Research Article

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Sources of Finite Speed Temperature Propagation

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Abstract: The relation between heat flux and temperature gradient has been considered as a constitutive structure or as a balance law in different approaches. Both views may allow a description of heat conduction characterized by finite speed propagation of temperature disturbances. Such a result, which overcomes Fourier's drawback of infinite speed propagation, can be obtained also by considering insufficient the representation of a conductor, even when it is considered to be rigid, rather than the sole relation between heat flux and temperature gradient. We comment this last view and describe the intersection with previous proposals. Eventually, we show how under Fourier's law we can have traveling-wave-type temperature propagation when thermal microstructures are accounted for.

Keywords: finite speed heat propagation, heat-driven phase transitions, non-Fourier heat conduction, the Maxwell-Cattaneo model

1 Introduction

A well-known problem in the classical theory of heat conduction is that Fourier's relation between heat flux q and temperature gradient ∇T , namely $q = -\kappa \nabla T$, with κ being the material conductivity, implies an infinite propagation speed for temperature variations. Notwithstanding this aspect, Fourier's law is largely used in technical applications as it appears to be a reasonable approximation of very wide phenomenology. However, the theoretical question remains. Also, records of phenomenology detaching from Fourier's law in a way showing such a law to be an unsatisfactory approximation are available (see, e. g., [1–6]).

Attempts at overcoming the physical drawback for the propagation speed of temperature variations in rigid conductors seem to follow distinct—although not mutually excluding, but rather intersecting—viewpoints. Roughly, they may be summarized in this way:

1. As a constitutive relation suggested by the second law written in terms of the Clausius–Duhem inequality [7], Fourier's law is insufficient for it does not account for memory and long-range spatial effects. So, other constitutive structures accompanied by the idea of refining the description of thermodynamic non-equilibrium states have been proposed (see [8–12]).
2. The relation between q and ∇T should be considered as a balance law, which renders hyperbolic the system with other standard balances (mass, momentum, energy), a system obtained, at least for monoatomic gases, through the Boltzmann equation as a cascade of balance laws for fields defined by averages of mass, velocity, and its progressive diads (see [13–15]).
3. The way we describe the conductor is too skeletal: matter in nature has a microstructure, and it is exactly the influence of microstructural interactions that is the source of the hyperbolic character in the phenomenology of heat conduction (see [16–18]).

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In what follows we discuss some aspects of these approaches, focusing more on the one in item 3 (a view introduced in references [16] and [17]) because for the others we already have at disposal a number of authoritative treatises, which we will mention at least in part.

As a specific technical result of the present work, we show how an approach falling within item 3 above, and introduced in reference [16], leads to traveling-wave-type classical real solutions; thermal waves are also foreseen in another view falling under item 3, the one pertaining to heat-driven microstructures, as introduced in reference [17]; the analysis of this last type of waves is in reference [18].

2 On items 1 and 2

2.1 Memory effects

The first attempt to overcome the drawback associated with Fourier's law has been C. Cattaneo's [8], who proposed in 1948 to consider the evolution law

$$\tau \dot{q} + q = -\kappa \nabla T, \quad (1)$$

with τ being a *relaxation time*, as a variant of Fourier's law (the superposed dot indicates time derivative). Cattaneo based his proposal on kinetic suggestions (in motivating his choice he considered explicitly a gas, each molecule of which carries a certain property not otherwise specified, and looked at the pertinent average, restricting eventually his attention to the case in which such a generic property coincides with the kinetic energy [8, pp. 87–90]), although he did not derive his results by analyzing directly the Boltzmann equation. His reasoning led to a variant of equation (1) augmented by a term proportional to $\nabla \dot{T}$ by square of the time delay, a term that he considered to be negligible assuming smallness for the time delay and the system to be near a stationary state. Cattaneo's proposal generated a large body of analyses and generalizations; the treatise [9] offers a sharp portrait of the pertinent scenario. Already J. C. Maxwell had obtained a similar expression [19, eq. (143)]; however, he cast out $\dot{q}$, considering it to be negligible. We refer to the combination of equation (1) and the balance of energy as *the Maxwell-Cattaneo scheme*, while we refer to relation (1) as *the Cattaneo equation*. When in analyzing the pure phonon gas undergoing a wave-type temperature perturbation, we adopt the Callaway approximation to the collision term and choose two relaxation times, as R. A. Guyer and J. A. Krumhansl did in 1968 ([20, 21]), we get a regularized variant of equation (1) that involves the Laplacian of q and its divergence.

The Cattaneo equation has a memory-type nature, as suggested by the presence of $\dot{q}$. In fact, if we assume a time delay τ such that $q(x, t + \tau) = -\kappa \nabla T(x, t)$ and we assume q to be differentiable with respect to time, Cattaneo's equation (1) emerges as a truncation at the first term of q series expansion with respect to t , so it is weakly non-local in time. This is a special case of the so-called *dual-phase-lag scheme* characterized by two different time delays, one for q , the other for T , namely $q(x, t + \tau_q) = -\kappa \nabla T(x, t + \tau_T)$, where τ_q and τ_T are time (or phase) lags [22, 23] (when we take $\tau := \tau_q - \tau_T$, we get the single-phase-lag equation involving just τ). Besides its formal acceptability, the dual-phase-lag scheme offered the hip to non-trivial criticisms (see references [24–26]).

