int_{\ell}^{\infty} e^{-\xi/\sqrt{c}}\sqrt{\xi^2-\ell^2}d\xi.$$

By calculating the Fourier Transform [14] of $g(x)$ and using Eq. (14) we obtain:

$$\tilde{u}_x=\frac{-\sigma_0\ell}{2\mu}\int_0^{+\infty}\frac{k_x^1\sin(\ell\xi)}{1+c\xi^2}\sin(\xi x)e^{-|y|\sqrt{\frac{1}{c}+\xi^2}}ds,$$

$$k_x^1=\frac{\ell}{\pi}(-1+2\nu)\int_1^{\infty}e^{\frac{\ell}{\sqrt{c}}(1-\xi)}\sqrt{\xi^2-1}d\xi.$$

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