Aperiodic logarithmic signatures

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Communicated by Simon R. Blackburn

Abstract. In this paper we propose a method to construct logarithmic signatures which are not amalgamated transversal and further do not even have a periodic block. The latter property was crucial for the successful attack on the system MST_3 by Blackburn, Cid and Mullan (2009). The idea for our construction is based on the theory in Szabó's book "Topics in Factorization of Abelian Groups".

Keywords. Logarithmic signatures, group theoretic cryptography, MST_3 .

2010 Mathematics Subject Classification. 94A60, 68P25.

1 Introduction

In the early 2000's Magliveras, Stinson and Tran van Trung introduced two public key cryptosystems, MST_1 and MST_2 , based on factorizations, covers and logarithmic signatures, of finite nonabelian groups [9]. Recently, Lempken, Magliveras, Tran van Trung and Wei [6] developed a third cryptosystem, MST_3 . Svaba and Tran van Trung published a newer, strengthened version of MST_3 [14].

A main question is how to produce covers and logarithmic signatures for a group. Blackburn, Cid and Mullan [1] suggested a construction of so-called amalgamated transversal logarithmic signatures from exact transversal logarithmic signatures (for the definition see Section 4.1). Based on the use of these amalgamated transversal logarithmic signatures they presented a successful attack on the system MST_3 .

In this paper we propose a method to construct logarithmic signatures which are not amalgamated transversal and further do not even have the property of being periodic, which was crucial for breaking the system MST_3 (see cases 2 and 3 in [1, Section 4.3]). The idea for this construction is based on the theory in Szabó's book about group factorizations [15].

The paper is organized as follows: In Section 2, covers and logarithmic signatures are introduced and some basic facts are presented. We briefly introduce the cryptosystem MST_3 , for further information see also [6] or [1]. Then we introduce the in [6] proposed platform groups, the Suzuki 2-groups. The question of how to

construct logarithmic signatures is the main issue of Section 4. In Section 5 we present the method for the construction of aperiodic logarithmic signatures. We close with some final thoughts and remarks on further research in Section 6.

2 Covers and logarithmic signatures

The cryptosystem MST_3 is based on the use of covers and logarithmic signatures. We introduce them in this section and give a short overview of necessary results. Further information can be found in [2, 6–9]. Throughout this paper, G denotes a finite group and every set is assumed to be finite.

Let $K \subseteq G$ and $\alpha = [A_1, \dots, A_s]$ be a sequence of sequences

$$A_i = [a_{i,1}, \dots, a_{i,r_i}] \quad \text{with } a_{i,j} \in G$$

such that $\sum_{i=1}^s |A_i|$ is bounded by a polynomial in $\lceil \log |K| \rceil$. Then α is a *cover* for $K \subseteq G$ if every product $a_{1,j_1} \cdots a_{s,j_s}$ lies in K and if every $g \in K$ can be written as

$$g = a_{1,j_1} \cdots a_{s,j_s} \tag{2.1}$$

with $j_i \in \{1, \dots, |A_i|\}$. We denote the set of all covers for $K \subseteq G$ by $\mathcal{C}(K|G)$. If, moreover, the tuple $(j_1, \dots, j_s)$ is unique for every $k \in K$, then α is called a *logarithmic signature* for K . The set of all logarithmic signatures for K is denoted by $\Lambda(K|G)$.

We call the product $a_{1,j_1} \cdots a_{s,j_s}$ in (2.1) a *factorization* of g w.r.t. α . Two factorizations $a_{1,j_1} \cdots a_{s,j_s}$ and $a_{1,h_1} \cdots a_{s,h_s}$ of g are *different* if

$$(j_1, \dots, j_s) \neq (h_1, \dots, h_s).$$

(Note that for $\alpha = [[a, a], [b, b]]$ the element ab has four different factorizations $a \cdot b$.)

If $\alpha = [A_1, \dots, A_s] \in \mathcal{C}(K|G)$ with $r_i := |A_i|$ for all $i \in \{1, \dots, s\}$, then the sequence A_i is called a *block* of α and the sequence $(r_1, \dots, r_s)$ the *type* of α . The *length* of α is

$$l(\alpha) := \sum_{i=1}^s r_i.$$

Covers of minimal length are noteworthy due to the fact that less memory capacity has to be used. The interested reader is referred to [7], [12] and [13] for information on this issue.

For the application in cryptography the following distinction is made. A logarithmic signature $\beta \in \Lambda(K|G)$ is *tame* if every $g \in K$ can be factorized in polynomial time (polynomial in $\lceil \log |K| \rceil$) w.r.t. to β , otherwise β is called *wild*.

The map $\check{\alpha}$, which we are going to define now, is used during the encryption and decryption procedure of the cryptosystem MST_3 . Later on we will identify factorizing w.r.t. a cover α with inverting $\check{\alpha}$.