The idea of a time delay between q and ∇T falls within the general format of materials with memory, as M. E. Gurtin and A. C. Pipkin pointed out in 1968 [11]. With T^t and G^t the histories of temperature and its gradient, respectively, and T the present instantaneous value of the temperature, they presumed that free energy, entropy, and heat flux have to be considered as functionals of a state defined for homogeneous bodies by the list (T, T^t, G^t) . After linearization under appropriate considerations, they got

$$q(x, t) = - \int_{-\infty}^t \mathcal{K}(t - t') \nabla T(x, t') dt', \quad (2)$$

a structure in which they assumed the kernel $\mathcal{K}(0)$ to be bounded. Cattaneo's equation can be derived as a special case. Such a structure allows a more refined description of temperature propagation with finite

speed when we consider, as D. D. Joseph and L. Preziosi did in 1989, that the kernel may involve a delta-type addendum [27]. In any case, we could ask about the origin of memory effects in heat conduction. An answer could be that they are an indirect effect of a material's microstructure. This answer looks reasonable but does not cover other questions concerning what justifies from a physical viewpoint the infinitely-extended-in-the-past memory or whether there is any general principle addressing us to choose the kernel explicit structure.

In 1926, V. A. Fock reached a hyperbolic-type representation of diffusion by considering a discrete random walk of particles with bounded velocity (see remarks in [28], and also [29–31]); when the density involved is considered as the one of heat carriers, one obtains an evolution equation for T involving the second time derivative of T [32]. A issue is, however, what we do intend for heat carriers.

2.2 Refining the description of non-equilibrium states

By looking at heat conducting viscous fluids with the aim of avoiding Fourier's paradox, in 1967 I. Müller suggested the introduction of fluxes, as q and the dissipative (viscous-type [33]) stress component are, in the list of variables specifying non-equilibrium states. Such a view implies a non-standard version of the Gibbs relation given by the differential of a non-equilibrium entropy, which needs to be necessarily introduced. For the resulting differential form, an integral factor plays the role of a non-equilibrium temperature [34, p. 171], and a question is then how it is connected with the equilibrium temperature. Evolution laws emerge for the heat flux and the dissipative component of the stress, which—we recall—is assumed to be in general the sum of energetic (in the sense of energy-dependent), dissipative, and reactive components, the last to be considered in the presence of internal constraints as, e. g., the one of incompressibility. The resulting system satisfies the entropy principle, agrees with approximations of the Boltzmann equation, and provides linear high frequency waves, which propagate with finite speed through a constant state. This point of view—non-equilibrium fluxes enter the description of state as independent variables—had already been indicated by C. Truesdell and R. Toupin in discussing M. Brillouin's general theory of specific and latent heats [35, Sect. 250]. Müller stressed the attention on the connection with the issue concerning the Fourier paradox. Then, a massive body of work, based on this point of view about the representation of non-equilibrium states, has been developed (the treatise [12] offers an extended overview on the scenario). However, Müller himself proved [36] that the constitutive structures that he proposed for the heat conduction in fluids fail to be objective (i. e., they are not covariant under the Euclidean group action over the ambient space); so he concluded rejecting the universal validity of the *objectivity principle* for constitutive equations (indeed, the heat flux fails to be objective, as we can show by deriving it from the Boltzmann equation, but the question raised in Müller's work is more general). A debate ensued; within it, A. I. Murdoch proposed to modify the principle in a form admitting a detachment from the constitutive structures covariant behavior under rigid-body-type changes of observers in the physical space, measured by a skew-symmetric tensor [37]; in fact, Müller's constitutive structures failed to be objective just by a spin term. A question is whether such a proposal may be generally accepted even when we deal with the description of complex materials (those with *active* microstructure, say liquid crystals, quasicrystals, ferroelectrics, etc.). Consider, for example, a ferroelectric material. In representing its mechanical behavior, besides its placement in space, we commonly introduce a vector field $\mathbf{p} := \tilde{\mathbf{p}}(x, t) \in B_p \subset \mathbb{R}^3$, where B_p is a ball with radius the polarization saturation $|\mathbf{p}_{\max}|$. Let $\mathbf{p}$ and $\mathbf{p}'$ be the polarization recorded at x and t by two observers differing by a rigid-body motion. The vector $\mathbf{p}$ is not influenced by rigid translations of frames. At variance, a smoothly time-depending rotation $Q(t) \in SO(3)$ connects the two vectors, namely $\mathbf{p}' = Q\mathbf{p}$. By differentiating with respect to time, we have $\dot{\mathbf{p}}' = \dot{Q}\mathbf{p} + Q\dot{\mathbf{p}}$. By pulling back $\dot{\mathbf{p}}'$ in the first observer, we determine a vector $\dot{\mathbf{p}}^\diamond := Q^T \dot{\mathbf{p}}' = W\mathbf{p} + \dot{\mathbf{p}}$, where W is the skew-symmetric spin tensor $Q^T \dot{Q}$. Consequently, the rule defining the behavior of $\mathbf{p}$ under rigid-body-type changes in observers is $\dot{\mathbf{p}} \mapsto \dot{\mathbf{p}} + W\mathbf{p}$. If we would admit changes in observers purified by a spin, should we erase W choosing a transformation of the type $\dot{\mathbf{p}} \mapsto \dot{\mathbf{p}}^\diamond - W\mathbf{p}$? If so, $\dot{\mathbf{p}}$ would be invariant under the action of $SO(3)$, and this is evidently not true. Or, else, should we consider and apply differently the proposal of considering changes in observers modulo a spin? We leave open the question here.