Let $\alpha = [A_1, \dots, A_s] \in \mathcal{C}(K|G)$ be a cover for $K \subseteq G$ of type $(r_1, \dots, r_s)$ with $A_i = [a_{i,1}, \dots, a_{i,r_i}]$ and let

$$m := \prod_{i=1}^s r_i, \quad m_1 := 1 \quad \text{and} \quad m_i := \prod_{l=1}^{i-1} r_l \quad \text{for all } i \in \{2, \dots, s\}.$$

Let τ_α be the canonic bijection from $\mathbb{Z}_{r_1} \times \dots \times \mathbb{Z}_{r_s}$ to $\mathbb{Z}_m$, i.e.

$$\tau_\alpha : \mathbb{Z}_{r_1} \times \dots \times \mathbb{Z}_{r_s} \rightarrow \mathbb{Z}_m, \quad \tau_\alpha(j_1, \dots, j_s) := \sum_{i=1}^s j_i m_i.$$

That is a generalization of n -ary representations. Let $\check{\alpha} : \mathbb{Z}_m \rightarrow K$ be the surjection

$$\check{\alpha}(x) := a_{1,j_1+1} \cdots a_{s,j_s+1}, \quad \text{where } (j_1, \dots, j_s) = \tau_\alpha^{-1}(x).$$

Note that τ_α^{-1} can be computed efficiently (using Euclid's algorithm) and, therefore, the same is true for $\check{\alpha}$. Moreover, the map τ_α does only depend on the type of α , i.e. for $\alpha, \beta \in \mathcal{C}(K|G)$ we have

$$\tau_\alpha = \tau_\beta \quad \text{if and only if} \quad \alpha \text{ and } \beta \text{ are of the same type.}$$

Let $g \in G$ and let α_g denote the number of pairwise different factorizations $a_{1,j_1} \cdots a_{s,j_s}$ of g w.r.t. α . Then g has exactly α_g different preimages w.r.t. $\check{\alpha} \circ \tau_\alpha$, namely the tuples $(j_1, \dots, j_s)$ with $g = a_{1,j_1} \cdots a_{s,j_s}$. That is the connection to equation (2.1). Therefore, a logarithmic signature $\beta \in \Lambda(K|G)$ is tame if we can compute β^{-1} in polynomial time (polynomial in $\lceil \log |K| \rceil$).

For $\mathcal{F} \in \{\mathcal{C}, \Lambda, \dots\}$ we use the notation

$$\mathcal{F}(G) := \mathcal{F}(G|G).$$

3 The cryptosystem MST_3

Alice chooses a public non-abelian group G with large center Z and generates

- a tame logarithmic signature $\beta = [B_1, \dots, B_s] \in \Lambda(Z)$ of type $(r_1, \dots, r_s)$,
- a random cover $\alpha = [A_1, \dots, A_s] \in \mathcal{C}(K|G)$ for a subset K of G with

$$a_{i,j_i} \in G \setminus Z \quad \text{for all } i \in \{1, \dots, s\} \text{ and } j_i \in \{1, \dots, r_i\},$$

which is of the same type as β .

Then she chooses random elements $t_0, \dots, t_s \in G \setminus Z$ and computes the following covers:

- $\tilde{\alpha} = [\tilde{A}_1, \dots, \tilde{A}_s]$, whereas $\tilde{A}_i = t_{i-1}^{-1} A_i t_i$ for all $i \in \{1, \dots, s\}$,
- $\gamma := [H_1, \dots, H_s]$ with $H_i := [b_{i,1} \tilde{a}_{i,1}, \dots, b_{i,r_i} \tilde{a}_{i,r_i}]$ for all $i \in \{1, \dots, s\}$.

The public key is (α, γ) and the private key is $(\beta, t_0, \dots, t_s)$.

To encrypt an element $x \in \mathbb{Z}_{|Z|}$, Bob computes $y_1 = \check{\alpha}(x)$ and $y_2 = \check{\gamma}(x)$ and sends $y = (y_1, y_2)$ to Alice.

Alice decrypts y by calculating $\check{\beta}^{-1}(y_2 t_s^{-1} y_1^{-1} t_0)$ which equals x . Since β is tame, the decryption-algorithm is efficient.

The cryptographic hypothesis is the hardness of factorizing w.r.t. the random cover α . Furthermore it has to be hard for the attacker to reconstruct the private key by using the public key. For information on these two issues we refer the reader to [1], [3] and [10].

Remark 3.1. Lempken et al. [6] demand two additional properties. Firstly the group G should not be a direct product of Z and a subgroup $U \leq G$, otherwise the system could be weakened using Schreier-trees [6]. The second assumption is

$$a_{i,j} a_{i,l}^{-1} \notin Z \quad \text{for all } i \in \{1, \dots, s\} \text{ and } j \neq l.$$

However, Blackburn, Cid and Mullan [1] did not use that property for their attacks, because it holds for a large number of public keys and it is not required during the encryption and decryption process.

Lempken et al. [6] suggested the use of Suzuki 2-groups (see also [4] and [5]) as platform-groups for the system: Let $\theta \neq \text{id}$ be an odd order field automorphism of $\mathbb{F}_q$ ($q = 2^n$). We then define the Suzuki 2-group as

$$G := \{S(c, d) : c, d \in \mathbb{F}_q\},$$

where

$$S(c, d) := \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ d & c^\theta & 1 \end{pmatrix}.$$

Lemma 3.2. *The center*

$$Z(G) = \{S(0, d) : d \in \mathbb{F}_q\}$$

is an elementary abelian 2-group.

4 Classes of logarithmic signatures

We will now concentrate on the construction of β and we will restrict us, motivated by Lemma 3.2, to elementary abelian 2-groups (denoted by 2^n), although all results in Section 5 hold for every abelian group. Note that β is supposed to be secret.

4.1 Exact transversal logarithmic signatures

A logarithmic signature $\beta = [B_1, \dots, B_s]$ for a group G is called *l-exact transversal* (*r-exact transversal*) if there is a subgroup chain

$$G = G_0 > G_1 > \dots > G_s = \{1\}$$

such that B_i is a left (right) transversal of G_i in G_{i-1} for all $i \in \{1, \dots, s\}$. A logarithmic signature is said to be *exact transversal* if it is l-exact transversal or r-exact transversal. We denote the set of all exact transversal logarithmic signatures for a group G by $\mathcal{ET}(G)$.

Remark 4.1. The block B_s of an exact transversal logarithmic signature β is a subgroup of G , more precisely $B_s = G_{s-1}$. Moreover, $[B_i, \dots, B_s]$ is an exact transversal logarithmic signature for G_{i-1} .

4.2 Amalgamated transversal logarithmic signatures

Let $\beta = [B_1, \dots, B_s]$ denote an exact transversal logarithmic signature of type $(r_1, \dots, r_s)$ for an abelian group G . Blackburn, Cid and Mullan [1] define the following operations on β :

- permute elements within each B_i ,
- permute the B_i ,
- replace B_i by a translate $B_i g$ for some $g \in G$,
- amalgamate two sets B_i and B_j by the single set

$$B_i \cdot B_j := \{gh : g \in B_i, h \in B_j\}.$$

The logarithmic signatures that are constructed from an exact transversal logarithmic signature by applying a finite number of the four previous maps are called *amalgamated transversal* logarithmic signatures, see [1]. We will denote the set of amalgamated transversal logarithmic signatures for a group G by $\mathcal{AT}(G)$.

The amalgamated transversal logarithmic signatures have the special property of being periodic, which Blackburn, Cid and Mullan [1] used to break MST_3 under the assumption that the platform-group G is a Suzuki 2-group.

A subset B of an abelian group G is called *periodic* if there exists a $g \in G \setminus \{1\}$ (the *period*) such that $gB = B$. Let $P(B) := \{g \in G \setminus \{1\} : gB = B\}$ be the *set of periods* of B .

Proposition 4.2 ([1, Lemma 2.1]). *Let G be an abelian group and $\beta \in \mathcal{AT}(G)$. Then at least one of the blocks B_i of β is periodic.*

Blackburn, Cid and Mullan showed that every amalgamated transversal logarithmic signature can be used as a private key in MST_3 . Their proof is based on Proposition 4.2, see [1].

Theorem 4.3 ([1, Lemma 2.2]). *Let G be an elementary abelian 2-group. Every logarithmic signature $\beta \in \mathcal{AT}(G)$ is tame.*

5 Constructing aperiodic tame logarithmic signatures

Since the usage of amalgamated transversal logarithmic signatures leaves the cryptosystem insecure, we are in need to find new ways of constructing tame logarithmic signatures, preferably some without periodic blocks. In this section we will introduce an algorithm to construct tame logarithmic signatures without periodic blocks for elementary abelian 2-groups G .

As in a logarithmic signature β every group element is at most once in a block, we will consider sets instead of sequences in the first two paragraphs of this section to simplify the notation.

We call a logarithmic signature $\beta \in \Lambda(G)$ *aperiodic* if none of the blocks B_i is periodic. The set of all aperiodic logarithmic signatures for a group G is denoted by $\mathcal{A}(G)$.

Theorem 5.1 ([15, Theorem 7.3.1]). *Let G be an elementary abelian 2-group. There exists an aperiodic logarithmic signature β of type $(r_1, \dots, r_s)$ with $r_1 \geq \dots \geq r_s \geq 2$ if one of the following cases holds:*

- $s = 2$ and $r_2 \geq 8$,
- $s \geq 3$ and $r_1 \geq 8, r_s \geq 4$.

There does not exist an aperiodic logarithmic signature of type $(r_1, \dots, r_s)$ with $r_1 \geq \dots \geq r_s \geq 2$ if one of the following cases holds:

- $r_s = 2$,
- $s = 1$,
- $s = 2$ and $r_2 \mid 4$,
- $s \geq 3$ and $r_1 \mid 4, \dots, r_s \mid 4$.

We are going to use the idea of the proof of this theorem to construct tame aperiodic logarithmic signatures for elementary abelian 2-groups, for example for the center of a Suzuki 2-group.