With independent works, A. Bressan [38] and T. Ruggeri [34] (see also [39]) suggested to consider the evolution laws for the heat flux and the dissipative component of the stress as true field equations, as such not necessarily subjected to the objectivity principle (although we have to remind that the balance of forces and the one of couples can be derived from the requirement of $SO(n)$ -invariance of the external power of actions on an n -dimensional body, also, in conservative setting, the Nöther theorem states that for every transformation leaving invariant the Lagrangian, we get a balance law; and the considered transformations can be intended properly as changes in observers). However, if we accept that these relations are field equations, we can ask about the physical origin of such a character. Also, for the stress, a question would concern the reasons for thinking that its energetic component pertains to a constitutive structure, while the dissipative one should be determined by an independent balance law for which we should also identify the nature of the interactions involved. Indeed, if we work from a kinetic viewpoint, we can derive a balance for the dissipative component of the stress [40], but in that view there are only collisions and we do not consider molecular bonds (we come back on the issue in the next section). Perhaps we could be inclined to think of a law describing the evolution of q as an independent balance law for reasons that will appear later and would justify the same interpretation for all fluxes free of an equilibrium counterpart, as q is, leaving for now suspended the question concerning those fluxes with equilibrium counterparts.

Moreover, in the scheme above summarized, the resulting equations are not always hyperbolic (e.g., Müller's proposal in [10] is not so). Then, they do not cover necessarily the whole scenario of shocks and second sound; regularity of initial data does not necessarily imply that solution be also regular in the same way. However, the approach offers a number of non-trivial results.

The introduction of (non-equilibrium) internal variables may allow one to get hyperbolic structures [41]. Here, with the syntagma *internal variables* we strictly refer to those that are not observable, meaning that they are insensitive to *any* class of changes in observers; also, they do not contribute to the body morphology description, rather they measure the detachment from thermodynamic equilibrium; their evolution is postulated on the basis of phenomenological information (exception is associated plasticity in which these flow rules emerge from a maximum dissipation principle). The scheme is characterized by a wide flexibility (the treatise [42] shows a pertinent panorama). A connected question is, however, whether such evolution laws for the internal variables can be derived from first principles or they reduce themselves as offspring of suggestions from phenomenology or matter of convenience.

2.3 A cascade of field equations

More drastically, we may decide to search a priori a system of balance equations with hyperbolic character. Such a point of view has been proposed first in 1983 by I-S. Liu and I. Müller, who looked at “a phenomenological theory of dilute gases” [13, p. 287]. At roots we find the Boltzmann equation. Densities of mass, momentum, energy, and their fluxes of progressively higher degree emerge as a *cascade* of averages weighted by the distribution function; they involve mass, velocity v of each particle, and its diads with itself. A hierarchy of hyperbolic field equations emerge with the use of Boltzmann's structure. They go far beyond Grad's 13-moments procedure. This view generated another massive body of work in both classical and relativistic settings (the treatises [14] and [15] offer a sharp and comprehensive review of the matter), which has been also extended to include possible internal structure of gas molecules [43].

In 2012, T. Ruggeri showed that the laws by Fourier, Fick, and Darcy, the stress dissipative component linear dependence on the velocity gradient appearing in the presence of viscous effects, as it occurs in the Navier–Stokes equations, and a diffusion equation for multi-temperature mixtures can be all derived from the hierarchy of balance equations emerging from the moment expansion associated with the velocity distribution, when some suitable relaxation time goes near zero and can be neglected. Such a result exploits Maxwell's iteration scheme, i. e., “an identification of relaxation times and a formal power expansion of the solution in terms of these: a sort of Chapman–Enskog procedure at the macroscopic level” [40, p. 601]. It “preserves the entropy principle at least for processes not far from equilibrium” [40, p. 609].

In his analysis, Ruggeri looks essentially at viscous fluids (and their mixtures) because he attributes the dissipative character to the whole deviatoric part of the stress [40, p. 598]. In this view, the pressure has a kinetic character, due to its derivation from the velocity distribution moment expansion (indeed, origins of pressure are various: it can have an energy-dependent character, i. e., it may emerge from the variations of energy with respect to the volume ones; it may also be an undetermined reactive stress corresponding to the incompressibility internal constraint).

The laws derived for the stress deviatoric component and heat flux are (we can say) *weakly non-local* because they depend on gradients. Also, the procedure does not involve nets of linked molecules or atomic lattices, so it does not furnish information on the energetic component of the stress, namely the stress–strain relation, which (in a sense) has also a weakly non-local character, because when the strain is compatible, it is determined by the symmetric factor in the polar decomposition of the deformation *gradient*; the standard stress, in fact, represents first-neighbor interactions; rather than to a point its locality in constitutive terms (see Noll’s axiom of *local action*) refers to an infinitesimal neighborhood of the point itself.

Indeed, the approach based on the cascade of moments already summarized has produced a number of remarkable results. We can use that procedure as a progressive approximation of the thermodynamic behavior, stopping the moment expansion when we describe in a way that we consider at least appreciable the aspects of a gas dynamics we are interested in. At any finite level of the cascade we have a hyperbolic system of balance laws, which allows us not only to describe the shock structures [44] but also to recover finite speed heat propagation. However, as shown in 1997 by G. Boillat and T. Ruggeri, in a classical space-time the characteristic wave velocities of the evolution laws for an increasing number of velocity distribution moments goes to infinity, so the system tends to recover a parabolic character [45].

3 On item 3 ... that is, “thermal material microstructures”

Leaving for a while gases apart and thinking of condensed matter, we may consider that real materials are endowed by interacting microstructures that may be sensitive to temperature variations (see phase transitions, e. g., between paraelectric and ferroelectric phases, or paramagnetic and ferromagnetic ones). Also their behavior may depend on the heat flux, as it occurs for the bubble structures in boiling processes.