5.1 The algorithm

Now we are presenting the algorithm which constructs a new logarithmic signature out of a subgroup and a left transversal of that subgroup. The realization of some rather vague steps in the algorithm, namely the construction of δ and all $\alpha^{(j_1, \dots, j_s)}$, will be discussed in the last part of the paper.

Algorithm 1: Construction of aperiodic LS

output: $\beta \in \mathcal{A}(G)$

Choose an abelian group G , a subgroup U of G and a transversal R of U in G ;

Generate

$$\delta = [D_1, \dots, D_s] \in \Lambda(R)$$

with

$$D_i = \{d_{i,1}, \dots, d_{i,r_i}\}$$

of type $(r_1, \dots, r_s)$ and logarithmic signatures

$$\alpha^{(j_1, \dots, j_s)} := [A_1^{(j_1)}, \dots, A_s^{(j_s)}] \in \Lambda(U)$$

for all $(j_1, \dots, j_s) \in \{1, \dots, r_1\} \times \dots \times \{1, \dots, r_s\}$;

Compute $\beta := [B_1, \dots, B_s]$ by

$$\begin{aligned} B_1 &:= d_{1,1}A_1^{(1)} \cup \dots \cup d_{1,r_1}A_1^{(r_1)}, \\ &\vdots \\ B_s &:= d_{s,1}A_s^{(1)} \cup \dots \cup d_{s,r_s}A_s^{(r_s)}. \end{aligned}$$

Notice that all logarithmic signatures $\alpha^{(j_1, \dots, j_s)}$ are used for the construction of β .

Example 5.2. We choose

$$G := \langle u, v, w, x, y, z \rangle = 2^6, \quad U := \langle u, v, w, x \rangle, \quad R := \{1, y, z, yz\}$$

and set

$$D_1 := \{1, z\}, \quad D_2 := \{1, y\},$$

and

$$\begin{aligned} A_1^{(1)} &:= \{1, u, v, uv\}, & A_1^{(2)} &:= \{1, w, x, wx\}, \\ A_2^{(1)} &:= \{1, uw, vx, uvwx\}, & A_2^{(2)} &:= \{1, ux, uvw, vwx\}. \end{aligned}$$

We get

$$\begin{aligned} B_1 &= \{1, u, v, uv, z, wz, xz, wxz\}, \\ B_2 &= \{1, uw, vx, uvwx, y, uxy, uvwy, vwx y\}. \end{aligned}$$

Neither of these two blocks is periodic. It follows that $\beta \in \mathcal{A}(G)$ of type (8, 8).

Theorem 5.3. *The sequence β constructed by Algorithm 1 is a logarithmic signature for G of type $(l_1, \dots, l_s)$, where $l_i := \sum_{j=1}^{r_i} |A_i^{(j)}|$.*

We will denote a logarithmic signature, which can be obtained from U and R by the construction above, *decomposed and reunited* out of U and R , shortly *d.r.*, and we denote the set of logarithmic signatures for a group G which are d.r. by $\mathcal{DR}_G(U, R, \mathcal{E}(U), \mathcal{F}(R))$ where $\mathcal{E}, \mathcal{F} \in \{\Lambda, \mathcal{ET}, \mathcal{AT}, \dots\}$.

Remark 5.4. Every logarithmic signature $\beta = [B_1, \dots, B_s] \in \Lambda(G)$ is d.r. out of $U = G$ and $R = \{1\}$: Set $D_i = \{1\}$ and $A_i^{(1)} = B_i$ for all $i = 1, \dots, s$.

The following questions turn up:

Question 1. How does the choice of the subgroup U and the transversal R influence the set $\mathcal{DR}_G(U, R, \mathcal{E}(U), \mathcal{F}(R))$?

Question 2. Which logarithmic signatures are constructible out of the pair (U, R) when we choose δ and $\alpha^{(j_1, \dots, j_s)}$ to be for example exact transversal only?

Notice that it is possible to construct an aperiodic logarithmic signature by using only *total exact transversals*, i.e. exact transversals where every block is a subgroup, see Example 5.2 above.

Proposition 5.5. *A logarithmic signature β which is d.r. is tame if δ and all logarithmic signatures $\alpha^{(j_1, \dots, j_s)}$ are tame and if $|R|$ is bounded by a polynomial in $[\log |G|]$ (then for every $g \in G$ the coset representative in R which lies in the same coset as g can be found efficiently).*

Remark 5.6. The last assumption of Proposition 5.5 is not required if G is given in form of a maximal set of generators $\{g_1, \dots, g_t\}$ with the property that every element can be represented uniquely, where $U = \langle g_1, \dots, g_i \rangle$, $R = \langle g_{i+1}, \dots, g_t \rangle$. In that case we get the desired coset representative by using a projection.

5.2 Aperiodicity of β

From now on we assume that $\beta = [B_1, \dots, B_s]$ is constructed by Algorithm 1 and we use the notation introduced there. Next we summarize some basic facts. After that we show how to choose the sets $A_i^{(j)}$ to force the non-periodicity of B_i .