A direct representation of material microstructures and their interactions involves geometric descriptors [46, 47]. They bring information about low-dimensional material structures at a macroscopic scale. As such they have to be considered as observable entities that play a role even at equilibrium and are covariant under changes in observers [48]. Thinking in terms of a unified theory (i. e., a meta-theory), we need just to say that such descriptors—say v —of the material microstructure are values of differentiable space-time fields over a finite-dimensional differentiable manifold $\mathcal{M}$. Specific cases dictate particular choices of $\mathcal{M}$. When we change frames in the physical space, we may alter the representation of microstructures over $\mathcal{M}$. We have already shown an example in Section 2.2: when v represents a polarization vector p , an observer differing from another by a rotation perceives v as rotated with respect to what is perceived by the first observer [49, 50]. In general, what we can say is that if $\dot{v}$ is the time rate of v with respect to a given observer and $\dot{v}'$ is the value as recorded by the second one, the two observers linked in physical space by a rigid-body motion, the pull-back $\dot{v}^\diamond$ of $\dot{v}'$ in the first observer is given by $\dot{v}^\diamond = \dot{v} + \mathcal{A}(v)q$, where q is the rigid rotation velocity pertaining to the change in observer, i. e., the characteristic vector of the skew-symmetric tensor $W = Q^T \dot{Q}$, where $\dot{Q}$ is the time derivative of the smooth map $t \mapsto Q(t) \in \mathbb{R}^3$ describing the relative rotation between the two observers. $\mathcal{A}(v)$ is a linear operator from $\mathbb{R}^3$ onto $T_v \mathcal{M}$, which is the tangent space to $\mathcal{M}$ at $v \in \mathcal{M}$ (see [16] for details and generalizations). We take the map $v \mapsto \mathcal{A}(v)$ to be smooth and bounded. For example, when v is a vector in (say) $\mathbb{R}^3$ or in a ball centered at zero, as in the example of Section 2.2, we have $\mathcal{A}(v) = -v \times = -\mathbf{e}v$, with $\mathbf{e}$ being Ricci’s alternating symbol. When v is a scalar, $\mathcal{A}(v)$ vanishes. In this case v might be sensitive to changes in observers determined by diffeomorphisms that are not isometric. So, generally speaking, $\dot{v}^\diamond = \dot{v} + v$, with $v \in T_v \mathcal{M}$ to be determined by defining each time the class of changes in observers taken into account. Also v is indifferent under rigid translations of observers in the physical space: e. g., the polarization vector or a

relative microdisplacement related to a material element does not change if we just translate rigidly the body in space.

Bulk and contact actions pertinent to the microstructure are defined by the power that they develop in the time rate of v , namely $\dot{v}$, exactly as it occurs for standard bulk forces and boundary traction that are defined by the power that they perform in the velocity field. To formalize the idea, consider a conductor occupying $\mathcal{B}$, which is a bounded, open, connected region of $\mathbb{R}^3$, endowed with piecewise Lipschitz boundary. We call *part* of $\mathcal{B}$ any its subset b endowed with the same properties attributed to $\mathcal{B}$. With the aim of showing just basic ideas, we exclude here gross motion and related possible deformation; we then restrict the analysis to a rigid conductor at rest. For any part b , we write *the external power* of actions that are *external* to b and are exerted on and among material microstructures, in the latter case across the boundary ∂b of b , as

$$\mathcal{P}_b(\dot{v}) := \int_b \beta \cdot \dot{v} \, d\mu(x) + \int_{\partial b} \tau_\partial \cdot \dot{v} \, d\mathcal{H}^2(x),$$

where $d\mu$ and $d\mathcal{H}^2$ are volume and surface measures, the latter along ∂b ; the interposed dot indicates duality pairing, which coincides with the scalar product when we consider orthonormal frames of references over $\mathcal{M}$. The contact microstructural action τ_∂ depends on x and t , as the external bulk action β , but also on the boundary ∂b , as indicated by the subscript ∂ .

We define *balanced* those actions for which $\mathcal{P}_b$ is invariant under changes in observers, i. e., those such that $\mathcal{P}_b(\dot{v}) = \mathcal{P}_b(\dot{v}^\diamond)$, for any part considered. Take v *not to be* a scalar. Consider, as above, $\dot{v}^\diamond = \dot{v} + \mathcal{A}(v)q$. The arbitrariness of q , which is the one of changes in observers, implies the integral balance

$$\int_b \mathcal{A}^* \beta \, d\mu(x) + \int_{\partial b} \mathcal{A}^* \tau_\partial \, d\mathcal{H}^2(x) = 0,$$

where the asterisk in $\mathcal{A}^*$ indicates formal adjoint, namely $\mathcal{A}^*(v)$ maps at each v elements of the cotangent space $T_v^* \mathcal{M}$ into $\mathbb{R}^3$; such a property allows the previous balance to be well-defined when the integrands are L^1 maps. If β is bounded over $\mathcal{B}$ and τ_∂ is continuous with respect to x , by standard techniques we find that τ_∂ depends on ∂b just through the normal n in all Lebesgue points with respect to $\mathcal{H}^2$, namely $\tau_\partial = \tilde{\tau}(x, t, n)$; it is such that $\mathcal{A}^*(\tilde{\tau}(x, t, n) + \tilde{\tau}(x, t, -n)) = 0$ (a generalized action–reaction principle); it is also linear with respect to n , namely there is a tensor field $(x, t) \mapsto S(x, t)$ such that $\tilde{\tau}(x, t, n) = S(x, t)n(x)$. When this latter tensor field is C^1 over $\mathcal{B}$, there exists a field $(x, t) \mapsto z(x, t) \in T_{\tilde{v}(x, t)}^* \mathcal{M}$ such that, on $\mathcal{B}$,

$$\beta^\dagger - z + \operatorname{div} S = 0, \quad \mathcal{A}^* z + ((D\mathcal{A}^*)^t S) = 0, \quad (3)$$

where the superscript t indicates minor right transposition of third-rank tensors and D means derivative (its contravariant counterpart is the standard gradient). Eventually, we get the identity