Lemma 5.7. *We have $d_{i,j}^{-1}d_{i,k} \notin U$ for all $i = 1, \dots, s$ and $j, k = 1, \dots, r_i$ with $j \neq k$.*

Proof. We assume that there are i and $j \neq k$ with $d_{i,j}^{-1}d_{i,k} \in U$. We consider the two factorizations

$$d_{1,1} \cdots d_{i-1,1} d_{i,j} d_{i+1,1} \cdots d_{s,1} \quad \text{and} \quad d_{1,1} \cdots d_{i-1,1} d_{i,k} d_{i+1,1} \cdots d_{s,1}.$$

These elements of R are in different cosets of U in G . On the other hand we have

$$\begin{aligned} (d_{1,1} \cdots d_{i-1,1} d_{i,j} d_{i+1,1} \cdots d_{s,1})^{-1} d_{1,1} \cdots d_{i-1,1} d_{i,k} d_{i+1,1} \cdots d_{s,1} \\ = d_{i,j}^{-1} d_{i,k} \in U, \end{aligned}$$

which is not possible. $\square$

Lemma 5.8. *Let $A, B \leq G$. Then $A = B$ if and only if there is an element $g \in G$ with $gA = B$.*

Lemma 5.9. *If B_i is periodic with period $g \in G$, then for every $d_{i,j} A_i^{(j)}$ there is a $k \in \{1, \dots, r_i\}$ such that*

$$g d_{i,j} A_i^{(j)} = d_{i,k} A_i^{(k)}.$$

If additionally $A_i^{(j)}, A_i^{(k)} \leq G$ holds, then $A_i^{(j)} = A_i^{(k)}$.

Proof. Assume there is no such k . Then we have $a_1, a_2 \in A_i^{(j)}$ with $a_1 \neq a_2$ and $b \in A_i^{(e)}, c \in A_i^{(l)}$ for $e \neq l$ such that $g d_{i,j} a_1 = d_{i,e} b$ and $g d_{i,j} a_2 = d_{i,l} c$. From that it follows

$$d_{i,l}^{-1} d_{i,e} = c a_2^{-1} a_1 b^{-1} \in U,$$

which is a contradiction to Lemma 5.7. This shows the first statement. The second part follows from Lemma 5.8. $\square$

Motivated by the second part of Lemma 5.9, we introduce the sets

$$D_i^{(j)} := \{d_{i,k} : A_i^{(k)} = A_i^{(j)}\}$$

of elements $d_{i,k}$ that have the same corresponding subset $A_i^{(k)}$. Then we immediately obtain the following lemma.

Lemma 5.10. *The block B_i is periodic if one of the following holds:*

- (i) $\bigcap_{j=1}^{r_i} P(A_i^{(j)}) \neq \emptyset$.
- (ii) $\bigcap_{j=1}^{r_i} P(D_i^{(j)}) \neq \emptyset$.

The special case $r_i = 2$ or 3 and pairwise different subgroups $A_i^{(j)}$ of the following proposition was proven in cooperation with Anja Nuss [11].

Proposition 5.11. *Let $A_i^{(j)} \leq G$ for all $j \in \{1, \dots, r_i\}$. If B_i is periodic with period $g \in G$, then*

$$g \in \bigcap_{j=1}^{r_i} \left[\{d_1 d_2^{-1} : d_1, d_2 \in D_i^{(j)}\} \cdot A_i^{(j)} \right].$$

Proof. Assume that B_i is periodic. Let $g \in G$ be a period of B_i . By Lemma 5.9 we have for every j and $D_i^{(j)} = \{d_{i,j_1}, \dots, d_{i,j_k}\}$ that

$$g(d_{i,j_1} A_i^{(j)} \cup \dots \cup d_{i,j_k} A_i^{(j)}) = d_{i,j_1} A_i^{(j)} \cup \dots \cup d_{i,j_k} A_i^{(j)}.$$

Moreover, every $d_{i,j_l} A_i^{(j)}$ is mapped to some $d_{i,j_c} A_i^{(j)}$ by multiplication with g , i.e.

$$g d_{i,j_l} A_i^{(j)} = d_{i,j_c} A_i^{(j)}.$$

From that we obtain

$$g d_{i,j_l} d_{i,j_c}^{-1} \in P(A_i^{(j)}) \cup \{1\} = A_i^{(j)}.$$

□

An immediate consequence is the following equivalence.

Corollary 5.12. *Let $A_i^{(j)} \leq G$ for all $j \in \{1, \dots, r_i\}$ such that $A_i^{(j)} \neq A_i^{(k)}$ for all $j, k \in \{1, \dots, r_i\}$ with $j \neq k$. Then B_i is periodic if and only if*

$$\bigcap_{j=1}^{r_i} A_i^{(j)} \neq \{1\}.$$

Next we generalize Proposition 5.11. For a group G and $A, B \subseteq G$ we say that A is a *multiple* of B if there is a $g \in G$ with $gA = B$. Notice if B is a subgroup of G and A a multiple of B , then A is a left coset of B in G . We say that a multiple A of B is *proper* if $A \neq B$.