$$\int_b \beta \cdot \dot{v} \, d\mu(x) + \int_{\partial b} \tau_\partial \cdot \dot{v} \, d\mathcal{H}^2 = \int_b (z \cdot \dot{v} + S \cdot D\dot{v}) \, d\mu(x), \quad (4)$$

and we call the right-hand side term an *internal power* (pertinent proofs are in references [48, 51]). When v is a scalar, since $\mathcal{A}(v)$ vanishes in this case, we can once again refer to the general setting, assuming v in an abstract manifold on which $\mathcal{A}$ is generically non-zero. Then we may reduce the resulting equations to the scalar case. Else, by taking into consideration at first the scalar choice, we can derive the pertinent balance equations from a variational principle in the conservative case, or from a d'Alembert–Lagrange principle when a dissipative potential plays a role, even from a requirement of covariance for the second law when even stresses may have dissipative components (we do not go into details here; we just remark that in all the last three procedures, which we can use also when v is not a scalar, we implicitly put on the same conceptual level the specific description of the state, i. e., the list of energy entries, and the balance equations, which emerge as independent of the constitutive structures when we call upon just the invariance of the external power, as above adopted). In the scalar case, equation (3)₂ does not play a role and the generalized action–reaction principle above reduces to the standard one.

The first law of thermodynamics writes now

$$\frac{d}{dt} \int_b e \, d\mu(x) = \mathcal{P}_b(\dot{v}) - \int_{\partial b} q \cdot n \, d\mathcal{H}^2(x) + \int_b r \, d\mu(x), \quad (5)$$

where e is the internal energy density, a differentiable function of its arguments (not yet specified at this level), and r the heat source. The second law reads

$$\frac{d}{dt} \int_b \eta \, d\mu(x) \geq - \int_{\partial b} \hat{h} \cdot n \, d\mathcal{H}^2(x) + \int_b s \, d\mu(x), \quad (6)$$

where η is the entropy density, also a differentiable function of its arguments, s the heat source, and h the entropy flux. This last two entities are related to q and r by the standard relations $\hat{h} = \frac{q}{T}$ and $s = \frac{r}{T}$. The arbitrariness of b , the relation (4), and the assumed differentiability of e and η imply an expression of the Clausius–Duhem inequality given by

$$\dot{\psi} - \eta \dot{T} - z \cdot \dot{v} - S \cdot D\dot{v} + \frac{1}{T} q \cdot DT \leq 0 \quad (7)$$

and assumed to be valid for *any* choice of the rate fields involved; in it $\psi := e - T\eta$ is the Helmholtz free energy density. As is well known, the inequality links conservative components of the *microstress* S and the *microstructural self-action* z with derivatives of the Helmholtz *free energy density* (see [46]). Precisely, when we take $\psi = \tilde{\psi}(x, T, v, Dv)$, with $\tilde{\psi}$ being a differentiable function, and also $\eta = \tilde{\eta}(x, T, v, Dv)$, $S = \tilde{S}(x, T, v, Dv)$, and $z = \tilde{z}(x, T, v, Dv)$, arbitrariness of the rates involved in inequality (7) implies $\eta = -\partial_T \psi$, $S = \partial_{Dv} \psi$, $z = \partial_v \psi$, and $\frac{1}{T} q \cdot DT \leq 0$, where $\partial_a := \frac{\partial}{\partial a}$. Of course, these are not the sole options at disposal. In fact, we may consider dissipative components of first-neighbor interactions (S) and self-actions (z) involved. In the absence of experimental data, explicit constitutive structures can be deduced by means of identification from (complex) lattice schemes collecting the essential features of the material behavior under specific analysis; the identification rests on appropriate variants of the Cauchy–Born rule (see, e. g., [52]).

This is the skeletal structure of the general model-building framework of the complex material mechanics.

A way to link this scheme with the question on heat propagation discussed here is to account for two possible internal constraints, depending on circumstances:

- We may assume, as in reference [16],

$$v = \ell(x, T), \quad (8)$$

with $\ell(\cdot, \cdot)$ being a differentiable function. (For example, polarization in various material classes depends on temperature.)

- We may also consider heat-driven microstructures, namely those characterized by an internal constraint of the type

$$v = f(x, t, q), \quad (9)$$

with f being a differentiable $\mathcal{M}$ -valued function [17]. (For example, when v describes the density of bubbles in a boiling process, it depends on the heat flux received by the liquid.) When $q = 0$, v reaches thermodynamic equilibrium values (see pertinent remarks in reference [17]).

With these choices we obtain different descriptions of temperature propagation, which admit finite speed. To enlighten better the idea justifying our choice to follow the present path, forget for a while the representation of microstructures, a circumstance in which the local dissipation inequality (7) would reduce to

$$\dot{\psi} - \eta \dot{T} + \frac{1}{T} q \cdot DT \leq 0, \quad (10)$$

assumed to hold for *any* choice of the rate fields involved. If we would take *ab initio* $\psi = \tilde{\psi}(x, T, v, Dv)$, computing its time derivative and inserting it in (10), the arbitrariness of $\dot{T}$ and $D\dot{T}$, which can be a priori independently chosen, would imply that ψ cannot depend on Dv unless we would prescribe a law for $D\dot{T}$, constraining it; in this case, we should justify the origin of such an imagined law. Analogously, if we would choose $\psi = \tilde{\psi}(x, q, v, Dq)$, also *ab initio*, for the arbitrariness of $\dot{q}$ and $D\dot{q}$ inequality (10) would exclude the dependence of ψ on q and Dq , unless we prescribe laws for $\dot{q}$ and $D\dot{q}$, once again with the need of justifying their origin.

In the setting leading to the dissipation inequality (7), we have naturally the necessary additional ingredients allowing structures of the free energy as those indicated above, provided that we introduce the internal constraints (8) and (9). Such ingredients, namely z and S , are true interactions, defined by the power that they develop in the microstructure rate of change. They satisfy balance equations derived by requirements of invariance under appropriate changes in observers.

Details below show how a hyperbolic description of heat conduction emerges in the present setting.

From now on we refer to orthonormal frames for the sake of simplicity so that we do not distinguish between the derivative operator D and its contravariant counterpart ∇ , the gradient.

3.1 Temperature-dependent microstructures

1. First, take $\beta = 0$. In this case the balance equation (3)₁ reduces to $z = \operatorname{div} S$ and, consequently, the density of internal power becomes $z \cdot \dot{v} + S \cdot \nabla \dot{v} = \operatorname{div}(S^*) \dot{v}$, where the asterisk in superscript position indicates formal adjoint.
2. Assume also that $\psi = \tilde{\psi}(T, v, \nabla v)$ and $z = z^e + z^d$, where $z^e := \tilde{z}^e(T, v, \nabla v)$ is the energetic component of z and is given by $\frac{\partial \psi}{\partial v}$ as a consequence of the Clausius–Duhem inequality, while $z^d := \tilde{z}^d(T, v, \nabla v, \dot{v})$ is a dissipative component such that $z^d \cdot \dot{v} \geq 0$ for any choice of $\dot{v}$, the equality sign holding when $\dot{v} = 0$. The arbitrariness of $\dot{v}$ in the previous inequality implies $z^d = a\dot{v}$, with $a \geq 0$ being the value at x and t of a positive definite function $\tilde{a}$.
3. Consider valid the Fourier law $q = -\kappa \nabla T$.
4. Take the pointwise form of the energy balance (5) by exploiting the arbitrariness of $\dot{b}$, namely

$$\dot{e} = z \cdot \dot{v} + S \cdot \nabla \dot{v} + r - \operatorname{div} q, \quad (11)$$

where we take the internal energy density to be $e := \tilde{e}(T, v, \nabla v)$.

5. Assume also the internal constraint (8), so that by the chain rule we find $\dot{v} = \lambda(x, T)\dot{T}$, with $\lambda(x, T) = \frac{\ell(x, T)}{\partial T} \in T_{\tilde{v}(x, t)} \mathcal{M}$, where $T_v \mathcal{M}$ is the tangent space of $\mathcal{M}$ at v .

Under these assumptions, we find that T satisfies the hyperbolic evolution equation

$$\zeta \cdot \nabla \dot{T} - k \Delta T + (c_v + \delta) \dot{T} + \xi \cdot \nabla T + (\hat{y} - r) = 0, \quad (12)$$

with $\zeta = -\frac{\partial(S^* \dot{v})}{\partial v} \lambda$, $c_v = \frac{\partial e}{\partial T}$, $\delta = \zeta \cdot \nabla \lambda$, $\xi = -\frac{\partial(S^* \dot{v})}{\partial T}$, $\hat{y} = -\frac{\partial(S^* \dot{v})}{\partial v} \cdot \nabla v$ (see proofs in reference [16]).

Equation (12) is a linearization of an expression derived in reference [16] under the same assumptions made here. We notice that an extra entropy flux could be accounted for as the one suggested by I. Müller [53] and used in various settings (even in the case of porous materials, as proposed by P. Giovine [54]). Here, we leave apart that additional flux on one side for the sake of simplicity and on the other side because in this way we are able to show that our setting allows finite speed propagation of temperature disturbances even under Fourier's law, as pointed out in Section 4.

3.2 Heat-flux-dependent microstructures

Consider now the internal constraint (9).

1. Take v to be a scalar, a phase field, indeed, and presume more specifically that $v = \varsigma q + g(x, t)$, with ς being a positive constant and $g(x, t)$ a differentiable function of space and time.
2. Assume as above that $z = z^e + z^d$ and that the free energy density ψ does not depend on ∇q and is quadratic with respect to q , so that $z = \varpi v + a\dot{v}$, with ϖ and a being positive constants and $S = 0$.
3. Take also $\beta = \zeta T_x$, with ζ being a positive constant.

Under these assumptions, the balance of energy and the one of microstructural interactions reduce respectively to

$$\frac{\partial e}{\partial T} \dot{T} + \left(\frac{\partial e}{\partial q} - \varpi \varsigma^2 q \right) \cdot \dot{q} + \operatorname{div} q - a \varsigma^2 \dot{q} \cdot \dot{q} = 0, \quad (13)$$

$$a \varsigma \dot{q} + \varpi \varsigma q = \zeta \nabla T, \quad (14)$$

where, for the sake of simplicity, we assume also $\frac{\partial e}{\partial T}$ to be constant. If we set $\tau := \varpi^{-1}a$ and $\kappa := -(\varpi \varsigma)^{-1}\zeta$ in equation (14), we recover Cattaneo's equation accompanied by a modified balance of energy.

Perturbed classical solutions of traveling wave type for such a system have been determined in reference [18]. They are of the form

$$T(t, x) = T_0(t, x) + \lambda \phi(t) e^{2\beta_2 x}, \quad q(t, x) = q_0(t, x) + \phi(t) e^{2\beta_2 x}, \quad (15)$$

with initial condition

$$\phi(0) = \varphi(0) = 0. \quad (16)$$

The two functions ϕ and φ are linked by the relation

$$\phi(t) = -2\beta_2 \frac{k}{\tau} \int_0^t e^{(s-t)/\tau} \varphi(s) ds. \quad (17)$$

Detailed analyses can be found in reference [18]. For this reason, below we discuss only traveling-wave-type solutions for equation (12). We also remark that when we presume the microstress S is different from zero, we may derive from the balance of microstructural actions the Guyer–Krumhansl equation coupled with a non-standard form of the energy balance.