Lemma 5.13. *If B_i is periodic with period $g \in G$, if $A_i^{(j)}$ is not a proper multiple of $A_i^{(k)}$ and if*

$$g d_{i,j} A_i^{(j)} = d_{i,k} A_i^{(k)},$$

then $A_i^{(j)} = A_i^{(k)}$.

□

Now we obtain analogously to Proposition 5.11 the following:

Proposition 5.14. *Suppose $A_i^{(j)}$ is not a proper multiple of $A_i^{(k)}$ for all indices $j, k \in \{1, \dots, r_i\}$. If B_i is periodic with period $g \in G$, then*

$$g \in \bigcap_{j=1}^{r_i} \left[\{d_1 d_2^{-1} : d_1, d_2 \in D_i^{(j)}\} \cdot \left(P(A_i^{(j)}) \cup \{1\} \right) \right].$$

Corollary 5.15. *Suppose $A_i^{(j)}$ is not a multiple of $A_i^{(k)}$ for all $j, k \in \{1, \dots, r_i\}$. Then B_i is periodic if and only if*

$$\bigcap_{j=1}^{r_i} P(A_i^{(j)}) \neq \emptyset.$$

5.3 Concrete construction for G elementary abelian of order 2^n

We construct aperiodic logarithmic signatures for elementary abelian 2-groups G . Such a logarithmic signature has already been constructed in Example 5.2 for $G = 2^6$. Now we generate one for $G = 2^7$ and then use these two logarithmic signatures to construct tame aperiodic logarithmic signatures for all groups 2^n with $n \geq 6$.

Example 5.16 (see also [15, Theorem 7.3.1]). We choose

$$G := \langle t, u, v, w, x, y, z \rangle = 2^7, \quad U := \langle u, v, w, x, y, z \rangle, \quad R := \{1, t\}$$

and set

$$\begin{aligned} A_1^{(1)} &:= \{1, v, wx, vwx\}, & A_1^{(2)} &:= \{1, w, vz, vwz\}, \\ A_2^{(1)} &:= \{1, x, y, xyz\}, \\ A_3^{(1)} &:= \{1, z, u, zuw\}. \end{aligned}$$

and

$$D_1 := \{1, t\}, \quad D_2 := \{1\}, \quad D_3 := \{1\}.$$

The resulting logarithmic signature β is aperiodic of type $(8, 4, 4)$.

General construction. Let $G = 2^n$ be an elementary abelian group of order $n \in \mathbb{N}_{\geq 7}$ and let $\mathcal{B} = (g_1, \dots, g_n)$ be a basis for G . Further, we choose an $s \in \mathbb{N}$ and a partition $v = (v_1, \dots, v_{2s})$ of n with the property

$$2 \leq v_{s+i} < \prod_{j=1}^{i-1} v_j$$

for $i \in \{2, \dots, s\}$. Moreover, we choose a tame logarithmic signature β' for an elementary abelian group of order $2^{v_1+v_{s+1}}$. Next, we take the lexicographical order on G w.r.t. $\mathcal{B}$, meaning that for example $g_1g_2 > g_1g_3$.

Then, using the notation $v_i := \sum_{k=1}^i v_k$, we consider the following decomposition:

$$G = \underbrace{\langle g_1, \dots, g_{v_1} \rangle \times \dots \times \langle g_{v_{s-1}+1}, \dots, g_{v_s} \rangle}_{U} \times \underbrace{\langle g_{v_s+1}, \dots, g_{v_{s+1}} \rangle \times \dots \times \langle g_{v_{2s-1}+1}, \dots, g_{v_{2s}} \rangle}_{R}.$$

$U_1 \qquad U_s \qquad D_1 \qquad D_s$

Note that v contains the information which elements of $\mathcal{B}$ generate which subgroup U_i resp. D_i . Further, the order on G encodes the position of the elements of every subgroup of G (this information is needed for the factorization).

From now on we want every subgroup U_i and D_i of G to be ordered in the way mentioned above (except for U_1 and D_1 , because we do not need them to be ordered) and we write

$$U_i = [u_{i,1}, \dots, u_{i,m_i}, 1] \quad \text{and} \quad D_i = [d_{i,1}, \dots, d_{i,r_i-1}, 1]$$

for $i = 2, \dots, s$ and $r_i := |D_i|$. For example $u_{i,1} = g_{v_{i-1}+1} \dots g_{v_i}$ for $i \geq 2$.

Then we choose subsets

$$K_i := \{k_i^{(1)}, \dots, k_i^{(r_i)}\} \subseteq (U_1 \times \dots \times U_{i-1})^\#$$

of size r_i for every $i \in \{2, \dots, s\}$. We construct the logarithmic signature

$$\beta = [\beta', B_2, \dots, B_s]$$

using Algorithm 1 by setting

$$\delta := [D_2, \dots, D_s],$$

$$A_i^{(j)} := [k_i^{(j)}u_{i,1}, \dots, k_i^{(j)}u_{i,m_i}, 1] \quad \text{for } i = 2, \dots, s \text{ and } j = 1, \dots, r_i.$$

Then $A_i^{(j)}$ is not a multiple of $A_i^{(l)}$ for $j \neq l$. Therefore, Corollary 5.15 implies that the resulting logarithmic signature β for G is aperiodic.