4 Classical traveling wave solutions in a rigid conductor under Fourier's law but with temperature-dependent microstructures

4.1 The character of equation (12)

Let us assume the coefficients in equation (12) to be real, constant, and non-zero. The pertinent characteristic matrix is given by

$$C = \begin{pmatrix} 0 & \zeta^1 & \zeta^2 & \zeta^3 \\ \zeta^1 & -k & 0 & 0 \\ \zeta^2 & 0 & -k & 0 \\ \zeta^3 & 0 & 0 & -k \end{pmatrix},$$

where ζ^i are the components of ζ . The characteristic polynomial $P(C) = \det(C - \lambda I)$ of C is

$$P(C) = (\lambda + k)^2 (\lambda^2 + k\lambda - |\zeta|^2).$$

Thus, the eigenvalues of C are

$$\lambda_{1,2} := \frac{1}{2}(-k \pm \sqrt{|\zeta|^2 + k^2}), \quad \lambda_3 := -k.$$

Clearly, $\lambda_1, \lambda_2, \lambda_3$ are distinct and non-zero because $k < \sqrt{|\zeta|^2 + k^2}$. Also, λ_1 and λ_2 , as simple roots of $P(C)$, have algebraic (and thus geometric) multiplicity 1. The eigenvalue λ_3 has algebraic multiplicity 2 and, as shown below, its geometric multiplicity is 2 as well. So, the matrix C can be expressed in diagonal form since algebraic and geometric multiplicities of its eigenvalues coincide.

As regards the geometric multiplicity of λ_3 , the vectors

$$w_1 = \begin{cases} (0, 0, \zeta^3, -\zeta^2)^T & \text{if } \zeta^3 \neq 0, \\ (0, \zeta^2, -\zeta^1, 0)^T & \text{otherwise,} \end{cases} \quad w_2 = \begin{cases} (0, \zeta^3, 0 - \zeta^1)^T & \text{if } \zeta^3 \neq 0, \\ (0, 0, 0, 1)^T & \text{otherwise} \end{cases}$$

are linearly independent eigenvectors of C that span the eigenspace of λ_3 whenever $|\zeta| \neq 0$.

In summary, when $|\zeta| \neq 0$ and $k \neq 0$, equation (12) is hyperbolic. Despite the presence of mixed derivatives, two bounded characteristic velocities are associated with it.

4.2 Equation (12) in the scalar case and pertinent traveling waves

Consider temperature-dependent microstructures as described above and, for the sake of simplicity, take v to be a scalar and the body to be one-dimensional. Equation (12) reads

$$\zeta \frac{\partial^2 T}{\partial t \partial x} - k \frac{\partial^2 T}{\partial x^2} + (c_v + \delta) \frac{\partial T}{\partial t} + \xi \frac{\partial T}{\partial x} + \hat{y} - r = 0. \quad (18)$$

Once again we assume all the coefficients to be constant and non-zero. We seek classical traveling wave solutions of (18), that is, we aim at determining a C^2 function $u: \mathbb{R} \rightarrow \mathbb{R}$ and a real constant $c \neq 0$ such that $T(t, x) = u(x - ct)$. By imposing the latter condition and setting $s = x - ct$, equation (18) boils down to

$$-(c\zeta + k)u''(s) + (\xi - c(c_v + \delta))u'(s) + \hat{y} - r = 0. \quad (19)$$

If we take $c = -k/\zeta$, equation (19) reduces to

$$\left(\xi + \frac{k}{\zeta}(c_v + \delta) \right) u'(s) + \hat{y} - r = 0,$$

so that, for arbitrary constant K ,

$$u(s) = -\frac{\hat{y} - r}{\xi + \frac{k}{\zeta}(c_v + \delta)} s + K,$$

which yields the family of traveling wave solutions

$$T(t, x) = \frac{r - \hat{y}}{\xi + \frac{k}{\zeta}(c_v + \delta)} \left(x + \frac{k}{\zeta} t \right) + K, \quad (20)$$

with “affine” initial condition

$$T(0, x) = \frac{r - \hat{y}}{\xi + \frac{k}{\zeta}(c_v + \delta)} x + K. \quad (21)$$

Similarly, when $c = \xi/(c_v + \delta)$, by direct integration of (19), we find for arbitrary constants K_1, K_2

$$T(t, x) = \frac{(\hat{y} - r)(c_v + \delta)}{\xi\zeta + kc_v + k\delta} \left(x - \frac{\xi t}{c_v + \delta} \right)^2 + K_1 \left(x - \frac{\xi t}{c_v + \delta} \right) + K_2, \quad (22)$$

so that $T(0, x) = \frac{(\hat{y}-r)(c_v+\delta)}{\xi\zeta+kc_v+k\delta}x^2 + K_1x + K_2$.

For $c \neq -k/\zeta$ and $c \neq \xi/(c_v + \delta)$, by setting $v(s) = u'(s)$, equation (19) becomes

$$-(c\zeta + k)v'(s) + (\xi - c(c_v + \delta))v(s) + \hat{y} - r = 0,$$

so that, for an arbitrary constant K_1 ,

$$v(s) = K_1 e^{\frac{\xi - c(c_v + \delta)}{c\zeta + k}s} - \frac{\hat{y} - r}{\xi - c(c_v + \delta)}.$$

By integrating and taking into account that K_1 is arbitrary, we get

$$u(s) = K_1 e^{\frac{\xi - c(c_v + \delta)}{c\zeta + k}s} - \frac{\hat{y} - r}{\xi - c(c_v + \delta)}s + K_2,$$

where K_2 is another constant. Thus, we find the following family of traveling wave solutions of (18):

$$T(t, x) = K_1 e^{\frac{\xi - c(c_v + \delta)}{c\zeta + k}(x - ct)} - \frac{\hat{y} - r}{\xi - c(c_v + \delta)}(x - ct) + K_2. \quad (23)$$

Clearly, we have

$$T(0, x) = K_1 e^{\frac{\xi - c(c_v + \delta)}{c\zeta + k}x} - \frac{\hat{y} - r}{\xi - c(c_v + \delta)}x + K_2. \quad (24)$$

Hence, assuming $c \notin \{-k/\zeta, \xi/(c_v + \delta)\}$, only initial conditions of the form $h(x) = Ae^{\sigma x} + Bx + C$, A, B, C, σ being constants, may lead to classical traveling wave solutions. Comparing (24) we see that, although A and C can be assigned freely, B and σ are not independent of each other; in fact, we must have

$$\sigma = \frac{\xi - c(c_v + \delta)}{c\zeta + k} \quad \text{and} \quad B = -\frac{\hat{y} - r}{\xi - c(c_v + \delta)}.$$

Whence, when $\zeta\sigma + c_v + \delta \neq 0$ and $\xi\zeta + kc_v + k\delta \neq 0$,

$$c = \frac{\xi - k\sigma}{\zeta\sigma + c_v + \delta} \quad \text{and} \quad B = -\frac{(\hat{y} - r)(\zeta\sigma + c_v + \delta)}{\sigma(\xi\zeta + kc_v + k\delta)}. \quad (25)$$

The relations (25) tell us that the choice of σ is not completely free: in order to get a finite value for c and B one must have $\sigma \neq -\frac{1}{\zeta}(c_v + \delta)$ and $\sigma \neq 0$. For such σ , by (25) we get

$$T(t, x) = Ae^{\sigma(x - \frac{\xi - k\sigma}{\zeta\sigma + c_v + \delta}t)} - \frac{(\hat{y} - r)(\zeta\sigma + c_v + \delta)}{\sigma(\xi\zeta + kc_v + k\delta)}\left(x - \frac{\xi - k\sigma}{\zeta\sigma + c_v + \delta}t\right) + C. \quad (26)$$

In summary, equation (18), with initial condition $T(0, x) = h(x)$, admits (classical) traveling wave solutions in the following three circumstances:

1. If

$$h(x) = \frac{r - \hat{y}}{\xi + \frac{k}{\zeta}(c_v + \delta)}x + K$$

for some constant K , the solution is given by formula (20) and the wave speed is

$$c = -k/\zeta.$$

2. If

$$h(x) = \frac{(\hat{y} - r)(c_v + \delta)}{\xi\zeta + kc_v + k\delta}x^2 + K_1x + K_2,$$

for some constants K_1, K_2 , the solution is given by formula (22) and the wave speed is

$$c = \xi/(c_v + \delta).$$

3. If

$$h(x) = Ae^{\sigma x} - \frac{(\hat{y} - r)(\zeta\sigma + c_v + \delta)}{\sigma(\xi\zeta + kc_v + k\delta)}x + C$$

for some constants A, C, σ , with $\sigma \neq -\frac{1}{\zeta}(c_v + \delta)$ and $\sigma \neq 0$, the solution is given by formula (26) and the wave speed is

$$c = (\xi - k\sigma)/(\zeta\sigma + c_v + \delta).$$

In all three cases we have to assume $\xi\zeta + kc_v + k\delta \neq 0$ for the sought traveling wave solutions to exist.

In the above item (1), unless $r = \hat{y}$ (a circumstance excluded here), the initial condition $T(0, x) = h(x)$ cannot be physically admissible because, regardless of the choice of K , $h(x)$ becomes negative for large $|x|$, i. e., for large x , be it positive or negative.

For what is described in item (2), it is possible to assign a non-negative initial condition (i. e., to choose the constants K_1 and K_2 so that $h(x) \geq 0$ for all $x \in \mathbb{R}$) if and only if

$$\frac{(\hat{y} - r)(c_v + \delta)}{\xi\zeta + kc_v + k\delta} > 0.$$

Conditions ensuring $h(x) > 0$ for all $x \in \mathbb{R}$ in item (3) emerge from a geometric analysis of the relative positions between the exponential curve $y = Ae^{\sigma x}$ and the straight line

$$y = \frac{(\hat{y} - r)(\zeta\sigma + c_v + \delta)}{\sigma(\xi\zeta + kc_v + k\delta)}x - C.$$

There are two cases of interest:

1. If

$$\frac{(\hat{y} - r)(\zeta\sigma + c_v + \delta)}{\sigma(\xi\zeta + kc_v + k\delta)} > 0,$$

we must take $\sigma > 0$ and $A > 0$. Then, there exists C_0 such that $h(x) > 0$ for all choices of $C > C_0$.

2. If

$$\frac{(\hat{y} - r)(\zeta\sigma + c_v + \delta)}{\sigma(\xi\zeta + kc_v + k\delta)} < 0,$$

we must take $\sigma < 0$ and $A > 0$. Then, there exists C_0 such that $h(x) > 0$ for all choices of $C > C_0$.

5 Additional remarks

Considering a direct description of material microstructures constitutes a consistent path along which we can obtain a description of heat transfer that foresees finite speed temperature propagation even if we accept Fourier's law. The setting that we propose offers various modeling possibilities: in fact, what is presented in Sections 3.1 and 3.2 can be generalized by varying appropriately the constitutive assumptions listed above.

The emerging specific models complement other theories, summarized in Section 2. Specific circumstances may suggest one option or another. The choice is often influenced by one's own knowledge and preferences.

Of course, consequences are not invariant with respect to choices, each requiring (at least ideally) awareness of the conceptual foundations it rests on with the pertinent limits.

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