Remark 5.17. For security and storage issues it seems to be reasonable to choose small subgroups U_i and D_i . Further, one should apply some of the operations from Section 4.2 to β to hide the subgroup $U_1 \times D_1$, more precisely, the blocks of the logarithmic signature β' , otherwise an attacker could obtain a periodic (and therefore tame) logarithmic signature for $G/(U_1 \times D_1)$.

In practice we want to store the following secret objects: $\mathcal{B}$, v and β (remember that β is part of the private key in MST_3).

Question 3. Can we store the group G represented in $\mathcal{B}$ without revealing β ? Then it would be possible to drop the currently necessary basis transformation.

We now give the algorithm for the factorization of a group element g w.r.t. the just constructed logarithmic signature β .

Algorithm 2: Factorization

input : An arbitrary element $g \in G$

output: $\tau_\beta^{-1}(\check{\beta}^{-1}(g))$

Let $y = (y_1, \dots, y_n) = \phi_{\mathcal{B}}(g)$ be the coordinate vector of g w.r.t $\mathcal{B}$;

Let $(h_2, \dots, h_s)$ be the $(s-1)$ -tuple consisting of only ones;

for $i \leftarrow s$ **to** 2 **do**

for $j \leftarrow 1 + v_{i+s-1}$ **to** v_{i+s} **do**

if $y_j = 0$ **then**

$h_i \leftarrow h_i + 2^{v_{i+s}+v_i-j}$

end

end

for $l \leftarrow 1 + v_{i-1}$ **to** v_i **do**

if $y_l = 0$ **then**

$h_i \leftarrow h_i + 2^{v_i-l}$

end

end

$y \leftarrow y + \phi_{\mathcal{B}}(b_{i,h_i})$

end

Factorize $\phi_{\mathcal{B}}^{-1}(y)$ w.r.t. β' which yields $\tau_{\beta'}^{-1}(\check{\beta}'^{-1}(\phi_{\mathcal{B}}^{-1}(y))) = (x_1, \dots, x_t)$;

Altogether we get

$$\tau_\beta^{-1}(\check{\beta}^{-1}(g)) = (x_1, \dots, x_t, h_2, \dots, h_s).$$

Correctness. First we have to recall the structure of a block of β :

$$B_i = \left[d_{i,1} k_i^{(1)} U_i^\#, d_{i,1}, \dots, d_{i,r_i-1} k_i^{(r_i-1)} U_i^\#, d_{i,r_i-1}, 1 \cdot k_i^{(r_i)} U_i^\#, 1 \right]$$

for $i = 2, \dots, s$. In lines 4–7 the D_i -part $d_{i,t}$ (for some t) of g is determined. This is done by looking at the components of y which correspond to the generators of D_i , namely $y_{1+v_{i+s-1}}, \dots, y_{v_{i+s}}$. Since the elements of D_i are or-

dered lexicographically, the position of the right multiple of U_i inside the block B_i can easily be determined by the computation in line 6. For example there are $\frac{|D_i|}{2}|U_i| = 2^{v_i+s-(1+v_i+s-1+v_i)}$ elements with $y_{1+v_i+s-1} = 1$, so one has to skip them in case $y_{1+v_i+s-1} = 0$, as done in line 6.

Lines 9–13 deal with the position inside U_i in the same way as before. Altogether the element from B_i in the factorization of g is b_{i,h_i} .

The computation in line 10 is necessary, since the factor $k_i^{(t)}$ has to be eliminated from y , otherwise one could not determine the right element from B_{i-1} (except for the special case that $k_i^{(t)}$ has no U_{i-1} -part).

We have to treat β' differently, which is done in line 16.

Complexity. Inside the loops in lines 6 and 11 we have less than n operations (in $\mathbb{Z}$), every loop has length less than n and the determination of the coordinate vector in line 14 can be done in polynomial time as well. As β' is tame, we have in total a polynomial runtime (polynomial in $\log_2 |G| = n$), which means that β is tame.

Example 5.18. We consider $G := \langle u, v, w, a, b, x, y, z, c, f \rangle = 2^{10}$ (for the sake of simplicity we do not use two different representations of G) and the known aperiodic logarithmic signature

$$\beta' := [[1, u, v, uv, z, wz, xz, wxz], [1, uw, vx, uvwx, y, uxy, uvwy, vwx y]]$$

for $\langle u, v, w, x, y, z \rangle$ from Example 5.2. We choose

$$\mathcal{B} := (u, v, w, x, a, b, y, z, c, f), \quad s := 3 \quad \text{and} \quad v := (4, 1, 1, 2, 1, 1).$$

Further, let $K_2 := \{u, vw\}$ and $K_3 := \{v, ua\}$. This yields the subgroups

$$\begin{aligned} U_1 &= \langle u, v, w, x \rangle, & U_2 &= \langle a \rangle, & U_3 &= \langle b \rangle, \\ D_1 &= \langle y, z \rangle, & D_2 &= \langle c \rangle, & D_3 &= \langle f \rangle \end{aligned}$$

(note that $U_1 \times D_1$ is the group from Example 5.2) and the aperiodic logarithmic signature

$$\beta = [\beta', [\underbrace{cua, c}_{d_{2,1}A_2^{(1)}}, \underbrace{vwa, 1}_{d_{2,2}A_2^{(2)}}], [\underbrace{f vb, f}_{d_{3,1}A_3^{(1)}}, \underbrace{uab, 1}_{d_{3,2}A_3^{(2)}}]]$$

for G . We now factorize for example $g = vwbc$, where

$$y = \phi_{\mathcal{B}}(g) = (0, 1, 1, 0, 0, 1, 0, 0, 1, 0).$$

We start with $h = (1, 1)$ and the iteration $i = 3$ in the outer loop. Both inner loops have only one iteration. It is $y_{10} = 0$, which means that the generator $f = d_{3,1}$ is

not used in the factorization of g . Since the first $\frac{|D_3|}{2}|U_3|$ elements of B_3 contain f as a factor, we have to skip these elements, which is done by the change of h to $(1, 1 + 2^1) = (1, 3)$. Inside the second inner loop h stays the same, because $y_6 = 1 \neq 0$. At the end of the outer loop we get a new y , namely

$$y = (0, 1, 1, 0, 0, 1, 0, 0, 1, 0) + \phi_{\mathcal{B}}(b_{3,3}) = (1, 1, 1, 0, 1, 0, 0, 0, 1, 0).$$

The last iteration $i = 2$ does not change h , since $y_9 \neq 0$ and $y_5 \neq 0$. Note that without the computation in line 14 of Algorithm 2 we would get a different, and therefore wrong, result. Again we carry out the computation in line 10 and get

$$y = (1, 1, 1, 0, 1, 0, 0, 0, 1, 0) + \phi_{\mathcal{B}}(b_{2,1}) = (0, 1, 1, 0, 0, 0, 0, 0, 0, 0).$$

The corresponding element is vw and this obviously has the factorization $uv \cdot uw$, meaning

$$\tau_{\beta'}^{-1}(\check{\beta}'^{-1}(\phi_{\mathcal{B}}^{-1}(y))) = (4, 2).$$

Altogether we get

$$\tau_{\beta}^{-1}(\check{\beta}^{-1}(vwbc)) = (4, 2, 1, 3).$$

Remark 5.19. In the following we suggest some possible improvements and generalizations:

- (i) By straightforward computation one can get a runtime of $O(s \cdot V)$ with V given by $V := \max_i v_i$ for lines 3–14.
- (ii) We could store β' and K_i ($i = 2, \dots, s$) instead of β , but we would have to make some modifications in line 14 of Algorithm 2.
- (iii) We might drop the strict total order on G , but then the factorization algorithm would be slower, although it would still have polynomial runtime (instead of the computation in lines 6 and 14 we would have to search the element of B_i in the factorization of g inside B_i). This could be necessary if the proposed construction turns out to be insecure.
- (iv) A generalization of the algorithm to abelian groups without an element of large (compared to $|G|$) prime power order is possible if all the steps are changed accordingly. One should choose a maximal generating set $\mathcal{B}$ with the property that every element in G can be represented uniquely (meaning, we take for example an element of order p and one of order q except for one of order pq ($p \neq q$)).
- (v) Other strict total orders on G can be used, but then one has to adjust the factorization algorithm.

6 Conclusion

We presented a new way to construct tame logarithmic signatures. The advantage of this method is the possibility to produce aperiodic logarithmic signatures which resist the attack proposed in [1]. As far as we know, one is in need to store δ and all $\alpha^{(j_1, \dots, j_s)}$ to factorize with respect to β . This is also an aspect of security, since these objects have to be reconstructed even if an attacker knows β . Therefore, it is an important question if β is tame whether or not one knows δ and $\alpha^{(j_1, \dots, j_s)}$.

Further, we showed how to get a huge number of aperiodic tame logarithmic signatures by using Algorithm 1. Although, those might not be enough, the fact that we mainly used exact transversal logarithmic signatures for the construction of our examples implies the assumption that many more aperiodic logarithmic signatures might be gained when using for example amalgamated transversal logarithmic signatures.

Still, it is not clear if Algorithm 1 has any weaknesses in view of the reconstruction of δ and the $\alpha^{(j_1, \dots, j_s)}$ from a given β because of the known structure of the algorithm, although in the special case (Section 5.3) we conjecture that keeping the used generating set $\mathcal{B}$ a secret makes it hard to extract any information.

Acknowledgments. We like to thank the referees for their helpful comments.

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Received April 11, 2011; revised February 27, 2012; accepted March 28, 2012.

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